Graph-Theoretic Approach to Symbolic Analysis of Linear Descriptor Systems

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ABSTRACT

Continuous descriptor systems $E\dot{x} = Ax + Bu$, $y = Cx$, where E is a possibly singular matrix, are symbolically analyzed by means of digraphs. Starting with four different digraph characterizations of square matrices and determinants, the author favors the Cauchy-Coates interpretation. Then, an appropriate digraph representation of the matrix pencil $(sE - A)$ is given, which is followed by a digraph interpretation of $\det(sE - A)$ and the transfer-function matrix $C(sE - A)^{-1}B$. Next, a graph-theoretic procedure is derived to reveal a possibly hidden factorizability of the determinant $\det(sE - A)$. This is very important for large-scale systems. Finally, as an application of the derived results, an electrical network is analyzed symbolically.

1. INTRODUCTION

Since the nineteen sixties, the state-space theory has been widely accepted as so-called "modern control theory" by the control engineers' community. Unfortunately, due to cumbersome matrix manipulations which are typical of this approach to plant analysis and controller synthesis, control engineers lose desirable "feeling" and visual insight. The numerical results may be greatly sensitive to small variations of the numerical values for the matrix entries. The practicing engineer, however, has to cope with more or less uncertain parameters.

The *graph-theoretic approach* has been developed as an attempt to overcome the disadvantages of the numerically oriented state-space theory

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(see the monographs of Franksen, Falster, and Evans, 1979; Siljak, Pichai, and Sezer, 1982; Andrei, 1985; Murota, 1987; Reinschke, 1988; Trave, Titli, and Tarras, 1989; Wend, 1993). Moreover, especially for large-scale systems, the state-space description in standard form

$$\dot{x} = Ax + Bu, \qquad y = Cx \tag{1}$$

may not be considered to be a natural system description. In many applications it is rather difficult and expensive to transform a given natural system description which appears as a mixture of differential equations and purely algebraic constraints into state-space equations (1). A system description of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t), \tag{2}$$

where E is allowed to be a singular matrix, is much better suited. Here the vectors $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^r$ denote the descriptor variables, input variables, and output descriptor systems (2)—synonymous terms: semistate systems, generalized state-space systems, implicit systems, differential-algebraic equations, singular systems—have attracted the increasing interest of many researchers (e.g. Luenberger, 1977; van Dooren, Verghese, and Kailath, 1979; Campbell, 1980; Yip and Sincovec, 1981; van der Weiden, 1983; Cobb, 1984; Yamada and Luenberger, 1985; Willems, Kitapci, and Silverman, 1986; Griepentrog and März, 1986; Bender and Laub, 1987; Murota, 1987; Gear, 1988; Shyaman, 1988; Fahmy and O'Reilly, 1989; Dai, 1989; Reinschke, 1989; Hairer, Lubich, and Roche, 1989; Brenan, Campbell, and Petzold, 1989; Mehrmann and Krause, 1989; Bunse-Gerstner, Mehrmann, and Nichols, 1991; Hairer and Wanner, 1991).

In most real-world applications, the numerical parameters influencing the nonzero entries of the matrices E, A, B, and C are more or less uncertain. That is why both theorists and practicing engineers are interested in methods which enable them to analyze descriptor systems (2) symbolically. The graph-theoretic approach paves the way for the symbolic analysis of descriptor systems.

2. DIGRAPH CHARACTERIZATIONS OF SQUARE MATRICES AND DETERMINANTS

Let Q be a square matrix of order n,

$$Q = (q_{ij}) \qquad \text{for} \quad i, j = 1, 2, \dots, n, \tag{3}$$

where the matrix elements q_{ij} are real numbers. There are several possibilities of constructing weighted digraphs that have a one-to-one correspondence with a given square matrix Q; see Reinschke (1988, Chapter A1.3). For example, consider a matrix Q of order 3,

$$Q = \begin{pmatrix} q_{11} & 0 & q_{13} \\ q_{21} & q_{22} & q_{23} \\ 0 & q_{32} & 0 \end{pmatrix}. \tag{4}$$

In Figure 1, the example matrix (4) has been characterized graph-theoretically in four different manners.

Characterization I (e.g. König, 1916, 1936, Ford and Fulkerson, 1962). There is a one-to-one correspondence between the given square matrix (3) and a bipartite graph, where

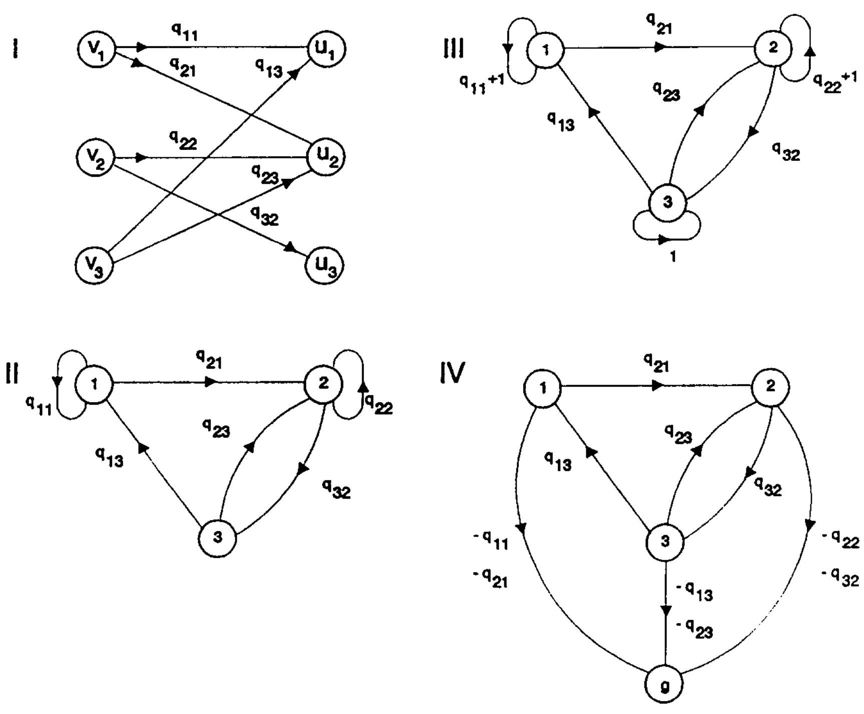

FIG. 1. Four different digraph characterizations.

(1) each row of Q corresponds to one of the vertices $u_1, u_2, \ldots, u_n$;
(2) each column of Q corresponds to one of the vertices $v_1, v_2, \ldots, v_n$;
(3) each entry $q_{ij} \neq 0$ corresponds to an edge from v_j to u_i with the weight q_{ij}.

Characterization II (Cauchy, 1815; Coates, 1959). There is a one-to-one correspondence between the square matrix (3) and a weighted digraph $G(Q)$ which has n vertices $v_1, v_2, \ldots, v_n$ and a directed edge from the initial vertex v_j to the final vertex v_i if the matrix element q_{ij} does not vanish $(i, j = 1, 2, \ldots, n)$. The edge weight is equal to the value of q_{ij}.

