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Weight modules over exp-polynomial Lie algebras ☆

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Abstract

In this paper, we generalize a result by Berman and Billig on weight modules over Lie algebras with polynomial multiplication. More precisely, we show that a highest weight module with an exp-polynomial "highest weight" over an exp-polynomial Lie algebra has finite dimensional weight spaces. We also get a class of irreducible weight modules with finite dimensional weight spaces over generalized Virasoro algebras which do not occur over the classical Virasoro algebra.

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0. Introduction

Representations of affine Lie algebras and the Virasoro algebra have many important applications in mathematics and physics. One of the main ingredients of these theories is the construction of the highest weight modules. Recently there has been substantial activity in developing representation theories for higher rank infinite dimensional Lie algebras, in particular toroidal Lie algebras, generalized Virasoro algebras and quantum torus Lie algebras (see [1–4,9,11–13,15,16,18,20–23]).

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Unlike rank one algebras (affine and Virasoro), the higher rank infinite dimensional Lie algebras do not possess the triangular decomposition (as defined in [19]). This is explained by the fact that these algebras are graded by $\mathbb{Z}^n$ with all graded components being non-trivial, and there is no natural way of dividing $\mathbb{Z}^n$ into a positive and a negative parts when $n > 1$. Because of this, the standard construction of the highest weight modules produces uninteresting representations. Nonetheless, there have been found several explicit realizations of nice representations, using the vertex operator approach.

In the vertex constructions the highest weight is replaced with a loop-like module for the subalgebra of degree zero (in general, this subalgebra is non-commutative and infinite dimensional). Let us describe in brief these representations from the perspective of the construction of the highest weight modules.

Let G be a $\mathbb{Z}\times\mathbb{Z}^n$-graded Lie algebra and let $G=G^-\oplus G^{(0)}\oplus G^+$ be a decomposition of G relative to the $\mathbb{Z}$-grading. The subalgebra $G^{(0)}$ is an infinite dimensional Lie algebra of rank n. We take some natural module V for $G^{(0)}$ (usually V is either $\mathbb{Z}^n$-graded or finite dimensional). Parallel to the construction of a highest weight module, we let G^+ act on V trivially, and introduce the induced module

$$\widetilde{M}(V) = \mathrm{Ind}_{G^{(0)}+G^+}^{G} V \simeq U(G^-) \otimes_{\mathbb{C}} V.$$

If V is $\mathbb{Z}^n$-graded then $\widetilde{M}(V)$ inherits a $\mathbb{Z}\times\mathbb{Z}^n$-grading, and it is $\mathbb{Z}$-graded when V is finite dimensional.

The difficulty here is that $\widetilde{M}(V)$ will have infinite dimensional homogeneous components (and thus will not have a character). Nonetheless the explicit vertex operator constructions show that in some cases $\widetilde{M}(V)$ has quotients with finite dimensional homogeneous components. This situation has been clarified in [1], where it was proved that $\widetilde{M}(V)$ has a graded factor-module $M(V)$ with finite dimensional components provided that G is a polynomial Lie algebra and V is a polynomial module. The polynomiality condition means that the structure constants of the Lie algebra and of the module are given by polynomial expressions (see Section 1 for the precise definitions). However this left out some important examples, notably quantum torus Lie algebras. In this paper we expand the class of Lie algebras and modules for which the theorem is applicable to quantum torus Lie algebras. We prove that $M(V)$ has finite dimensional homogeneous components when G is an exp-polynomial Lie algebra and V is an exp-polynomial module.

We illustrate our definitions and theorems with a sequence of 24 examples.

The paper is organized as follows. In Section 1, we define exp-polynomial Lie algebras, exp-polynomial modules and give the statement of the main results (Theorems 1.5 and 1.7). We provide numerous examples (old and new) of such Lie algebras and modules as well as give examples of applications of the main theorems. In Section 2, we first establish the extended Vandermonde determinant formula, and then give the proof of Theorems 1.5 and 1.7. In Section 3, we show a similar but stronger result for generalized Virasoro algebras to give a class of irreducible weight modules with finite dimensional weight spaces over generalized Virasoro algebras which do not occur over the classical Virasoro algebra. These irreducible weight modules are recently proved to be the only irreducible weight modules with finite dimensional weight spaces over higher rank Virasoro algebras besides modules of intermediate series [16].

1. Exp-polynomial Lie algebras and exp-polynomial modules

Let $\mathbb{C}$ be the complex number field. We assume that all Lie algebras and vector spaces are over $\mathbb{C}$ in this paper, although $\mathbb{C}$ can be replaced by any field of characteristic 0.

Definition 1.1. The algebra of exp-polynomial functions in r variables, $n_1, \ldots, n_r$, is the algebra of functions $f(n_1, \ldots, n_r) : \mathbb{Z}^r \to \mathbb{C}$ generated as an algebra by functions n_j and a^{n_j}, where $a \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $j = 1, \ldots, r$.

