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$(k, 1)$-coloring of sparse graphs

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ABSTRACT

A graph G is $(k, 1)$-*colorable* if the vertex set of G can be partitioned into subsets V_1 and V_2 such that the graph $G[V_1]$ induced by the vertices of V_1 has maximum degree at most k and the graph $G[V_2]$ induced by the vertices of V_2 has maximum degree at most 1. We prove that every graph with maximum average degree less than $\frac{10k+22}{3k+9}$ admits a $(k, 1)$-coloring, where $k \geq 2$. In particular, every planar graph with girth at least 7 is $(2, 1)$-colorable, while every planar graph with girth at least 6 is $(5, 1)$-colorable. On the other hand, when $k \geq 2$ we construct non-$(k, 1)$-colorable graphs whose maximum average degree is arbitrarily close to $\frac{14k}{4k+1}$.

1. Introduction

A graph G is $(d_1, \ldots, d_k)$-*colorable* if the vertex set of G can be partitioned into subsets $V_1, \ldots, V_k$ such that the graph $G[V_i]$ induced by the vertices of V_i has maximum degree at most d_i for all $1 \leq i \leq k$. This notion generalizes those of proper k-coloring (when $d_1 = \cdots = d_k = 0$) and d-improper k-coloring (when $d_1 = \cdots = d_k = d \geq 1$).

Proper and d-improper colorings have been widely studied. As shown by Appel and Haken [1,2], every planar graph is $(0, 0, 0, 0)$-colorable. Cowen et al. [10] proved that every planar graph is $(2, 2, 2)$-colorable (a list version of this theorem was given by Eaton and Hull [11] and independently Škrekovski [14]). This latter result was extended by Havet and Sereni [13] to not necessarily planar sparse graphs as follows: for every $k \geq 0$, every graph G with $\mathrm{mad}(G) < \frac{4k+4}{k+2}$ is (k, k)-colorable (in fact (k, k)-choosable), where

$$\mathrm{mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|}, H \subseteq G\right\}$$

is the maximum average degree of a graph G.

Let $g(G)$ denote the girth of graph G (the length of a shortest cycle in G). Glebov and Zambalaeva [12] proved that every planar graph G is $(1, 0)$-colorable if $g(G) \geq 16$. This was strengthened by Borodin and Ivanova [5] by proving that every graph G is $(1, 0)$-colorable if $\mathrm{mad}(G) < \frac{7}{3}$, which implies that every planar graph G is $(1, 0)$-colorable if $g(G) \geq 14$.

Borodin and Kostochka [9] proved that every graph G with $\mathrm{mad}(G) \leq \frac{12}{5}$ is $(1, 0)$-colorable. In particular, it follows that every planar graph G with $g(G) \geq 12$ is $(1, 0)$-colorable. On the other hand, they constructed graphs G with $\mathrm{mad}(G)$ arbitrarily close (from above) to $\frac{12}{5}$ that are not $(1, 0)$-colorable.

This was extended by Borodin et al. [7] by proving that every graph with a maximum average degree smaller than $\frac{3k+4}{k+2}$ is $(k, 0)$-colorable if $k \geq 2$. The proof in [7] extends that in [5] but does not work for $k = 1$.

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Table 1
The relationship between the girth of G and its (k, j)-colorability.

$g(G)$	$(k, 0)$	$(k, 1)$	$(k, 2)$
6	$\times$ [7]	$(5, 1)$	$(2, 2)$ [13]
7	$(8, 0)$ [7]	$(2, 1)$	
8	$(4, 0)$ [7]	$(1, 1)$ [13]	

In this paper, we focus on $(k, 1)$-colorability of a graph. A graph G is $(k, 1)$-colorable if its vertices can be partitioned into subsets V_1 and V_2 such that in $G[V_1]$ every vertex has degree at most k, while in $G[V_2]$ every component has at most two vertices. Our main result is:

Theorem 1. *For $k \geq 2$, every graph G with $\text{mad}(G) < \frac{10k+22}{3k+9}$ is $(k, 1)$-colorable.*

On the other hand, we construct non-$(k, 1)$-colorable graphs whose maximum average degree is arbitrarily close to $\frac{14k}{4k+1}$. Since every planar graph G satisfies $\text{mad}(G) < \frac{2g(G)}{g(G)-2}$, from Theorem 1 we have:

Corollary 1. *Planar graphs with girth at least 7 are $(2, 1)$-colorable; planar graphs with girth at least 6 are $(5, 1)$-colorable.*

On the other hand, there is (see [7]) a planar graph with girth 6 that is not $(k, 0)$-colorable for any k, whereas planar graphs with girth at least 7 are $(8, 0)$-colorable, and those with girth at least 8 are $(4, 0)$-colorable (see [7]). Also note that planar graphs G with girth at least 6 are $(2, 2)$-colorable, while those with girth at least 8 are $(1, 1)$-colorable (see [13]). The results are summarized in Table 1.