Characterization III (Mason, 1953, 1956). There is a one-to-one correspondence between the square matrix (3) and a weighted digraph which has n vertices $v_1, v_2, \ldots, v_n$ and, for $i \neq j$, a directed edge from v_j to v_i with weight q_{ij} if $q_{ij} \neq 0$, and, for $i = j$, a self-cycle at v_i with weight $q_{ii} + 1$ if $q_{ii} \neq -1$.

Characterization IV (Kirchhoff, 1847; Reinschke, 1985). There is a one-to-one correspondence between the square matrix (3) and a weighted digraph $G'(A)$ which has $n + 1$ vertices $v_1, v_2, \ldots, v_n, g$ and, for $i \neq j = 1, 2, \ldots, n$, an edge from v_j to v_i with weight q_{ij} if $q_{ij} \neq 0$, and, for $j = 1, 2, \ldots, n$, an edge from v_j to g with weight

$$\left(-\sum_{i=1}^{n} q_{ij}\right) \qquad \text{provided that} \quad \sum_{i=1}^{n} q_{ij} \neq 0.$$

The determinant $\det Q$ of an $n \times n$ matrix Q may be defined by

$$\det Q = \sum_{\substack{\text{even} \\ \text{permutations}}} \prod_{i=1}^{n} q_{it_i} - \sum_{\substack{\text{odd} \\ \text{permutations}}} \prod_{i=1}^{n} q_{it_i}, \tag{5}$$

where $\{t_1, t_2, \ldots, t_n\}$ is a permutation of $\{1, 2, \ldots, n\}$. For the example matrix (4) one obtains

$$\det Q = q_{13}q_{21}q_{32} - q_{11}q_{23}q_{32}. \tag{6}$$

Each of the four characterizations of a square matrix Q introduced above may be used as a starting point for a graph-theoretic interpretation of the determinant $\det Q$. The main problem is how to interpret the nonvanishing expressions

$$q_{1t_1}q_{2t_2} \cdots q_{nt_n} \tag{7}$$

graph-theoretically.

INTERPRETATION I. Each summand (7) of det Q corresponds to a set of n edges incident with all the $2n$ vertices of the bipartite graph. (A subgraph of this kind was called a factor of first degree by D. König.)

For the example (4), the bipartite digraph of Figure 1 contains two factors of first degree, which are shown in Figure 2.

In the following, a few graph-theoretic concepts are needed: path, length, cycle, cycle family, and grounded tree. A *path* is a sequence of edges $\{e_1, e_2, \dots\}$ such that the initial vertex of each succeeding edge is the final vertex of the preceding edge. The number of edges contained in the sequence $\{e_1, e_2, \dots\}$ is called the *length of the path*. The initial vertex of the first edge and the final vertex of the last edge of $\{e_1, e_2, \dots\}$ are called the initial vertex of the path and final vertex of the path, respectively. A closed path is a path whose initial and final vertices are the same.

A closed path is said to be a *cycle* if, going along the path, one reaches no vertex, other than the initial-final vertex, more than once.

A set of vertex disjoint cycles is said to be a *cycle family*.

A *tree* is a connected subgraph whose number of edges is one less than the number of vertices.

A *grounded tree* is a tree that has a ground vertex to which there is a unique path from every other vertex.

Now, the remaining three graph-theoretic interpretations of determinants may be easily formulated.

INTERPRETATION II. Each summand (7) of det Q corresponds to a cycle family of length n in the weighted digraph $G(Q)$. The value of (7) is given by the products of the weights of the n edges involved. If this cycle family

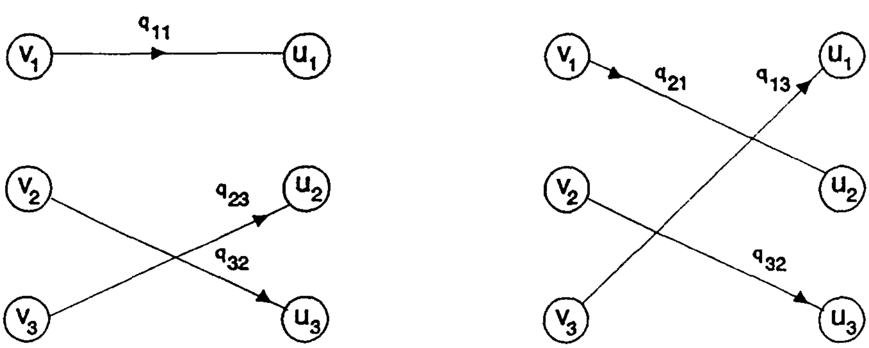

FIG. 2. Factors of first degree representing the determinant (6).

consists of d disjoint cycles, then the sign factor of (7) to be taken into account in (5) is $(-1)^{n-d}$.

For the example (4), the digraph $G(Q)$ (see Figure 1, characterization II) contains two cycle families of length $n = 3$. The sign factor of the cycle family drawn on the left of Figure 3 is $(-1)^{3-2} = -1$, whereas the cycle family on the right has a sign factor $(-1)^{3-1} = (-1)^2 = 1$.

INTERPRETATION III. Mason's rule for evaluating the determinant of a square matrix (3) says:

$$\det Q = (-1)^n + \sum_{k=1}^{n} (-1)^{n-k} S^{(k)}, \tag{8}$$

where $S^{(k)}$ denotes the sum of weights of all the subgraphs (within the digraph introduced as characterization III) consisting of k vertex disjoint cycles.

For the example matrix (4), the digraph characterization III (compare Figure 1) contains the five individual cycles shown in Figure 4. From this figure it is immediately seen that

$$S^{(1)} = (q_{11}+1) + (q_{22}+1) + 1 + q_{23}q_{32} + q_{12}q_{21}q_{32},$$

$$S^{(2)} = (q_{11}+1)\left[(q_{22}+1) + 1 + q_{23}q_{32}\right] + (q_{21}+1)\cdot 1,$$

$$S^{(3)} = (q_{11}+1)(q_{22}+1)\cdot 1,$$

whence

$$\begin{aligned}\det Q &= -1 + S^{(1)} - S^{(2)} + S^{(3)}\\ &= q_{13}q_{21}q_{32} - q_{11}q_{23q}q_{32}.\end{aligned}$$

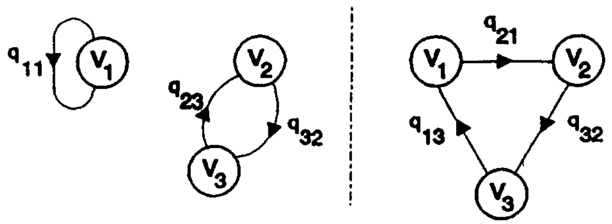

FIG. 3. Cycle families representing the determinant (6).