An exp-polynomial function may be written as a finite sum

$$f(n_1, \ldots, n_r) = \sum_{k \in \mathbb{Z}_+^r} \sum_{a \in (\mathbb{C}^*)^r} c_{k,a} n_1^{k_1} \ldots n_r^{k_r} a_1^{n_1} \ldots a_r^{n_r},$$

where $c_{k,a} \in \mathbb{C}$, $k = (k_1, \ldots, k_r)$ with $k_j \geqslant 0$, and $a = (a_1, \ldots, a_r)$.

It will be important for us that an exp-polynomial function $f(n_1, \ldots, n_r)$ has the property that the function $f(n_1 + m_1, \ldots, n_r + m_r)$ is also exp-polynomial as a function in $2r$ variables.

Definition 1.2. Let $G = \bigoplus_{\alpha \in \mathbb{Z}^n} G_\alpha$ be a $\mathbb{Z}^n$-graded Lie algebra and K be an index set. Then G is said to be an *exp-polynomial Lie algebra* if G has a homogeneous spanning set $\{g_k(\alpha) | k \in K, \ \alpha \in \mathbb{Z}^n\}$ with $g_k(\alpha) \in G_\alpha$, and there exists a family of exp-polynomial functions $\{f_{k,r}^s(\alpha, \beta) | k, r, s \in K\}$ in the $2n$ variables α_j, β_j and where for each k, r the set $\{s | f_{k,r}^s(\alpha, \beta) \neq 0\}$ is finite, such that

$$[g_k(\alpha), g_r(\beta)] = \sum_{s \in K} f_{k,r}^s(\alpha, \beta) g_s(\alpha + \beta) \quad \text{for } k, r \in K, \ \alpha, \beta \in \mathbb{Z}^n. \tag{1.1}$$

This homogeneous spanning set $\{g_k(\alpha) | k \in K, \alpha \in \mathbb{Z}^n\}$ is called *a distinguished spanning set*.

If the functions $f_{k,r}^s(\alpha, \beta)$ in (1.1) are in fact polynomials, we say that G is a *polynomial* Lie algebra (cf. Definition 1.6 in [1]).

Example 1. Let $\mathfrak{g}$ be a finite dimensional Lie algebra with a basis $B = \{g_k\}_{k \in K}$. Then the toroidal Lie algebra $G = \mathbb{C}[t_1^{\pm}, \ldots, t_n^{\pm}] \otimes \mathfrak{g}$ is a polynomial Lie algebra with the distinguished spanning set $g_k(\alpha) = t^\alpha g_k$, $\alpha \in \mathbb{Z}^n$, where here and elsewhere in this paper $t^\alpha = t_1^{\alpha_1} \ldots t_n^{\alpha_n}$. The Lie bracket in G is

$$[g_k(\alpha), g_r(\beta)] = [g_k, g_r](\alpha + \beta).$$

Example 2. Let $R_n = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. The *Witt algebra* (or Cartan type W Lie algebra) is the Lie algebra $W_n = \mathrm{Der}(R_n) = \mathrm{Span}\{t^\alpha \partial_i | \alpha \in \mathbb{Z}^n, 1 \leqslant i \leqslant n\}$, where $\partial_i = t_i(\partial/\partial t_i)$. The bracket in W_n is given by

$$[t^\alpha \partial_i, t^\beta \partial_j] = \beta_i t^{\alpha+\beta} \partial_j - \alpha_j t^{\alpha+\beta} \partial_i.$$

Thus W_n is a polynomial Lie algebra (see [6]).

Example 3. The following family of algebras plays an important role in the representation theory of toroidal Lie algebras (see [1]). Let Ω_n^1 be the space of 1-forms on a torus: $\Omega_n^1 = \sum_{p=1}^n R_n k_p$, where $k_p = t_p^{-1}\,\mathrm{d}t_p$. We define a 2-parameter family of algebras

$$V(\mu, v) = W_n \oplus \Omega_n^1/\mathrm{d}R_n.$$

The distinguished spanning set is

$$\{d_j(\alpha), k_j(\alpha) | \alpha \in \mathbb{Z}^n, j = 1, \ldots, n\},$$

where $d_j(\alpha) = t^\alpha \partial_j$, $k_j(\alpha) = t^\alpha k_j$. The Lie bracket in $V(\mu, v)$ is given by

$$[d_i(\alpha), k_j(\beta)] = \beta_i k_j(\alpha + \beta) + \delta_{ij} \sum_{p=1}^n \alpha_p k_p(\alpha + \beta),$$

$$[d_i(\alpha), d_j(\beta)] = \beta_i d_j(\alpha + \beta) - \alpha_j d_i(\alpha + \beta) + (\mu \beta_i \alpha_j + v \alpha_i \beta_j) \sum_{p=1}^n \beta_p k_p(\alpha + \beta),$$

$$[k_i(\alpha), k_j(\beta)] = 0.$$

Note that the distinguished spanning set for this polynomial Lie algebra is not a basis because of the linear dependencies between the $k_j(\alpha)$: $\alpha_1 k_1(\alpha) + \cdots + \alpha_n k_n(\alpha) = 0$.

Next we will give an example of a Lie algebra which is exp-polynomial, but not polynomial.