A distinctive feature of the discharging in the proof of Theorem 1 for $2 \leq k \leq 4$ is its "global nature": a charge for certain vertices is collected from arbitrarily large "feeding areas", which is possible due to the existence of reducible configurations of unlimited size in the minimum counter-examples, called "soft components". Such global discharging first appears in [3] and is used, in particular, in [4,6,8,5,7,13]. The terms "feeding area" and "soft component" are introduced in [5] and also used in our recent paper [7].

2. Non-$(k, 1)$-colorable graphs with small maximum average degree

Let $H^i_{a,b}$ be the graph consisting of two adjacent vertices a and b and vertices $c_1, \ldots, c_i$ each having neighborhood $\{a, b\}$. We take one copy of $H^{k+1}_{a,b}$ and $k-1$ copies of $H^2_{a,b}$ and identify all the vertices a to a single vertex a^*. Let H_{a^*} be the resulting graph. Finally, we take an odd cycle $C_{2n-1} = a_1a_2 \ldots a_{2n-1}$ and n copies of H_{a^*}, and we identify each vertex a_i with odd index with the vertex a^* of a copy of H_{a^*}. Let $G_{n,k}$ be the resulting graph. An example is given in Fig. 1.

One can observe that $G_{n,k}$ is not $(k, 1)$-colorable. Indeed, observe first that no two consecutive vertices x, y on C_{2n-1} belong to V_2: otherwise all the vertices except x of the subgraph $H^{k+1}_{a,b}$ associated to x must belong to V_1; it follows that the degree of b (of $H^{k+1}_{a,b}$) in $G[V_1]$ is $k+1$, a contradiction. Due to the parity of C_{2n-1}, it follows that two consecutive vertices x, y on C_{2n-1} belong to V_1. We can suppose that x is of odd index on C_{2n-1}. If $G_{n,k}$ is $(k, 1)$-colorable, then one more vertex in each $H^i_{a,b}$ associated to x must belong to V_1; it follows that the degree of x in $G[V_1]$ is $k+1$, a contradiction.

It is easy to check that the maximum average degree of $G_{n,k}$ is equal to its average degree. We have:

$$\text{mad}(G_{n,k}) = \frac{2|E(G_{n,k})|}{|V(G_{n,k})|} = \frac{2(2n-1+5(k-1)n+n(2k+3))}{2n-1+3(k-1)n+n(k+2)} = \frac{2(7nk-1)}{n(4k+1)-1}$$

$$\lim_{n\to\infty} \text{mad}(G_{n,k}) = \frac{14k}{4k+1}.$$

3. Proof of Theorem 1

Let G be a counterexample to Theorem 1 on the fewest number of vertices. Clearly, G is connected and its minimum degree is at least 2. By definition, we have:

$$\frac{2|E(G)|}{|V(G)|} \leq \text{mad}(G) < \frac{10k+22}{3k+9}$$

$$2|E(G)| - |V(G)|\frac{10k+22}{3k+9} = \sum_{v\in V}\left(d(v) - \frac{10k+22}{3k+9}\right) < 0,$$

where $d(v)$ is the degree of a vertex v.

Thus, we have:

$$\sum_{v\in V(G)}\left(\frac{3(k+3)}{2(k+1)}d(v) - \frac{5k+11}{k+1}\right) < 0. \quad (1)$$

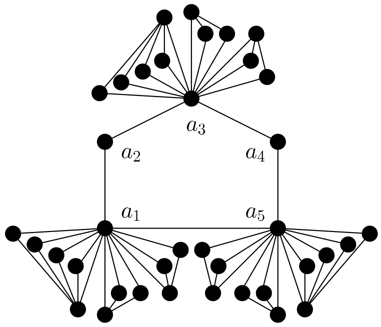

Fig. 1. An example of $G_{n,k}$ with $n = 3$ and $k = 3$.