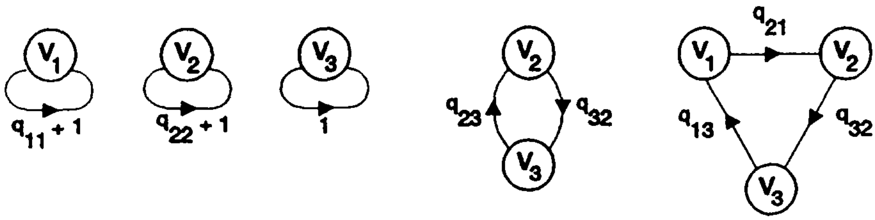

FIG. 4. Cycles contained in digraph III of Figure 1.

INTERPRETATION IV. The determinant $\det(-Q)$ can be determined in the digraph introduced by characterization IV as the sum of the weights of all grounded trees.

For the example matrix (4), there exist nine grounded trees, shown in Figure 5. One obtains

$$\begin{aligned}\det(-Q) &= q_{13}q_{31}(-q_{11}-q_{21}) + q_{13}q_{21}(-q_{22}-q_{32})\\ &\quad + q_{21}q_{32}(-q_{13}-q_{23})\\ &\quad + q_{21}(-q_{22}-q_{32})(-q_{13}-q_{23})\\ &\quad + q_{21}q_{23}(-q_{22}-q_{32})\\ &\quad + q_{13}(-q_{11}-q_{21})(-q_{22}-q_{32})\\ &\quad + q_{23}(q_{11}-q_{21})(-q_{22}-q_{32})\\ &\quad + q_{32}(-q_{11}-q_{21})(-q_{13}-q_{23})\\ &\quad +(-q_{11}-q_{21})(-q_{13}-q_{23})(-q_{22}-q_{32})\\ &= -q_{32}q_{21}q_{13} + q_{11}q_{23}q_{32}.\end{aligned}$$

From the mathematical point of view, each of the four representations could be used with the same propriety. In the analysis of real systems, however, special matrix structures may be typical of the applications under consideration. Depending on the given matrix structure, different representations may be more or less suited. For example, the nodal analysis of electrical networks leads to matrices for which representation IV may often be regarded as most natural. As for control systems, due to the influence of

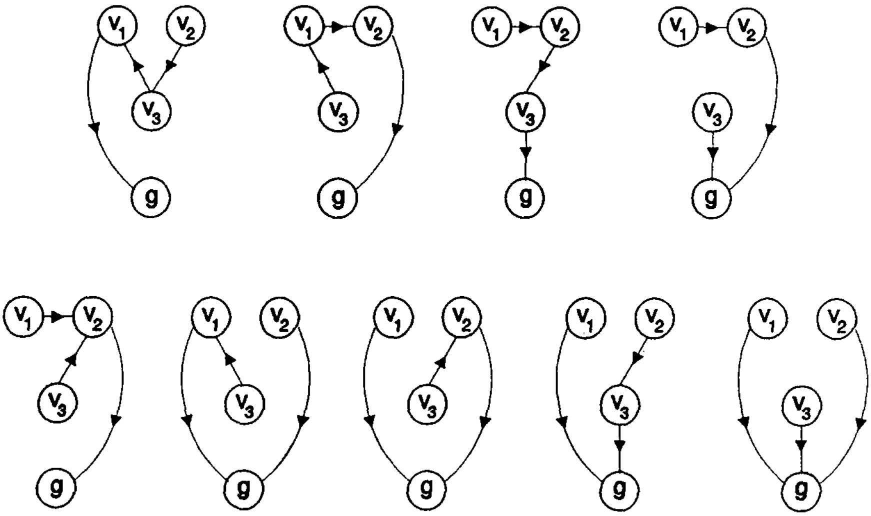

FIG. 5. Grounded trees contained in digraph IV of Figure 1.

popular textbooks, many control engineers favor representation III. Control systems are characterized by feedback loops and a rather general matrix structure. Therefore, representation II, which is based on the concept of cycle families in the simplest manner, seems to be yet more advantageous. In the sequel, we shall restrict ourselves to digraph representation II. Other authors prefer other digraph representations; see, for example, Murota (1987). The decomposition algorithm in Section 5 gives a representation-II version of the decomposition idea published by Dulmage and Mendelsohn for representation I as early as in 1959. In a more general setting, recent results in combinatorial approaches to dynamical systems have been obtained by Murota (1989).

3. DIGRAPH INTERPRETATION OF $\det(sE - A)$

For descriptor systems (2), the matrix pencil

$$sE - A, \qquad \text{where} \quad s \in \mathbb{C}, \tag{9}$$

plays a crucial role. Based on the graph-theoretic characterization II of square matrices, the matrix pencil (9) may be interpreted graphically as follows:

There is a one-to-one correspondence between the matrix pencil (9) and a digraph $G(sE - A)$ that consists of n vertices denoted by $1, 2, \dots, n$, and edges from vertex j to vertex i with weight $-a_{ij}$ if $a_{ij} \neq 0$ as well as edges from j to i with weight se_{ij} if $e_{ij} \neq 0$, where $i, j = 1, 2, \dots, n$.

For short, in drawing $G(sE - A)$ we can use full lines for nonvanishing A-entries and dotted lines for nonvanishing E-entries. Then, the complex scalar s may be omitted in the drawings. For example, let

$$n = 3, \qquad E = \begin{pmatrix} 0 & 0 & e_{13} \\ 0 & e_{22} & e_{23} \\ 0 & 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix}. \tag{10}$$

All the information contained in (10) is reflected by the digraph $G(sE - A)$ shown in Figure 6.

Based on the Cauchy-Coates interpretation of determinants (compare Section 2, characterization II), the characteristic polynomial

$$\det(sE - A) = \sum_{i=0}^{n} p_i s^{n-i} \tag{11}$$

may be interpreted graph-theoretically.

THEOREM 1. *The coefficients p_i $(0 \leqslant i \leqslant n)$ of the characteristic polynomial* (11) *are determined by those cycle families of length n in $G(sE - A)$ which involve exactly i A-edges (or, equivalently, $n - i$ E-edges): Each such cycle family corresponds to a summand of $p_i s^{n-i}$. The value of the summand results from the product of weights of edges involved in the cycle family,*

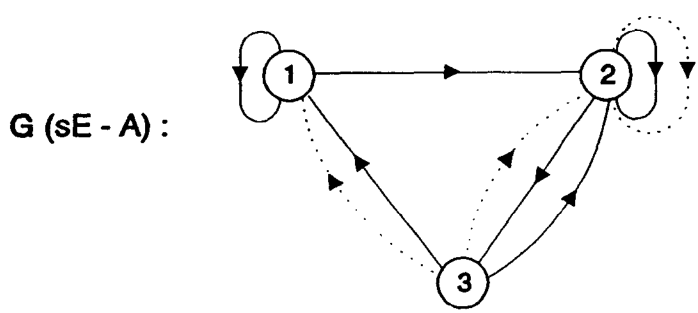

FIG. 6. Digraph $G(sE - A)$ belonging to the example system (10).

multiplied by a sign factor $(-1)^{n-i-d}$, *where d denotes the number of individual cycles forming the cycle family under consideration.*

For the example system (10), all cycle families of length n = 3, and the corresponding sign factors have been drawn in Figure 7. In this case we get the coefficients p_i (i = 0, 1, 2, 3) as

$$p_0 = 0, \qquad p_1 = 0, \qquad p_2 = e_{13}a_{32}a_{21} - e_{23}a_{23}a_{32}a_{11},$$
$$p_3 = -a_{13}a_{32}a_{21} + a_{11}a_{23}a_{32}.$$

Theorem 1 implies interesting consequences for all descriptor systems (2) with the same structure determined by the boolean structure matrices [A], [E].