Example 4. Consider an associative quantum torus

$$\mathbb{C}_q = \bigoplus_{i \in \mathbb{Z}, \alpha \in \mathbb{Z}^n} \mathbb{C} t_0^i t^\alpha$$

generated by the variables $t_0^\pm, t_1^\pm, \ldots, t_n^\pm$, where $t_1, \ldots, t_n$ commute and t_0 does not commute with $t_1, \ldots, t_n$, but satisfies the relations: $t_j t_0 = q_j t_0 t_j$, for some $q_1, \ldots, q_n \in \mathbb{C}^*$. The Lie algebra which is obtained from the associative algebra C_q is an exp-polynomial Lie algebra with the distinguished set $t_0^i(\alpha) = t_0^i t^\alpha$, and the Lie bracket given by $[t_0^i(\alpha), t_0^j(\beta)] = (q^{j\alpha} - q^{i\beta}) t_0^{i+j}(\alpha + \beta)$. Here $q^{j\alpha} = q_1^{j\alpha_1} \ldots q_n^{j\alpha_n}$.

Definition 1.3. Let $G = \bigoplus_{\alpha \in \mathbb{Z}^n} G_\alpha$ be an exp-polynomial Lie algebra. A G module $V = \bigoplus_{\alpha \in \mathbb{Z}^n} V_\alpha$ is called *a* $\mathbb{Z}^n$*-graded exp-polynomial module* if V has a basis $\{v_j(\alpha)\}_{j \in J, \alpha \in \mathbb{Z}^n}$, and there exists a family of exp-polynomial functions $h_{k,j}^s(\alpha, \beta)$ for $k \in K$, $j, s \in J$ such that

$$g_k(\alpha) v_j(\beta) = \sum_{s \in J} h_{k,j}^s(\alpha, \beta) v_s(\alpha + \beta),$$

where $\{g_k(\alpha)\}_{k\in K}$ is the distinguished spanning set for G, and for each k,j the set $\{s|h^s_{k,j}(\alpha,\beta)\neq 0\}$ is finite. The homogeneous components of the $\mathbb{Z}^n$-grading on $V=\bigoplus_{\alpha\in\mathbb{Z}^n} V_\alpha$ are given by $V_\alpha=\mathrm{Span}\,\{v_j(\alpha)\}_{j\in J}$.

Note that if G is a Lie algebra with polynomial multiplication and all $h^s_{k,j}$ are polynomial functions, then the module V is actually a polynomial module defined in Definition 1.8 of [1].

Example 5. Let $V_1,\dots,V_k$ be finite dimensional modules for a finite dimensional Lie algebra $\mathfrak{g}$. Fix $q_1,\dots,q_k\in(\mathbb{C}^*)^n$. We define on the space $V=R_n\otimes V_1\otimes\cdots\otimes V_k$ the structure of an exp-polynomial module for the toroidal Lie algebra $G=R_n\otimes\mathfrak{g}$ by

$$g(\alpha)v_1\otimes\cdots\otimes v_k(\beta)=\sum_{p=1}^{k} q_p^{\alpha_p} v_1\otimes\cdots\otimes(gv_p)\otimes\cdots\otimes v_k(\alpha+\beta),$$

where $v_1\otimes\cdots\otimes v_k(\beta)=t^\beta v_1\otimes\cdots\otimes v_k$. It is easy to see that in general V is an exp-polynomial module and not a polynomial module.

This module V was studied in several papers like [8].

Example 6. A tensor module for the Witt algebra W_n is a polynomial module. Let V_0 be a finite dimensional gl_n module. Then the tensor module $V=R_n\otimes V_0$ is a module for W_n under the action

$$t^\alpha\partial_j v(\beta)=\beta_j v(\alpha+\beta)+\sum_{p=1}^{n}\alpha_p(E_{pj}v)(\alpha+\beta),$$

where $v(\beta)=t^\beta\otimes v$, and E_{pj} are the elementary matrices in gl_n. The families $\{v_i(\beta)\}$ with $\{v_i\}$ being a basis of V_0, form a distinguished basis in V, for which the structure constants are polynomials.

This module V was initially defined in [21] and later studied in several references like [10].

We extend Definition 1.6 from [1] to give the following

Definition 1.4. Let G be a $\mathbb{Z}^n$-graded exp-polynomial Lie algebra. We call this algebra *$\mathbb{Z}^n$-extragraded* if G has another $\mathbb{Z}$-gradation

$$G=\bigoplus_{i\in\mathbb{Z}} G^{(i)} \tag{1.2}$$

and the set K is a disjoint union of *finite* subsets K_i,

$$K=\bigcup_{i\in\mathbb{Z}} K_i,$$

such that the elements of the homogeneous spanning set $\{g_k(\alpha)|k\in K_i,\alpha\in\mathbb{Z}^n\}$ are homogeneous of degree i under this new $\mathbb{Z}$-gradation and span $G^{(i)}$.

Many important infinite dimensional Lie algebras are in fact $\mathbb{Z}^n$-extragraded exp-polynomial Lie algebras. Here we give some examples.