Let the *charge* $\mu(v)$ of each vertex v of G be $\frac{3(k+3)}{2(k+1)}d(v) - \frac{5k+11}{k+1}$. We shall describe a number of structural properties of G (Section 3.1) which make it possible to vary the charges so that the new charge μ^* of every vertex becomes nonnegative for $k \geq 5$ (Section 3.2). For $2 \leq k \leq 4$ there is a difference: some vertices have a non-negative μ^* individually (Section 3.3), while the others are partitioned into disjoint subsets, called *feeding areas*, and the total charge of each feeding area is proven to be non-negative (Lemma 1 in Section 3.3). Since the sum of charges does not change, in both cases we get a contradiction with (1), which will complete the proof of Theorem 1.

A vertex of degree d (resp. at least d, at most d) is a *d-vertex* (resp. *d^+-vertex*, *d^--vertex*). A $(k+1)^-$-vertex is *minor*; a $(k+2)^+$-vertex is *senior*. A *weak vertex* is a minor vertex adjacent to exactly one senior vertex. A *light vertex* is either a 2-vertex or a weak vertex. A 3_i-vertex is a 3-vertex adjacent to i 2-vertices.

Claims 2 and 3 below lead us to the following definition. A d-vertex, where $d \geq k+3$, is *soft* if it is adjacent to $d-1$ weak vertices. For $d = k+2$ the notion of soft vertex is broader: a $(k+2)$-vertex is *soft* if it is adjacent to $k+1$ light vertices.

We will color the vertices of the subgraph of maximum degree at most k by color k and the other vertices by color 1.

3.1. Structural properties of G

Claim 1. *No 2-vertex in G is adjacent to a 2-vertex.*

Proof. Suppose G has two adjacent 2-vertices t and u, and let s (resp. v) be the other neighbor of t (resp. u). By the minimality of G, the graph $G - \{t, u\}$ has a $(k, 1)$-coloring c. It suffices to color t and u with a color different from those of s and v respectively to extend c to the whole graph G, a contradiction. □

Claim 2. *Every minor vertex in G is adjacent to at least one senior vertex.*

Proof. Suppose G has a minor vertex x adjacent only to minor vertices. Take a $(k, 1)$-coloring c of $G - x$. If none of the neighbors of x has color 1, then we simply color x with 1. So suppose that at least one neighbor of x is colored with 1. We then color x with k. There is now a problem only if there exists a neighbor of x, say y, colored with k and surrounded by $k+1$ neighbors colored with k. In this case, we recolor y with 1. We iterate this operation while such a y exists. The coloring obtained is a $(k, 1)$-coloring of G, a contradiction. □

Claim 3. *If a senior d-vertex is adjacent to $d-1$ weak vertices, then it is adjacent to a non-light vertex.*

Proof. Suppose G has a d-vertex x adjacent to vertices $x_1, \ldots, x_d$, where $x_1, \ldots, x_{d-1}$ are weak while x_d is either weak or has $d(x_d) = 2$. We take a $(k, 1)$-coloring of $G - x$. We recolor each weak vertex x_i with color k. There is now a problem only if there exists a neighbor of a x_i, say y, colored with k and surrounded by $k+1$ neighbors colored with k. In this case, we recolor y with 1. We iterate this operation until such a y exists. If x_d is a 2-vertex, then we recolor it properly. Now it suffices to color x with 1, a contradiction. □

Claim 4. *No 3-vertex is adjacent to two soft vertices and to a minor vertex.*

Proof. Suppose G has a 3-vertex x adjacent to vertices x_1, x_2, x_3, where x_1 and x_2 are $(k+2)^+$-vertices while $d(x_3) \leq k+1$. Let $y^1_1, \ldots, y^1_{d(x_1)-1}$ (resp. $y^2_1, \ldots, y^2_{d(x_2)-1}$) be the other neighbors of x_1 (resp. x_2). We take a $(k, 1)$-coloring of $G - \{x, x_1, x_2\}$. We first recolor the vertices y^i_j as follows: if y^i_j has $d(y^i_j) = 2$ and is not weak, then we recolor y^i_j properly; otherwise if y^i_j is weak, then we recolor y^i_j with k (followed by recoloring if necessary the neighbors of y^i_j's; see the proof of Claim 3). Now if $d(x_1) \geq k+3$, then we color x_1 with 1 (observe that all colored neighbors of x_1 are colored with k). Assume $d(x_1) = k+2$. If the color 1 appears at least twice on the y^i_j, then we color x_1 with k and with 1 otherwise. We do the same for x_2. Finally, if an identical color appears three times in the neighborhood of x, then we color x properly. Otherwise we color x with k (followed by recoloring x_3 if necessary). This gives an extension of c to the whole graph G, a contradiction. □

An edge xy is *soft* if one of the following holds:

- $d(x) = k + 2$ while y is light, i.e. is a 2-vertex or a weak vertex, or
- x is a minor vertex while $d(y) = 2$.