DEFINITION (See Reinschke, 1988). The elements of a *boolean structure matrix* $[Q]$ are either fixed at zero or indeterminate values which are assumed to be independent of one another.

A numerical matrix Q is called an *admissible numerical realization* of $[Q]$ (for short, $Q \in [Q]$) if Q can be obtained by fixing all indeterminate entries of $[Q]$ at some particular values.

Two matrices Q' and Q'' are said to be *structurally equivalent* if both $Q' \in [Q]$ and $Q'' \in [Q]$.

Now, many structural properties of (2) may be checked graphically by looking at the digraph $G(sE - A)$.

COROLLARIES TO THEOREM 1. *The given descriptor system* (2) *is structurally degenerate, i.e.,*

$$\det(sE - A) = 0 \qquad \forall s \in \mathbb{C},\ A \in [A],\ E \in [E], \tag{12}$$

iff there exists no cycle family of length n in $G(sE - A)$.

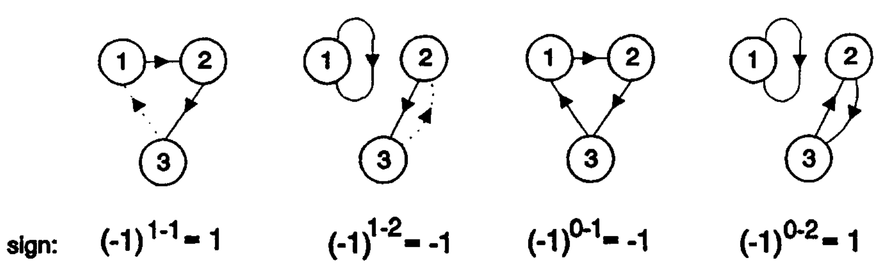

FIG. 7. Cycle families of length 3 contained in the digraph $G(sE - A)$ of Figure 6.

The matrix E is structurally singular, i.e.,

$$\text{term-rank}[E] < n, \tag{13}$$

iff there exists no cycle family of length n consisting of E-edges only.

The matrix A is structurally singular, i.e.,

$$\text{term-rank}[A] < n, \tag{14}$$

iff there exists no cycle family of length n consisting of A-edges only.

The generic degree g of the polynomical (11), *i.e.*

$$g = \textit{generic degree of } \det(sE - A), \tag{15}$$

is given by the maximal number of E-edges occurring in a cycle family of length n.

4. DIGRAPH INTERPRETATION OF THE TRANSFER-FUNCTION MATRIX

The $r \times m$ transfer-function matrix is defined by

$$T(s) = C(sE - A)^{-1}B. \tag{16}$$

The entry $t_{ji}(s)$ is the transfer function from the ith input to the jth output:

$$t_{ji}(s) = c_j'(sE - A)^{-1}b_i = \frac{c_j'(sE - A)_{\text{adj}}b_i}{\det(sE - A)} = \sum_{k=1}^{n} \frac{p_{k,ji}s^{n-k}}{\det(sE - A)}$$

$$= [\det(sE - A)]^{-1} \det\begin{pmatrix} 0 & c_j' & 0 \\ 0 & sE - A & b_i \\ -1 & 0 & 0 \end{pmatrix}. \tag{17}$$

The denominator polynomial has been treated in Section 3. The numerator polynomial may be associated with the supplemented digraph sketched in Figure 8.

The digraph $G(sE - A)$ has been supplemented by

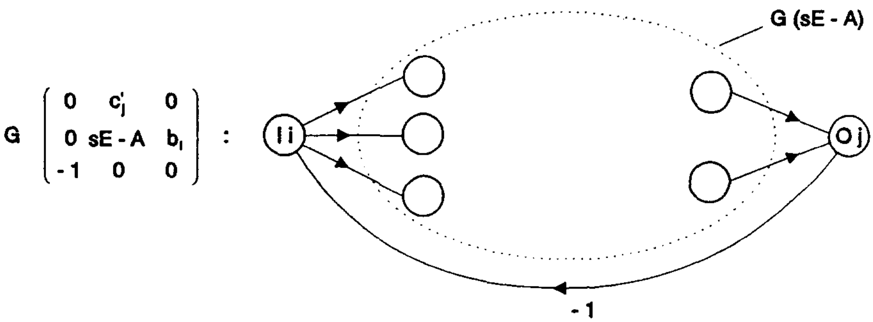

FIG. 8. Supplemented digraph to interpret the numerator polynomial of the transfer function $t_{ji}(s)$.

(1) two new vertices: an input vertex Ii and an output vertex Oj;

(2) input edges leading from Ii to state vertices according to the structure of the column vector b_i;

(3) output edges leading from state vertices to Oj according to the structure of the row vector c_j';

(4) a feedback edge leading form Oj to Ii with weight -1.

On the supplemented digraph, all cycle families of length $n + 2$ must contain the feedback edge from Oj to Ii. For brevity's sake, they are called (j, i)-feedback-cycle families.

THEOREM 2. *The coefficients $p_{k,ji}$ of the numerator polynomial $c_j'(sE - A)_{\mathrm{adj}} b_i$ in (17) are determined by the (j, i)-feedback-cycle families which contain exactly $n - k$ E-edges: Each such cycle family corresponds to a summand of $p_{k,ji} s^{n-k}$. The value of the summand results from the product of weights of edges involved, multiplied by a sign factor $(-1)^{n-k-d}$, where d denotes the number of individual cycles forming the cycle family under consideration.*

To give an example, let us supplement the system (10) by $m = 1$ input and $r = 1$ output, where

$$b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad c' = (1 \quad 1 \quad 0). \tag{18}$$

The supplemented digraph and the (1, 1)-feedback-cycle families are shown in Figure 9. The rules of Theorem 2 yield the numerator polynomial as

$$\begin{aligned}
&(1 \quad 1 \quad 0)(sE - A)_{\mathrm{adj}}\begin{pmatrix}0\\0\\1\end{pmatrix}\\
&\quad = -(-1)(se_{13} - a_{13})(se_{22} - a_{22})\\
&\qquad +(-1)(se_{13} - a_{13})(-a_{21}) - (-1)(se_{23} - a_{23})(-a_{11})\\
&\quad = e_{13}e_{22}s^2 + (-e_{13}a_{22} - e_{22}a_{13} + e_{13}a_{21} - e_{23}a_{11})s\\
&\qquad + (a_{13}a_{22} - a_{13}a_{21} + a_{11}a_{23}).
\end{aligned}$$