Example 8. We slightly modify Example 1 to get an extragraded algebra. Adding an extra variable t_0, we get an $n+1$ toroidal Lie algebra $R_{n+1} \otimes \mathfrak{g}$, where $R_{n+1} = \mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_n^{\pm}]$. We consider a $\mathbb{Z}^n$-grading on this algebra by degrees in $t_1, \ldots, t_n$, and a $\mathbb{Z}$-grading by degree in t_0. The distinguished spanning set is

$$t_0^i g_k(\alpha) = t_0^i t^\alpha g_k$$

with $i \in \mathbb{Z}$, $\alpha \in \mathbb{Z}^n$, and $\{g_k\}$ being a basis of $\mathfrak{g}$.

The previous example can be generalized in two ways:

Example 9. Let $G^{(0)}$ be an exp-polynomial Lie algebra with $G_\alpha^{(0)}$ being finite dimensional for all $\alpha \in \mathbb{Z}^n$. Then $G = \mathbb{C}[t_0, t_0^{-1}] \otimes G^{(0)}$ is an extragraded exp-polynomial Lie algebra.

Example 10. Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie algebra with finited dimensional homogeneous components. Then $G = \mathfrak{g} \otimes R_n$ is an extragraded exp-polynomial Lie algebra.

Remark. In the definition of an exp-polynomial Lie algebra we can relax the requirement that each family $\{g_k(\alpha)\}$ be defined for all $\alpha \in \mathbb{Z}^n$. We may instead require that $\{g_k(\alpha)\}$ is defined for α in some sublattice $L_k \subset \mathbb{Z}^n$. Of course in this case we should have a restriction so that the expression in the right-hand side of (1.1) is well defined.

Example 11. With this relaxed definition we may consider the following as an extragraded exp-polynomial Lie algebra:

$$G = \mathfrak{g} \otimes R_{n+1} \oplus \operatorname{Der} \mathbb{C}[t_0, t_0^{-1}]$$

for a finite dimensional Lie algebra $\mathfrak{g}$ with the distinguished spanning set $\{t_0^i g_k(\alpha), t_0^i \partial_0 \mid i \in \mathbb{Z}, \alpha \in \mathbb{Z}^n, g_k \in B\}$. Here the sublattice that corresponds to the elements $t_0^i \partial_0$ is $L=(0)$.

Example 12. Introducing an extra variable t_0 we can construct an extragraded version of the Lie algebra from Example 3:

$$V(\mu, \nu) = \operatorname{Der} R_{n+1} \oplus \Omega_{n+1}^1 / \mathrm{d}R_{n+1}.$$

The $\mathbb{Z}$-grading is by degree in t_0, and the distinguished spanning families are $t_0^i d_j(\alpha) = t_0^i t^\alpha \partial_j$ and $t_0^i k_j(\alpha) = t_0^i t^\alpha k_j$.

Example 13. The exp-polynomial Lie algebra from Example 4 is actually extragraded. The $\mathbb{Z}$-grading on it is by degree in t_0.

Example 14. Let $R_{n+1} = \mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, $W_{n+1} = \operatorname{Der}(R_{n+1}) = \operatorname{Span}\{t_0^i t^\alpha \partial_k, t_0^i t^\alpha \partial_0 \mid i \in \mathbb{Z}, \alpha \in \mathbb{Z}^n, 1 \leqslant k \leqslant n\}$. The Lie bracket in W_{n+1} is given by

$$[t_0^i t^\alpha \partial_k, t_0^j t^\beta \partial_s] = t_0^{i+j} t^{\alpha+\beta} (\beta_k \partial_s - \alpha_s \partial_k), \quad 1 \leqslant k, s \leqslant n,$$

$$[t_0^i t^\alpha \partial_0, t_0^j t^\beta \partial_s] = t_0^{i+j} t^{\alpha+\beta}(j\partial_s - \alpha_s \partial_0), \quad 1 \leqslant s \leqslant n,$$

$$[t_0^i t^\alpha \partial_0, t_0^j t^\beta \partial_0] = (j-i) t_0^{i+j} t^{\alpha+\beta} \partial_0.$$

It can be easily seen that W_{n+1} is a $\mathbb{Z}^n$-extragraded Lie algebra.

Cartan type S Lie algebras S_{n+1} are also $\mathbb{Z}^n$-extragraded polynomial Lie algebras. However we do not know whether Cartan type H or K Lie algebras are $\mathbb{Z}^n$-extragraded polynomial Lie algebras.

Example 15. The *Virasoro-like algebra* L over $\mathbb{C}$ is the Lie algebra with a $\mathbb{C}$-basis $\{L_x | x \in \mathbb{Z}^2\}$ and subject to the following commutator relations:

$$[L_x, L_y] = \det \begin{pmatrix} y \\ x \end{pmatrix} L_{x+y} \quad \forall x, y \in \mathbb{Z}^2,$$

where $x = (x^{(1)}, x^{(2)})$, $y = (y^{(1)}, y^{(2)})$, $\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} y^{(1)} & y^{(2)} \\ x^{(1)} & x^{(2)} \end{pmatrix}$. For more details, see [7]. It can be easily seen that L is a $\mathbb{Z}^1$-extragraded Lie algebra.