The vertex x is called the *good end* of the soft edge xy.

A *soft component* SC is a subgraph of G such that:

- $\Delta(SC) \leq k + 2$;
- each edge joining SC to $G \setminus SC$ is soft and each good end of the soft edges belongs to SC;
- in addition, a 2-vertex having its two neighbors in SC is in SC.

Claim 5. *G does not contain soft components.*

Proof. Assume that G contains a soft component S. By minimality of G, the graph $G - V(S)$ has a $(k, 1)$-coloring c. We will show that we can extend c to the whole graph G, a contradiction. First, for each edge xy with $x \in S$ and $y \notin S$, we will recolor (if necessary) the vertex y such that the choice of any color for x will not create any problem on y. If y is a 2-vertex, then we just recolor y properly. Assume now that y is a weak vertex with degree at least 3. We first consider successively all the weak vertices y having a neighbor colored with 1: if y is colored with 1, then we recolor y with k (followed by recoloring iteratively the neighbors of y colored with k which are surrounded by $k + 1$ neighbors colored with k). We then consider all the weak vertices y having k neighbors colored with k: we recolor all such y with color 1. Observe that if x is later colored with 1 or k, then that will not create a conflict for y. Now we extend the coloring c to the whole graph G as follows: we choose a coloring ϕ of S that minimizes $\sigma = k \cdot E_{11} + E_{kk}$ where E_{ii} denotes the number of edges whose both ends are colored with i in G. Clearly, such a coloring exists. Moreover we will show that c and ϕ is a $(k, 1)$-coloring to the whole graph G. Assume that the coloring ϕ of S and c of $G - V(S)$ is not a $(k, 1)$-coloring of G. So suppose that there exists a vertex u of S colored with 1 that has two neighbors colored with 1. We just recolor u with k and obtain a coloring with a smaller σ which contradicts the choice of ϕ. Similarly, assume that there exists a vertex v of S colored with k that has $k + 1$ neighbors colored with k. We just recolor v with 1 and obtain a coloring with a smaller σ, which contradicts the choice of ϕ. □

Corollary 2. *No $(k + 2)$-vertex can be adjacent to $k + 2$ light vertices.*

3.2. Discharging procedure when $k \geq 5$

Set $\alpha = \frac{3k+1}{2(k+1)}$, $\gamma = \frac{k-1}{k+1}$, $\epsilon = \frac{k-5}{2(k+1)}$. Note that $2 - \alpha = \frac{k+3}{2(k+1)}$. When $k \geq 5$, we have:

$$0 \leq \epsilon < \frac{1}{2} < 2 - \alpha \leq \frac{2}{3} \leq \gamma < 1 \quad \text{and} \quad \frac{4}{3} \leq \alpha < \frac{3}{2}.$$

Our rules of discharging are as follows:

R1. Every d-vertex with $3 \leq d \leq k + 1$ gives $2 - \alpha$ to each adjacent 2-vertex.
R2. Every weak vertex gets α from its adjacent senior vertex.
R3. Every non-weak 2-vertex gets 1 from each neighbor.
R4. Every minor non-light vertex gets γ from each non-soft adjacent $(k+2)$-vertex, ϵ from each soft adjacent $(k+2)$-vertex and $2 - \alpha$ from each adjacent $(k + 3)^+$-vertex.

We now show that $\mu^*(v) \geq 0$ for all v in $V(G)$. Let v be a d-vertex, where $d \geq 2$. Set

$$\mu_d = \frac{3(k+3)}{2(k+1)}d - \frac{5k+11}{k+1}.$$

In particular, $\mu_2 = -2$, $\mu_3 = \frac{-k+5}{2(k+1)} = -\epsilon$, and $-\frac{1}{2} < \mu_3 \leq 0$.

Case 1. $d \geq k + 3$.