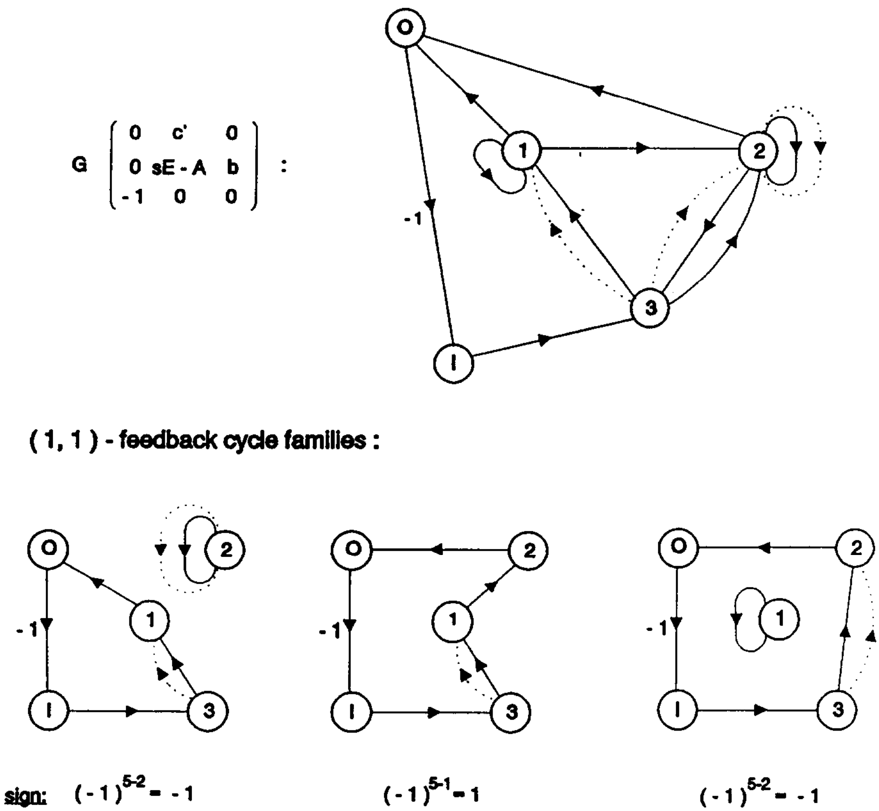

FIG. 9. Supplemented digraph and (1, 1)-feedback-cycle families for the example system defined by (10) and (18).

Results related to Theorem 2 may be found in Ohta and Komada (1985) and van der Woude (1991). There, the concepts of cycle families and feedback-cycle families are not used.

5. LOOKING FOR FACTORIZATIONS OF LARGE-SCALE SYSTEMS

It is well known from the theory of determinants that the absolute value of det Q is invariant with respect to permutations of rows and columns of Q. The topological properties of the digraph $G(Q)$, however, may depend heavily on such line permutations.

Practical experience with large-scale determinants shows that the determinant may split up into a product of k (> 1) subdeterminants of order $n_1, \dots, n_k$, i.e.,

$$\det Q = \pm \prod_{i=1}^{k} \det Q_{ii}, \qquad \sum_{i=1}^{k} n_i = n. \tag{19}$$

Unfortunately, such a factorization may not be recognizable at first glance from $G(Q)$. After appropriate permutations of rows and/or columns we can get a suitable representation of Q where the relation (19) becomes evident.

The desired representation of Q may be obtained by permutation of the rows (apply a permutation matrix P_r from the left) and by permutation of the columns (apply a permutation matrix P_c from the right). So our next aim is to find permutation matrices P_r and P_c such that

$$P_r Q P_c = \tilde{Q} = \begin{pmatrix} \tilde{Q}_{11} & * & \cdots & * \\ 0 & \tilde{Q}_{22} & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \tilde{Q}_{kk} \end{pmatrix}. \tag{20}$$

Because det Q = |det $\tilde{Q}$|, the hypermatrix structure of $\tilde{Q}$ ensures the factorization (19) immediately; compare Murota (1989).

Graph-theoretic procedure to obtain suitable permutation matrices P_r and P_c.

Step 1. Look for a cycle family of length n on $G(Q)$. Assume rank$[Q] = n$. Consequently, the existence of at least one cycle family of length n on $G(Q)$ is guaranteed. The chosen cycle family defines a permutation

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ i_1 & i_2 & i_3 & \cdots & i_n \end{pmatrix}$$

The associated row permutation matrix is

$$\underline{P}_r = (e_{i_1}, e_{i_2}, e_{i_3}, \ldots, e_{i_n})' \tag{21}$$

where e_{i_j} denotes the i_jth unit column vector. $\underline{P}_r$ transforms Q into a matrix

$$\underline{Q} = \underline{P}_r Q. \tag{22}$$

Note that all main-diagonal elements of $\underline{Q}$ are occupied.

Step 2. Consider $G(\underline{Q})$, and look for the equivalence classes of strongly connected vertices in $G(\underline{Q})$. If there is only one equivalence class in $G(\underline{Q})$, then $\det Q$ is not properly factorizable in the sense of Equation (19). Otherwise, due to the partial order between the classes, they can be enumerated in such a way that transitions from equivalence classes of lower indices to equivalence classes of higher indices are impossible. This reordering process may be interpreted as a similarity transformation of $\underline{Q}$ with a permutation matrix P_c, i.e.

$$\tilde{Q} = P_c^{-1}\underline{Q}P_c = P_c'\underline{Q}P_c. \tag{23}$$

Taking into account (22), one gets

$$\tilde{Q} = P_c'\underline{P}_r Q P_c = P_r Q P_c. \tag{24}$$

Furthermore,

$$\begin{aligned} \det \tilde{Q} &= (\det P_c)^2 (\det \underline{P}_r) \det Q \\ &= (\det \underline{P}_r) \det Q \end{aligned} \tag{25}$$

and

$$\det \underline{P}_r = (-1)^{n-c}, \tag{26}$$

where c is equal to the number of individual cycles contained in the cycle family which was chosen from $G(Q)$ in order to determine the permutation matrix $\underline{P}_r$.

For elaborate procedures to get the strong components of a digraph and the permutation matrix P_c, see, for example, Kemeny and Snell (1960), Kaufmann (1968), Kevorkian (1975), and Evans, Schizas, and Chan (1981).