From now on we assume that G is a $\mathbb{Z}^n$-extragraded Lie algebra with gradations (1.1) and (1.2), i.e., $G = \bigoplus_{i \in \mathbb{Z}, \alpha \in \mathbb{Z}^n} G_\alpha^{(i)}$ is a $\mathbb{Z}^{n+1}$-graded Lie algebra which has a homogeneous spanning set $\{g_k^{(i)}(\alpha) | k \in K_i, (i, \alpha) \in \mathbb{Z}^{n+1}\}$ with $g_k^{(i)}(\alpha) \in G_\alpha^{(i)}$, and there exists a family of exp-polynomial functions $\{f_{k,m,i,j}^s(\alpha, \beta)\}$ in the $2n$ variables α_p, β_p where $k \in K_i$, $m \in K_j$, $s \in K_{i+j}$ and where for each k, m, i, j the set $\{s | f_{k,m,i,j}^s(\alpha, \beta) \neq 0\}$ is finite, such that

$$[g_k^{(i)}(\alpha), g_m^{(j)}(\beta)] = \sum_{s \in K_{i+j}} f_{k,m,i,j}^s(\alpha, \beta) g_s^{(i+j)}(\alpha + \beta), \quad \text{for } \alpha, \beta \in \mathbb{Z}^n. \tag{1.3}$$

Let $G^+ = \bigoplus_{i \geqslant 1} G^{(i)}$, $G^- = \bigoplus_{i \leqslant -1} G^{(i)}$. Then we have the decomposition

$$G = G^- \oplus G^{(0)} \oplus G^+. \tag{1.4}$$

Note that $G^{(0)}$ is a $\mathbb{Z}^n$-graded exp-polynomial Lie algebra.

Following the construction in [1], we now introduce our $\mathbb{Z}^{n+1}$-graded module over $\mathbb{Z}^{n+1}$-graded Lie algebra G.

Assume $V = \bigoplus_{\alpha \in \mathbb{Z}^n} V_\alpha$ is a $\mathbb{Z}^n$-graded $G^{(0)}$ module with exp-polynomial action as defined in Definition 1.3. We can define the action of G^+ on V by $G^+ V = 0$ and then consider the induced module

$$\widetilde{M}(V) = \mathrm{Ind}_{G^{(0)}+G^+}^{G} V \simeq U(G^-) \otimes_{\mathbb{C}} V. \tag{1.5}$$

It is clear that $\widetilde{M}(V)$ is a $\mathbb{Z}^{n+1}$-graded module over G and

$$\widetilde{M}(V) = \bigoplus_{i \leqslant 0, \alpha \in \mathbb{Z}^n} \widetilde{M}(V)_\alpha^{(i)}, \tag{1.6}$$

where $\widetilde{M}(V)_\alpha^{(i)}$ is naturally defined, for example, $\widetilde{M}(V)_\alpha^{(0)} = V_\alpha$. In general, the homogeneous components $\widetilde{M}(V)_\alpha^{(i)}$ with $i < 0$ are infinite dimensional.

It is easy to see that $\widetilde{M}(V)$ has a unique maximal proper $\mathbb{Z}^{n+1}$-graded submodule $\widetilde{M}^{\rm rad}$ which intersects trivially with V. Let

$$M(V) = \widetilde{M}(V)/\widetilde{M}^{\rm rad}. \qquad (1.7)$$

Then we have the induced $\mathbb{Z}^{n+1}$-gradation

$$M(V) = \bigoplus_{i \leqslant 0, \alpha \in \mathbb{Z}^n} M(V)^{(i)}_{\alpha}. \qquad (1.8)$$

The main result of this paper is the following theorem.

Theorem 1.5. *Assume that G is a $\mathbb{Z}^n$-extragraded Lie algebra with grading (1.3), $V = \bigoplus_{\alpha \in \mathbb{Z}^n} V_\alpha$ is a $\mathbb{Z}^n$-graded exp-polynomial $G^{(0)}$ module as defined in Definition* 1.3 *with J being finite. Then the $\mathbb{Z}^{n+1}$-graded G module $M(V)$ defined in (1.7) has finite dimensional homogeneous spaces, i.e.*, $\dim M(V)^{(i)}_{\alpha} < \infty$, *for all $i \in \mathbb{Z}$, $\alpha \in \mathbb{Z}^n$.*

If $G^{(0)}$ is a polynomial Lie algebra and V is a polynomial $\mathbb{Z}^n$-graded $G^{(0)}$ module, the claim of Theorem 1.5 was proved in [1], Theorem 1.12.

Example 16. Let us consider an $n+1$-toroidal Lie algebra $G = R_{n+1} \otimes \mathfrak{g}$ as defined in Example 8. Its component $G^{(0)}$ is an n-toroidal Lie algebra. We consider a module $V = R_n \otimes V_1 \otimes \cdots \otimes V_k$ for $G^{(0)}$ which was described in Example 5. By the above theorem, the G module $M(V)$ has finite dimensional homogeneous components. Such modules were studied by Chari [5] and Rao [8].