Claim 6. *If $d \geq k + 3$, then $\mu_d \geq \alpha(d - 2) + 2$; in particular, $\mu_{k+3} = \alpha(k + 1) + 2$.*

Proof.

$$\mu_d - \alpha(d-2) - 2 = \frac{3(k+3)}{2(k+1)}d - \frac{5k+11}{k+1} - \frac{3k+1}{2(k+1)}(d-2) - 2$$
$$= \frac{4(d-(k+3))}{k+1} \geq 0. \quad \square$$

By Claim 3, v is adjacent to at most $d - 1$ weak vertices. If v is adjacent to at most $d - 2$ weak vertices, then $\mu^*(v) \geq \mu_d - \alpha(d - 2) - 2 \times 1 \geq 0$ by R1–R4, due to Claim 6. Suppose now that v is adjacent to exactly $d - 1$ weak vertices. By Claim 3, v is adjacent to a non-light vertex. So we have $\mu^*(v) \geq \mu_d - \alpha(d - 1) - (2 - \alpha) \geq 0$ by R1–R4, due to Claim 6.

Case 2. $d = k + 2$.

By Corollary 2, the vertex v is adjacent to at most $k + 1$ light vertices. By Claim 6, we have:

$$\begin{aligned}\mu_{k+2} &= \mu_{k+3} - \frac{3(k+3)}{2(k+1)} \\ &= \alpha(k+1) + 2 - \frac{3(k+3)}{2(k+1)} \\ &= \alpha k + 2\gamma.\end{aligned}$$

If v is adjacent to at most k light vertices, then $\mu^*(v) \geq \mu_{k+2} - k\alpha - 2\gamma \geq 0$ by R1–R4.

If v is adjacent to exactly $k+1$ light vertices, then v is soft. By Claim 3 and R1–R4, we have $\mu^*(v) \geq \mu_{k+2} - \alpha(k+1) - \epsilon = 2\gamma - \alpha - \epsilon = 0$.

Case 3. $2 \leq d \leq k + 1$.

By Claim 1, a 2-vertex is adjacent to 3^+-vertices. By Claim 2, a d-vertex with $3 \leq d \leq k + 1$ is adjacent to at most $d - 1$ vertices of degree 2, each of which gets $2 - \alpha$ from v by R1.

Subcase 3.1. *v is weak.*

If $d = 2$, then $\mu^*(v) = -2 + (2 - \alpha) + \alpha = 0$ by R1 and R2. Suppose $d \geq 3$.

Claim 7. *For each $d \geq 3$, it holds that $\mu_d - (d-1)(2-\alpha) + \alpha = \frac{(k+3)(d-3)}{k+1}$.*

Proof.

$$\begin{aligned}\mu_d - (d-1)(2-\alpha) + \alpha &= \frac{3(k+3)}{2(k+1)}d - \frac{5k+11}{k+1} - (d-1)\frac{k+3}{2(k+1)} + \frac{3k+1}{2(k+1)} \\ &= \frac{(k+3)(d-3)}{k+1}. \quad \square\end{aligned}$$

The vertex v is weak. By R2, it gets α from its adjacent senior vertex and gives $2 - \alpha$ to at most $d - 1$ adjacent 2-vertices, it follows from Claim 7 that $\mu^*(v) \geq \frac{(k+3)(d-3)}{k+1} \geq 0$, when $d \geq 3$.

Subcase 3.2. *v is not weak.*

The vertex v is adjacent to two senior vertices.

If $d = 2$, then $\mu^*(v) = -2 + 2 \cdot 1 = 0$ by R3.

If $d = 3$, then $\mu_3 = \frac{5-k}{2(k+1)}$. If v is adjacent to a 2-vertex, then v gives $2 - \alpha$ by R1. By Claim 4, v is adjacent to a non soft $(k+2)^+$-vertex. Note that $\gamma \geq 2 - \alpha > \epsilon$. By R1 and R4, we have $\mu^*(v) \geq \mu_3 - (2 - \alpha) + 2 - \alpha + \epsilon = 0$. On the other hand, if v is not adjacent to a 2-vertex, then $\mu^*(v) \geq \mu_3 + 2\epsilon = \epsilon \geq 0$.

If $d \geq 4$, then by R1, $\mu^*(v) \geq \mu_d - (d-2)(2-\alpha) = \frac{k(d-4)+3d-8}{k+1} \geq 0$.