EXAMPLE. Figure 10 shows a 7×7 matrix $[Q]$ and the corresponding digraph $G(Q)$. A cycle family of length $n = 7$ is marked with a dotted line. The associated row permutations are given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 7 & 1 & 3 & 5 & 6 \end{pmatrix}.$$

Equation (22) yields

$$\underline{Q} = \begin{pmatrix} e_2 & e_4 & e_7 & e_1 & e_3 & e_5 & e_6 \end{pmatrix}' Q.$$

The structure matrix $[\underline{Q}]$ and the digraph $G(\underline{Q}]$ are shown in Figure 11. The maximal sets of strongly connected vertices are encircled by dotted lines. The natural order between these sets is evident:

$$\{1, 6\} \to \{5\} \to \{4\} \to \{2, 3, 7\}.$$

$$[Q] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} & & & \times & \times & & \\ \times & & & & & \times & \\ \times & & & & \times & & \\ & \times & & \times & & & \times \\ \times & & & & & \times & \\ & & \times & & & & \times \\ & \times & \times & & & \times & \end{bmatrix} \end{array}$$

G(Q):

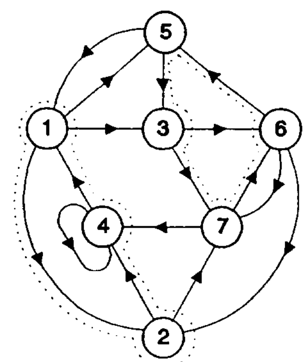

FIG. 10. A 7×7 structure matrix $[Q]$ and its associated digraph $G(Q)$.

$$[\underline{Q}] = \begin{array}{c} \\ 2 \\ 4 \\ 7 \\ 1 \\ 3 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{ccccccc} x & & & & & x & \\ & x & & x & & & x \\ & x & x & & & x & \\ & & & x & x & & \\ x & & & & x & & \\ x & & & & & x & \\ & & x & & & & x \end{array}\right] \end{array}$$

G(Q):

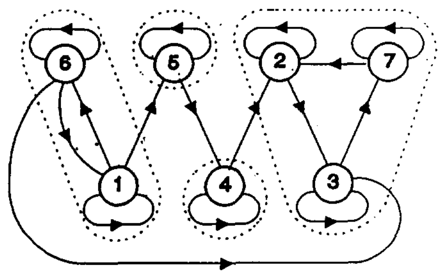

FIG. 11. The structure matrix $[\underline{Q}]$ derived from $[Q]$ in Figure 10 and $G(\underline{Q})$.

The permutation matrix P_c transforming $\underline{Q}$ into Q reads as

$$P_c = (e_2, e_3, e_7; e_4; e_5; e_1, e_6).$$

The resulting matrix $\tilde{Q}$ [see (24)] has the quasitriangular structure shown in Figure 12. It should be noted that the column heads and the row heads in Figure 11 and in Figure 12 refer to the lines of the given original matrix Q introduced in Figure 10.

Finally, let us return to descriptor systems (2). We have to investigate $\det(sE - A)$ instead of $\det Q$. In looking for hidden structural factorizations of the characteristic polynomial $\det(sE - A)$, new problems do not arise. The same procedure explained for a square matrix Q may be applied to the matrix pencil $(sE - A)$ and the associated digraph $G(sE - A)$. We simply put $Q = sE - A$ and subject the structure matrix $[Q]$ to the permutation procedure discussed above. The procedure yields

$$\tilde{Q} = s\tilde{E} - \tilde{A} = P_r(sE - A)P_c. \tag{27}$$

$$[\tilde{Q}] = \begin{array}{c} \\ 4 \\ 7 \\ 6 \\ 1 \\ 3 \\ 2 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccccc} 2 & 3 & 7 & 4 & 5 & 1 & 6 \end{array} \\ \left[\begin{array}{ccc|c|c|cc} x & & x & x & & & \\ x & x & & & & & x \\ & x & x & & & & \\ \hline & & & x & x & & \\ \hline & & & & x & x & \\ \hline & & & & & x & x \\ & & & & & x & x \end{array}\right] \end{array}$$

FIG. 12. The structure matrix $[\tilde{Q}]$ derived from $[\underline{Q}]$ by a permutation transform.

6. APPLICATION

Let us consider an example taken from electrical engineering. Figure 13 shows an electrical circuit, a so-called active RC filter. It consists of 12 resistors, 2 capacitors, and 4 ideal operational amplifiers. The circuit is excited by one voltage source $u^e(t)$ as input. The system output is $u_{\text{out}}(t)$.

The following observation is fundamental for linear circuits: If the source has a time dependence which is sinusoidal, then—in the stationary state—all currents and voltages occurring in the circuit show a sinusoidal time dependence with the same frequency. Therefore, the differential equations describing the circuit behavior may be replaced by algebraic equations which arise by applying the Laplace transform. This means, roughly speaking, that the differential operator d/dt is replaced by the complex factor s. So the voltage-current relations defining resistors and capacitors, i.e.

$$i(t) = \frac{1}{R}u(t) = Gu(t) \quad \text{and} \quad i(t) = C\frac{du}{dt},$$

are transformed into

$$I(s) = GU(s) \quad \text{and} \quad I(s) = sCU(s),$$

respectively.

Figure 14 illustrates the modeling of an ideal operational amplifier: the input branch is a so-called nullator, characterized by both branch voltage $u = 0$ and branch current $i = 0$; the output branch is a so-called norator, whose branch current and branch voltage are determined by the "surrounding" circuit elements. In Figure 15, the given active RC filter has been prepared for the analysis of this circuit. Nodes $1, 2, \ldots, 9$ have been introduced. The corresponding nodal voltages $U_1, U_2, \ldots, U_9$ denote the potential difference between the nodes and the ground. The input and output currents of the operational amplifiers are symbolized by $I_1, I_2, \ldots, I_8$. Now, one may start the analysis by writing down the well-known Kirchhoff current laws for the nodes $1, 2, \ldots, 9$ (the total current entering each node is zero):

Node 1:

$$G_1(U_1 - U_8) + (G_2 + sC_2)(U_1 - U_6) + I_1 + G_7(U_1 - U^e) = 0.$$

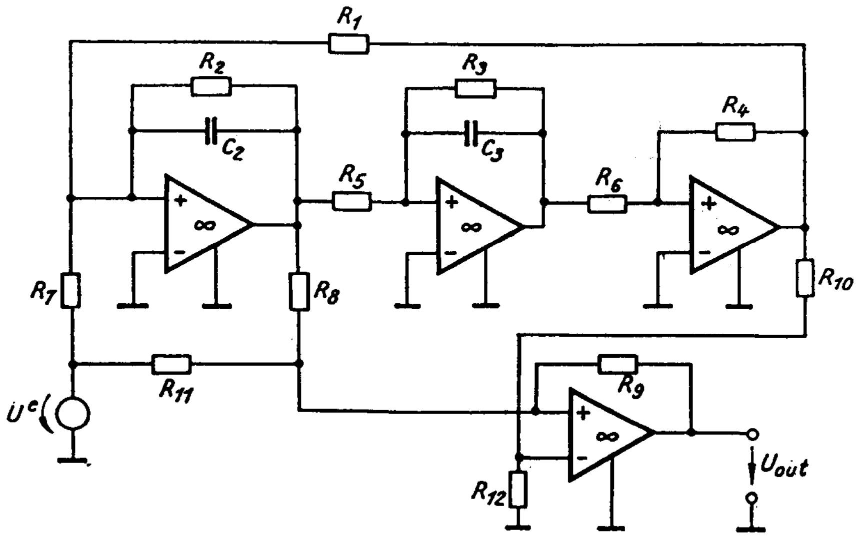

FIG. 13. Active RC filter.