Example 17. Let us consider the Virasoro-like algebra L as defined in Example 15. Let $L^{(i)} = \bigoplus_{k \in \mathbb{Z}} \mathbb{C}L_{i,k}$. Fix an exp-polynomial function $f(k)$. Then $V = \bigoplus_{j \in \mathbb{Z}} \mathbb{C}v_j$ becomes an exp-polynomial $L^{(0)}$ module via

$$L_{0,k} v_j = f(k) v_{j+k}.$$

Then $M(V)$ has finite dimensional homogeneous spaces.

Example 18. Consider a Witt algebra $W_{n+1} = \operatorname{Der} \mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_n^{\pm}]$, which we view as an extragraded polynomial Lie algebra with $\mathbb{Z}$-grading given by degree in t_0. The zero component with respect to this grading is $W_n \oplus R_n \partial_0$. Consider a tensor module $V = R_n \otimes V_0$ for W_n as discussed in Example 6. We let $R_n \partial_0$ act upon it by shifts $t^\alpha \partial_0 \cdot t^\beta \otimes v = {\rm d} t^{\alpha+\beta} \otimes v$ for some fixed constant $d \in \mathbb{C}$. By Theorem 1.5 the module $M(V)$ is a weight module with finite dimensional weight spaces.

Example 19. Let $V(\mu, v)$ be the extragraded Lie algebra from Example 12. Its zero component with respect to the $\mathbb{Z}$-grading is

$$W_n \oplus R_n \partial_0 \oplus \left(\sum_{p=0}^{n} R_n k_p / {\rm d} R_n \right).$$

Consider a tensor module V from the previous example on which we define the action of 1-forms of degree zero (in t_0) as follows:

$$t^\alpha k_p \cdot t^\beta \otimes v = 0, \quad p = 1, \ldots, n,$$

$$t^\alpha k_0 \cdot t^\beta \otimes v = c t^{\alpha+\beta} \otimes v, \qquad \text{for some } c \in \mathbb{C}.$$

Again by Theorem 1.5 the module $M(V)$ is a weight module with finite dimensional weight spaces. These modules and their irreducible quotients were studied in [1,3,4,9,15].

Example 20. Let G be the quantum torus Lie algebra from Example 4. We consider a module $V = \mathbb{C}[t_1^\pm, \ldots, t_n^\pm]$ for $G^{(0)} = \mathbb{C}[t_1^\pm, \ldots, t_n^\pm]$ with the action defined as follows: $t^\alpha \cdot t^\beta = f(\alpha, \beta) t^{\alpha+\beta}$, where $f(\alpha, \beta)$ is some fixed exp-polynomial function. By Theorem 1.5 the module $M(V)$ has finite dimensional homogeneous components.

Example 21. Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra with the triangular decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$. We consider a finite $\mathbb{Z}$-grading on $\mathfrak{g}$ compatible with this decomposition. As explained in Example 10 above, the algebra $G = R_n \otimes \mathfrak{g}$ is an extragraded polynomial Lie algebra. Its zero component is the abelian algebra $G^{(0)} = R_n \otimes \mathfrak{h}$. Consider a $G^{(0)}$ module $V = \mathbb{C}[t_1^\pm, \ldots, t_n^\pm]$ with the action $t^\alpha h_k \cdot t^\beta = f_k(\alpha, \beta) t^{\alpha+\beta}$, where $\{h_k\}$ forms a basis of $\mathfrak{h}$ and $\{f_k(\alpha, \beta)\}$ are some fixed exp-polynomial functions. Applying again Theorem 1.5 we conclude that $M(V)$ is a weight module with finite dimensional weight spaces.

We would like to discuss now a finite dimensional version of the exp-polynomial modules.

Definition 1.6. Let $G = \bigoplus_{\alpha \in \mathbb{Z}^n} G_\alpha$ be an exp-polynomial Lie algebra as defined in Definition 1.2. A G module V is called *a finite dimensional exp-polynomial module* if V has a finite basis $\{v_j\}_{j \in J}$, and there exists a family of exp-polynomial functions $h_{k,j}^s(\alpha)$ for $k \in K$, $j, s \in J$ such that

$$g_k(\alpha) v_j = \sum_{s \in J} h_{k,j}^s(\alpha) v_s,$$

where $\{g_k(\alpha)\}_{k \in K, \alpha \in \mathbb{Z}^n}$ is the distinguished spanning set for G.

Let G be an extragraded exp-polynomial Lie algebra and let V be a finite dimensional exp-polynomial module for $G^{(0)}$. Just as in (1.5) and (1.7) we define a $\mathbb{Z}$-graded G module $\widetilde{M}(V)$ and its $\mathbb{Z}$-graded factor-module $M(V) = \widetilde{M}(V)/\widetilde{M}^{\mathrm{rad}}$.