3.3. *Discharging procedure when $2 \leq k \leq 4$*

3.3.1. *Preliminaries*

A *weak edge* between vertices x and y is either an ordinary edge xy or a path xzy with $3 \leq d(z) \leq k + 1$, where z is called the *intermediate vertex of the weak edge xy*. A *feeding area*, abbreviated to *FA*, is a maximal subgraph of G consisting of $(k + 2)$-vertices mutually accessible from each other along weak edges and of the intermediate vertices of the weak edges of the feeding area. An edge xy with $x \in FA$ and $y \notin FA$ is a *link*. By Claim 5, at least one of the links for *FA* is not soft; such links will be called *rigid*. An *FA* is a *weak feeding area*, denoted by *WFA*, if it has just one rigid link xy; in this case, the vertex y is called the *sponsor* of *WFA*. See Fig. 2. Sometimes a *WFA* with $d(x) = i$ will be denoted by *WFA*(i), where $3 \leq i \leq k+2$. An *FA* with at least two rigid links is *strong* and denoted by *SFA*. By definition (more precisely by maximality), no *WFA*$(k + 2)$ can be joined by its rigid link to an *FA*, and no *WFA*$((k + 1)^-)$ can be joined by its rigid link to a $(k + 2)$-vertex in an *FA*. An immediate consequence of Claim 5 is that no two *WFA*$((k + 1)^-)$ can be joined by their rigid link.

3.3.2. *Discharging for $2 \leq k \leq 4$ and its consequences*

Set $\alpha = \frac{3k+1}{2(k+1)}$, $\gamma = \frac{k-1}{k+1}$, $\beta = \frac{5-k}{2(k+1)}$. Observe that $2 - \alpha = \frac{k+3}{2(k+1)}$. We have:

k	2	3	4
α	7/6	5/4	13/10
γ	1/3	1/2	3/5
β	1/2	1/4	1/10
$2-\alpha$	5/6	3/4	7/10

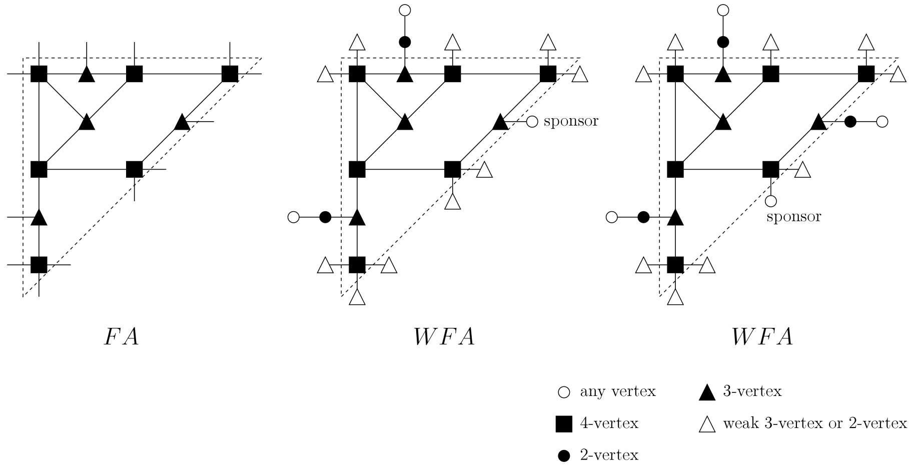

Fig. 2. Examples of feeding areas for $k = 2$.

$$\alpha > 1 > 2 - \alpha > \beta \quad \text{and} \quad 2 - \alpha \geq \gamma.$$

Moreover, $\mu_2 = -2$ and $\mu_3 = \beta$.

A 3^*-vertex is a 3-vertex adjacent to exactly one minor vertex.

The discharging rules for $2 \leq k \leq 4$ are almost the same as for $k \geq 5$. Our rules of discharging are as follows:

R1. Every d-vertex with $3 \leq d \leq k + 1$ gives $2 - \alpha$ to each adjacent 2-vertex.
R2. Every weak vertex gets α from its adjacent senior vertex.
R3. Every non-weak 2-vertex gets 1 from each neighbor.
R4. Every 3^*-vertex gets $2 - \alpha$ from each adjacent $(k + 3)^+$-vertex.
R5. Every *WFA* gets β along the rigid link from its sponsor.

By the definition of *FA*, a minor vertex can belong to at most one feeding area. We cannot prove that each vertex v belonging to an *FA* has $\mu^*(v) \geq 0$; however, it turns out that the total new charge $\mu^*(FA)$ of a feeding area *FA* is nonnegative (see Lemma 1). This is also a way to arrive at a contradiction with (1).