Node 2:

$$G_5(U_2 - U_6) + (G_3 + sC_3)(U_2 - U_7) + I_3 = 0.$$

Node 3:

$$G_6(U_3 - U_7) + G_4(U_3 - U_8) + I_5 = 0.$$

Node 4:

$$G_8(U_4 - U_6) + G_9(U_4 - U_9) + I_7 + G_{11}(U_4 - U^e) = 0.$$

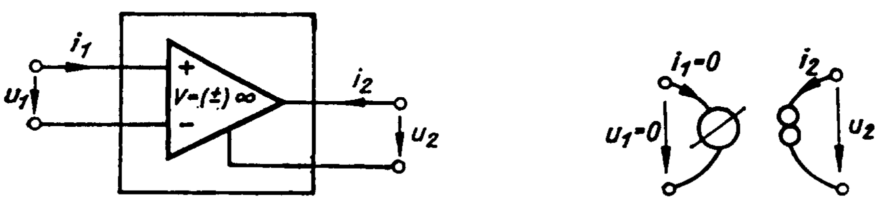

FIG. 14. Modeling of an operational amplifier.

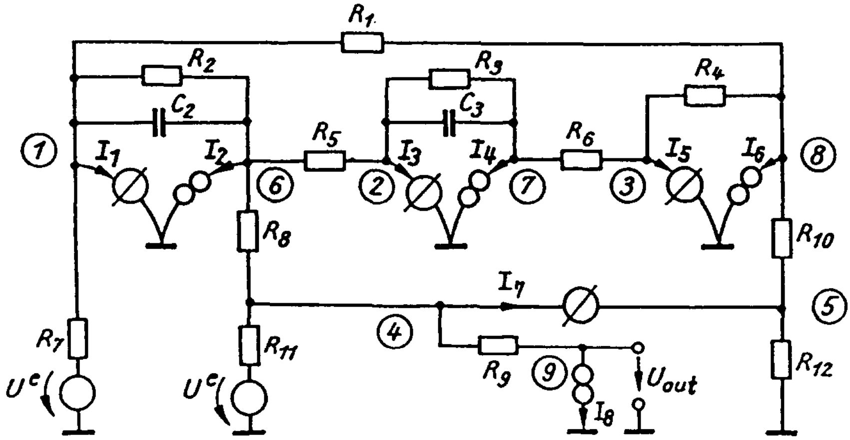

FIG. 15. Model of the electrical circuit shown in Figure 13.

Node 5:

$$G_{10}(U_5 - U_8) - I_7 + G_{12}(U_5 - 0) = 0.$$

Node 6:

$$(G_2 + sC_2)(U_6 - U_1) + I_2 + G_5(U_6 - U_2) + G_8(U_6 - U_4) = 0.$$

Node 7:

$$(G_3 + sC_3)(U_7 - U_2) + I_4 + G_6(U_7 - U_3) = 0.$$

Node 8:

$$G_1(U_8 - U_1) + G_4(U_8 - U_3) + I_6 + G_{10}(U_8 - U_5) = 0.$$

Node 9:

$$G_9(U_9 - U_4) + I_8 = 0.$$

Altogether, we have obtained 9 equilibrium conditions between 17 unknowns. Due to the above-mentioned modeling of each operational amplifier by a pair of nullator-norator branches, one may assume that

$$U_1 = U_2 = U_3 = 0, \qquad U_4 = U_5, \qquad I_1 = I_3 = I_5 = I_7 = 0.$$

There remain 9 unknowns in a system of 9 equations:

$$-G_1U_8 - (G_2 + sC_2)U_6 = G_7U^e, \tag{28a}$$

$$- G_5U_6 - (G_3 + sC_3)U_7 = 0, \tag{28b}$$

$$- G_3U_7 - G_4U_8 = 0, \tag{28c}$$

$$(G_8 + G_9 + G_{11})U_4 - G_8U_6 - G_9U_9 = G_{11}U^e, \tag{28d}$$

$$(G_{10} + G_{12})U_4 - G_{10}U_8 = 0, \tag{28e}$$

$$(G_2 + sC_2 + G_5 + G_8)U_6 + I_2 - G_8U_4 = 0, \tag{28f}$$

$$(G_3 + sC_3 + G_6)U_7 + I_4 = 0, \tag{28g}$$

$$(G_1 + G_4 + G_{10})U_8 + I_6 - G_{10}U_4 = 0, \tag{28h}$$

$$G_9U_9 - G_9U_4 + I_8 = 0. \tag{28i}$$

Ordering the unknowns according to the indices, we get a descriptor vector

$$X = \begin{pmatrix} I_2 & U_4 & I_4 & U_6 & I_6 & U_7 & U_8 & I_8 & U_9 \end{pmatrix}' \tag{29}$$

and a Laplace transformed descriptor system

$$(sE - A)X(s) = BU(s) = bU(s),$$

where the column vector

$$b = \begin{pmatrix} G_7 & 0 & 0 & G_{11} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}',$$

the scalar input

$$U(s) = U^e,$$

and the coefficient matrix $sE - A$ has the structure

$$[sE - A] = \begin{bmatrix} 0 & 0 & 0 & \times & 0 & 0 & \times & 0 & 0 \\ 0 & 0 & 0 & \times & 0 & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \times & \times & 0 & 0 \\ 0 & \times & 0 & \times & 0 & 0 & 0 & 0 & \times \\ 0 & \times & 0 & 0 & 0 & 0 & \times & 0 & 0 \\ \times & \times & 0 & \times & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \times & 0 & 0 & \times & 0 & 0 & 0 \\ 0 & \times & 0 & 0 & \times & 0 & \times & 0 & 0 \\ 0 & \times & 0 & 0 & 0 & 0 & 0 & \times & \times \end{bmatrix}. \tag{30}$$

At this point, it should be realized that we have got a descriptor system in the most natural way, simply by writing down the current equilibria at the nodes of the given active RC filter. It would demand considerable additional efforts to derive the state-space equations for systems of this kind. Moreover, in contrast to state-space representation, the coefficients of the obtained system of descriptor equations reflect the electrical parameters in the clearest way.