Theorem 1.7. *Let G be a $\mathbb{Z}^n$-extragraded Lie algebra with grading* (1.3), *V be a finite dimensional exp-polynomial $G^{(0)}$ module. Then the $\mathbb{Z}$-graded G module $M(V)$ has finite dimensional homogeneous spaces*, *i.e.*, $\dim M(V)^{(i)} < \infty$, *for all $i \in \mathbb{Z}$.*

Let us now give some examples of the applications of this theorem.

Example 22. Let G be an $n+1$-toroidal Lie algebra $G=R_{n+1}\otimes\mathfrak{g}$ as defined in Example 8. Its component $G^{(0)}$ is an n-toroidal Lie algebra. Let $V_1,\dots,V_k$ be finite dimensional modules for $\mathfrak{g}$, and let $q_1,\dots,q_k\in(\mathbb{C}^*)^n$. We define a structure of a $G^{(0)}$ module on the finite dimensional space $V=V_1\otimes\cdots\otimes V_k$ in the following way:

$$g(\alpha)v_1\otimes\cdots\otimes v_k=\sum_{p=1}^{k}q_p^{\alpha_p}v_1\otimes\cdots(gv_p)\otimes\cdots\otimes v_k.$$

It is easy to see that V is a finite dimensional exp-polynomial module for $G^{(0)}$. The induced module $\widetilde{M}(V)$ will have infinite dimensional homogeneous components, however, by Theorem 1.7, the homogeneous components of its factor $M(V)$ are finite dimensional.

The next two examples are modifications of Examples 20 and 21.

Example 23. Let G be the quantum torus Lie algebra from Example 4. We consider a one-dimensional module $\mathbb{C}$ for $G^{(0)}=\mathbb{C}[t_1^{\pm},\dots,t_n^{\pm}]$ with the action defined as follows: $t^\alpha\cdot 1=f(\alpha)1$, where $f(\alpha)$ is some fixed exp-polynomial function (highest weight). By Theorem 1.7 the module $M(\mathbb{C})$ has finite dimensional homogeneous components.

Example 24. Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra with the triangular decomposition $\mathfrak{g}=\mathfrak{g}_-\oplus\mathfrak{h}\oplus\mathfrak{g}_+$. As explained in Example 21 above, the algebra $G=R_n\otimes\mathfrak{g}$ is an extragraded polynomial Lie algebra. Its zero component is the abelian algebra $G^{(0)}=R_n\otimes\mathfrak{h}$. Let $\{h_1,\dots,h_\ell\}$ be a basis of $\mathfrak{h}$. Fix exp-polynomial functions $f_1(\alpha),\dots,f_\ell(\alpha)$ (highest weight) and consider a one-dimensional $G^{(0)}$ module $\mathbb{C}$ with the action $t^\alpha h_k\cdot 1=f_k(\alpha)1$. Applying again Theorem 1.7 we conclude that $M(\mathbb{C})$ is a $\mathbb{Z}$-graded module with finite dimensional homogeneous components.

2. Proof of the main theorems

In this section we shall prove Theorems 1.5 and 1.7. The key step in the proof of Theorem 1.12 in [1] was the Vandermonde determinant argument. Here we would need a generalization of the Vandermonde determinant formula for the exp-polynomial functions.

Lemma 2.1. *Let $a_1,\dots,a_m$ be elements of a field, $s_1,s_2,\dots,s_m\in\mathbb{N}$ with $s_1+\cdots+s_m=s$. Consider the following sequence of s exp-polynomial functions in one integer variable: $f_1(n)=a_1^n,\ f_2(n)=na_1^n,\dots,f_{s_1}(n)=n^{s_1-1}a_1^n,\ f_{s_1+1}(n)=a_2^n,\dots,f_{s_1+s_2}(n)=n^{s_2-1}a_2^n,\dots,f_s(n)=n^{s_m-1}a_m^n$. Let $V=(v_{pk})$ be the square $s\times s$ matrix where $v_{pk}=f_k(p-1),\ p,k=1,\dots,s$. Then*

$$\det(V)=\prod_{j=1}^{m}(s_j-1)!!\,a_j^{s_j(s_j-1)/2}\prod_{1\leqslant i<j\leqslant m}(a_j-a_i)^{s_is_j}.\qquad(2.1)$$

Here we use the notation $m!! = m! \times (m-1)! \times \cdots \times 2! \times 1!$ with the convention $0!! = 1$.

Proof. This elementary lemma may be proved by induction on s using elementary row and column transformations of the matrix. We will give here just an outline of the proof. The basis of induction case $s = 1$ is trivial. To establish the inductive step, one begins by applying elementary row operations to the matrix, the same as in the proof of the ordinary Vandermonde determinant formula: subtract from the last row the preceding row number $s-1$ multiplied by a_1, then subtract from row $s-1$ the preceding row $s-2$ multiplied by a_1, and so on. This will produce a matrix that has 1 as the top entry of the first column, and the rest of the entries in the first column being zeros.