We now prove $\mu^*(v) \geq 0$ assuming that v is not in an *FA*.

Case 1. $d = d(v) \geq k + 3$.

By Claim 3, the vertex v is adjacent to at most $d - 1$ weak vertices. If v is adjacent to exactly $d - 1$ weak vertices $z_1, \ldots, z_{d-1}$, then its dth neighbor z_d (which is not a 2-vertex by Claim 3) may be a 3^*-vertex or a vertex belonging to a *WFA*. Hence v gives α to each adjacent weak vertex by R2 and may give $2 - \alpha$ by R4, or β by R5 ($2 - \alpha > \beta$); it follows that $\mu^*(v) \geq \mu_d - (d-1)\alpha - (2-\alpha) = \mu_d - (d-2)\alpha - 2 \geq 0$ (see Claim 6). Now if v is adjacent to at most $d-2$ weak vertices, then its two last neighbors may be 2-vertices and so $\mu^*(v) \geq \mu_d - (d - 2)\alpha - 2 \geq 0$ by R2–R5 ($\alpha > 1 > 2 - \alpha > \beta$).

Case 2. $d = k + 2$.

Since every $(k + 2)$-vertex belongs to an *FA* by definition, this case does not occur.

Case 3. $2 \leq d \leq k + 1$.

We consider two cases depending on whether or not v is weak.

Subcase 3.1. *v is weak.*

If $d = 2$, then by R1 and R2, v receives $2 - \alpha$ from its minor neighbor and α from its senior neighbor, so $\mu^*(v) = -2 + 2 - \alpha + \alpha = 0$.

Suppose that $d \geq 3$. The vertex v is adjacent to $d - 1$ minor vertices, say $z_1, \ldots, z_{d-1}$, and to a senior vertex, say z_d. By Claim 5, the edge vz_d cannot be the rigid link of a *WFA*. By R2, v receives α from z_d. Now, each edge vz_i may lead to a 2-vertex, and in this case, v gives $2 - \alpha$ to z_i, or may lead to a l-vertex with $3 \leq l \leq k + 1$ belonging to a *WFA* (vz_i is a rigid link), and in this case, v gives β to the corresponding *WFA*. Since $2 - \alpha > \beta$, it follows that $\mu^*(v) \geq \mu_d - (d - 1)(2 - \alpha) + \alpha \geq 0$ (see Claim 7).

Subcase 3.2. *v is not weak.*

If $d = 2$, then $\mu^*(v) = -2 + 2 \cdot 1 = 0$ by R3.

Assume that $d \geq 3$. Observe that v is adjacent to at least two senior vertices (v is not weak) and at most one of them belongs to an *FA* (otherwise, v would belong to an *FA*, contradicting our assumption).

Suppose $d = 3$. If v is not a 3^*-vertex, then v is adjacent to three senior vertices and $\mu^*(v) \geq \mu_3 - \beta = 0$ by R5. If v is a 3^*-vertex, then v is adjacent to a $(k+3)^+$-vertex which gives $2-\alpha$ to v by R4. Hence, $\mu^*(v) \geq \mu_3 - (2-\alpha) - \beta + (2-\alpha) = 0$ by R1, R4, and R5 ($2 - \alpha > \beta$).

Suppose $d \geq 4$. By R1 and R5, v gives nothing to at least one $(k + 3)^+$-vertex; hence $\mu^*(v) \geq \mu_d - (d-2)(2-\alpha) - \beta = \frac{(2d-7)(k+3)}{2(k+1)} \geq 0$ when $d \geq 4$.

Hence we proved that for every vertex v not in an *FA*, $\mu^*(v) \geq 0$. Since the *FA*'s in G are disjoint, to complete the proof of Theorem 1 it suffices to prove the following:

Lemma 1. *For each FA in G,*

$$\mu^*(FA) = \sum_{v \in V(FA)} \mu^*(v) \geq 0.$$

Proof. Consider a feeding area F and let v be a vertex of F. Let $f(v)$ be the number of neighbors of v in F, $s(v)$ the number of rigid links incident with v over that v does not send charge by R5, and $r(v)$ the number of all other rigid links incident with v.