Figure 16 shows the digraph $G(sE - A)$. It is a strongly connected digraph. Nevertheless, we should look for hidden structure in the descriptor equations, taking into account that the enumeration sequence both of the columns and of the rows of the matrix $sE - A$ has been chosen completely arbitrarily until now. Let us apply the factorization procedure of Section 5. First, we have to look for a cycle family of length 9 in the digraph of Figure 16. There are several possibilities, e.g. $1 \to 6 \to 3 \to 7 \to 1$, $2 \to 5 \to 8 \to 9 \to 4 \to 2$, or $2 \to 5 \to 8 \to 9 \to 4 \to 1 \to 6 \to 2$, $3 \to 7 \to 3$. Using the second one, we get a row permutation matrix

$$\underline{P}_r = \begin{pmatrix} e_6 & e_5 & e_7 & e_1 & e_8 & e_2 & e_3 & e_9 & e_4 \end{pmatrix}',$$

$$\det \underline{P}_r = (-1)^{9-2} = -1.$$

The structure matrix $[\underline{P}_r(sE - A)] = [\overline{sE - A}]$ corresponds to the digraph $G(\overline{sE - A})$ sketched in Figure 17. The equivalence classes of vertices may be easily seen from Figure 17:

$$\{4,6,7\}, \quad \{1\}, \quad \{2\}, \quad \{3\}, \quad \{5\}, \quad \{8\}, \quad \{9\}.$$

$$
[sE - A] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ \uparrow \\ \text{row} \end{array}
\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \longleftarrow \text{column} \\
\left[\begin{array}{ccccccccc}
 & & & x & & & x & & \\
 & & & x & & x & & & \\
 & & & & & x & x & & \\
 & x & & x & & & & & x \\
 & x & & & & & x & & \\
x & x & & x & & & & & \\
 & & x & & & x & & & \\
 & x & & & x & & x & & \\
 & x & & & & & & x & x
\end{array}\right] \\ \\ \\ \end{array}
$$

G(sE-A):

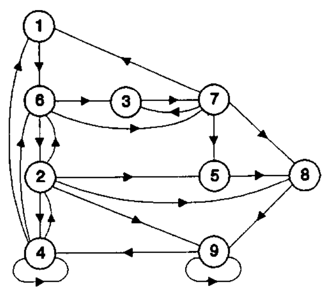

FIG. 16. Structure matrix $[sE - A]$ and its associated digraph $G(sE - A)$.

The partial order relations determined by the possibility of transition are the following ones:

$$\{4,6,7\} \to \{2\} \to \{9\} \to \{8\},$$

$$\{4,6,7\} \to \{2\} \to \{5\}, \{1\},$$

$$\{4,6,7\} \to \{3\}.$$

The permutation matrix P_c may be chosen as

$$P_c = \begin{pmatrix} e_1 & e_3 & e_5 & e_8 & e_9 & e_2 & e_4 & e_6 & e_7 \end{pmatrix}.$$

$$[\underline{sE-A}] = \begin{array}{c} \\ 6 \\ 5 \\ 7 \\ 1 \\ 8 \\ 2 \\ 3 \\ 9 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} \times & \times & & \times & & & & & \\ & \times & & & & & \times & & \\ & & \times & & & \times & & & \\ & & & \times & & & \times & & \\ & \times & & & \times & & \times & & \\ & & & \times & & \times & & & \\ & & & & & \times & \times & & \\ & \times & & & & & & \times & \times \\ & \times & & \times & & & & & \times \end{bmatrix} \end{array}$$

G(sE - A):

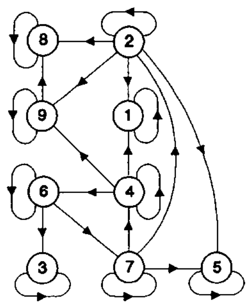

FIG. 17. Structure matrix $[\underline{P}_r(sE - A)] = [\underline{sE - A}]$ and the digraph $G(\underline{sE - A})$.

Thus, we obtain the structure matrix

$$[\tilde{Q}] = [P_c'(\underline{sE - A})P_c] = \begin{bmatrix} \times & 0 & 0 & 0 & 0 & \times & \times & 0 & 0 \\ 0 & \times & 0 & 0 & 0 & 0 & 0 & \times & 0 \\ 0 & 0 & \times & 0 & 0 & \times & 0 & 0 & \times \\ 0 & 0 & 0 & \times & \times & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \times & \times & \times & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \times & 0 & 0 & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & \times & 0 & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & \times & \times & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \times & \times \end{bmatrix}$$

compare Figure 18.

It should be noted that in Figures 16, 17 and 18 the row and column heads indicate the original equation indices and the original unknown indices, respectively, as introduced in (29) and (30). Now, for the example system, the

$$
P_c'(\underline{sE-A})P_c = \begin{array}{c} \\ 6\\7\\8\\9\\4\\5\\1\\2\\3 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 1 & 3 & 5 & 8 & 9 & 2 & 4 & 6 & 7 \end{array}\\
\left[\begin{array}{ccccccccc}
x & & & & & x & x & & \\
 & x & & & & & & x & \\
 & & x & & & x & & & x \\
 & & & x & x & x & & & \\
 & & & & x & x & x & & \\
 & & & & & x & & \cdot & x \\
 & & & & & & x & & x \\
 & & & & & & x & x & \\
 & & & & & & & x & x
\end{array}\right]
\end{array}
$$

$G(P_c'(\underline{sE-A})P_c)$:

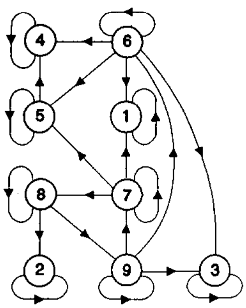

FIG. 18. Structure matrix $P_c'(\underline{sE - A})P_c = P_r(sE - A)P_c$ and its associated digraph.

factorization has been completed. Equation (28) has been transformed into

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -G_8 & +G_2 + sC_2 + G_5 + G8 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & G_3 + G_6 + sC_3 & \\
0 & 0 & 1 & 0 & 0 & -G_{10} & 0 & 0 & G_1 + G_4 + G_{10} \\
0 & 0 & 0 & 1 & G_9 & -G_9 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -G_9 & G_8 + G_9 + G_{11} & -G_8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & G_{10} + G_{12} & 0 & 0 & -G_{10} \\
0 & 0 & 0 & 0 & 0 & 0 & -G_2 - sC_2 & 0 & -G_1 \\
0 & 0 & 0 & 0 & 0 & 0 & -G_5 & -(G_3 + sC_3) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -G_3 & -G_4
\end{pmatrix}
$$

$$
\times \begin{pmatrix} I_2 \\ I_4 \\ I_6 \\ I_8 \\ U_9 \\ U_4 \\ U_6 \\ U_7 \\ U_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ G_{11} \\ 0 \\ G_7 \\ 0 \\ 0 \end{pmatrix} U^e.
$$

Obviously, the determinant of the example system splits up into a product of one subdeterminant of order 3 and six trivial subdeterminants of order 1:

$$\det(sE - A) = -\det\tilde{Q} = -G_9(G_{10} + G_{12}) \times \left[G_4(G_2 + sC_2)(G_3 + sC_3) + G_1G_3G_5\right]. \tag{32}$$

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