Suppose first v has degree $k+2$. By maximality of the feeding area, v does not send charge by R5. It follows that $r(v) = 0$. Hence v sends charge only by R2 and R3 to adjacent light vertices (the charge is sent only over incident soft edges). The final charge of v is at least

$$\mu(v) - (k + 2 - f(v) - s(v) - r(v))\alpha = (f(v) + s(v) + r(v))\alpha - 2(2 - \alpha).$$

Observe that this charge is non-negative if $s(v) + r(v) \geq 2$. If $s(v) + r(v) = 1$, then the charge is equal to $-\beta$, but, in that case, F is a weak feeding area containing only v, and receives β by R5. Hence $\mu^*(F) \geq 0$. Thus we can assume that F has more than one vertex.

Suppose now that $3 \leq d(v) \leq k + 1$. Vertex v sends charge over soft edges by R1 and over rigid links by R5, and its charge becomes at least

$$\mu(v) - (d(v) - f(v) - s(v) - r(v))(2 - \alpha) - r(v)\beta = (2 - \alpha)(2d(v) + f(v) + s(v)) + \gamma r(v) - \frac{5k + 11}{k + 1}.$$

Since $d(v) \geq 3$, we have $(2 - \alpha)2d(v) - \frac{5k+11}{k+1} \geq -2$. The final charge of v is at least

$$(2 - \alpha)(f(v) + s(v)) + \gamma r(v) - 2.$$

Let $s = \sum_{v \in V(F)} s(v)$ and $r = \sum_{v \in V(F)} r(v)$. For a vertex $v \in F$, let us define $w(v) = \alpha$ if $d(v) = k+2$ and $w(v) = 2-\alpha$ otherwise. Let n_1 be the number of vertices of F of degree $k + 2$ and n_2 the number of minor vertices of F. Summing the estimates obtained in the previous two paragraphs, we conclude that the total charge of the vertices of F is at least

$$(2 - \alpha)s + \gamma r - 2(2 - \alpha)n_1 - 2n_2 + \sum_{v \in V(F)} w(v)f(v). \tag{2}$$

For an edge $e = uv$ of F, let us define $w(e) = w(u) + w(v)$. Observe that $\sum_{v \in V(F)} w(v)f(v) = \sum_{e \in E(F)} w(e)$. Let an edge of F be *good* if at least one of its incident vertices has degree $k + 2$. We have $w(e) \geq 2$ if e is good and $w(e) = 2(2 - \alpha)$ otherwise. Let m be the number of good edges of F. Since F contains a spanning tree consisting of only good edges, we have $m \geq n_1 + n_2 - 1$. Let $\delta = m - (n_1 + n_2 - 1)$.

Observe that if F is weak, then F has a unique rigid edge (by definition) and by R5 a charge β is transferred inside F along this edge. If F is strong, then at least one rigid link does not lead to a weak feeding area by Claim 5, and no charge is transferred along this link by R5. Hence $s \geq 1$. Applying these inequalities, we conclude that the total charge of the vertices of F is at least

$$(2 - \alpha - \gamma) + \gamma(r + s) - 2(2 - \alpha)n_1 - 2n_2 + 2(n_1 + n_2 - 1 + \delta) = \beta + \gamma(r + s) + \gamma n_1 - 2 + 2\delta.$$

Recall that F contains at least two vertices. Hence $n_1 \geq 2$.

Let us first consider the case $r + s = 1$, i.e. F is weak. Then F receives β by R5 and its final charge is at least

$$\beta + \gamma + \gamma n_1 - 2 + 2\delta + \beta \geq 3\gamma + 2\beta - 2 \geq 0.$$

Consider now the case $r+s \geq 2$. The charge of F is at least $\beta+2\gamma+\gamma n_1-2+2\delta$, which is only negative if $k = 2$, $n_1 = 2$, and $\delta = 0$ (in this case, the charge is at least $\beta + 4\gamma - 2$). Since $\delta = 0$, F contains at most one minor vertex, and since $k = 2$, such a vertex has degree 3 and can be incident with at most one rigid link. Therefore, at least one vertex of degree $k+2$ is incident with a rigid link. However, this rigid link contributes α to the charge of F instead of $2-\alpha$ that we accounted for it in (2). Therefore, the charge of F is by $2(\alpha - 1)$ greater than we estimated, and thus the final total charge of F is $\beta + 4\gamma - 2 + 2(\alpha - 1) \geq 0$.

This completes the proofs of Lemma 1 and Theorem 1. □

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