Next we expand this determinant along the first column, which will yield an $(s-1) \times (s-1)$ matrix with the same determinant. Finally, applying elementary column operations, it is possible to bring this $(s-1)\times(s-1)$ matrix to the form corresponding to the sequence of functions $a_1 a_1^n$, $2a_1 n a_1^n$, $3a_1 n^2 a_1^n, \ldots, (s_1-1)a_1 n^{s_1-2} a_1^n$, $(a_2 - a_1)a_2^n$, $(a_2-a_1)na_2^n, \ldots, (a_2-a_1)n^{s_2-1}a_2^n$, $(a_3-a_1)a_3^n, \ldots, (a_m-a_1)n^{(s_m-1)}a_m^n$.

Pulling out the factors $a_1, 2a_1, 3a_1, \ldots, (s_1-1)a_1$ from the first $s_1 - 1$ columns, and $(a_j - a_1)$ from the remaining columns, we bring the matrix to the extended Vandermonde form of rank $s-1$. To establish the inductive step, we only need to multiply the expression for the extended Vandermonde determinant of rank $s-1$ given by the induction assumption by the factors we pulled out of the columns. □

Corollary 2.2. *In the notation of the previous lemma, let $a_1, \ldots, a_m$ be distinct non-zero elements of a field of characteristic greater or equal to the maximum of $s_1, \ldots, s_m$, or of characteristic 0. Then the set of exp-polynomial functions $f_1(n) = a_1^n, \ldots, f_s(n) = n^{s_m-1}a_m^n$ is linearly independent.*

Proof. From the determinant formula (2.1) we see that the vectors of values of the functions $(f_j(0), f_j(1), \ldots, f_j(s-1))$, $j = 1, \ldots, s$, are linearly independent. □

Corollary 2.3. *Let the exp-polynomial functions $f_1(n), \ldots, f_s(n)$ satisfy the conditions of Lemma 2.1 and Corollary 2.2. Let $\{c_k\}$ be a sequence with only finitely many non-zero terms. The sequence $\{c_k\}$ satisfies an infinite system of linear equations*

$$\sum_k \left(\sum_{j=1}^{s} d_{kj} f_j(n) \right) c_k = 0 \quad for\ all\ n \in \mathbb{Z} \tag{2.2}$$

if and only if it satisfies a finite system

$$\sum_k d_{kj} c_k = 0 \quad for\ all\ j = 1, \ldots, s. \tag{2.3}$$

Proof. Since

$$\sum_k \left(\sum_{j=1}^{s} d_{kj} f_j(n) \right) c_k = \sum_{j=1}^{s} f_j(n) \sum_k d_{kj} c_k,$$

we see that (2.3) implies (2.2).

Suppose now that (2.2) holds. Evaluating (2.2) at $n = 0, \ldots, s-1$, we get that

$$\sum_{j=1}^{s} f_j(n) \sum_k d_{kj} c_k = 0 \quad \text{for } n = 0, \ldots, s-1. \tag{2.4}$$

Now s equations of (2.4) are linear combinations of s equations of (2.3). The change of basis matrix from (2.3) to (2.4) is an extended Vandermonde matrix and is invertible, since by Lemma 2.1 it has a non-zero determinant. Hence, the equations of (2.3) are linear combinations of equations of (2.4) and thus (2.3) holds. □

The last corollary also admits a straightforward multi-variable generalization.

Corollary 2.4. *Let $f_1(n_1, \ldots, n_r), \ldots, f_s(n_1, \ldots, n_r)$ be a set of distinct exp-polynomial functions of the form $f_j(n_1, \ldots, n_r) = n_1^{p_1} \ldots n_r^{p_r} b_1^{n_1} \ldots b_r^{n_r}$, such that exponents p_i's are less than the characteristic of the field, if the field has finite characteristic. Let $\{c_\beta\}_{\beta \in \mathbb{Z}^m}$ be a set with only finitely many non-zero terms. The set $\{c_\beta\}$ satisfies an infinite system of linear equations*:

$$\sum_{\beta \in \mathbb{Z}^m} \left(\sum_{j=1}^{s} d_{\beta j} f_j(n_1, \ldots, n_r) \right) c_\beta = 0 \quad \textit{for all } (n_1, \ldots, n_r) \in \mathbb{Z}^r \tag{2.5}$$

if and only if it satisfies a finite system

$$\sum_{\beta \in \mathbb{Z}^m} d_{\beta j} c_\beta = 0 \quad \textit{for all } j = 1, \ldots, s. \tag{2.6}$$

Now we are ready to give a proof of Theorem 1.5.

Proof of Theorem 1.5. For the $\mathbb{Z}^n$-graded $G^{(0)}$ module $V = \bigoplus_{\alpha \in \mathbb{Z}^n} V_\alpha$ as defined in Definition 1.3, we have used $g_i(\alpha)$ to denote $g_i^{(0)}(\alpha)$. We stress here that both $\{k | g_k^{(i)}(\alpha) \neq 0\}$ for any fixed $(i, \alpha) \in \mathbb{Z}^{n+1}$ and J in Definition 1.3 are finite sets. The proof of the theorem will amount to proving two claims.

Claim 1. *Let us fix $i_1, i_2, \ldots, i_s \in \mathbb{N}$, $k_1, k_2, \ldots, k_s$, with $k_p \in K_{-i_p}$, $k \in J$ and $\alpha \in \mathbb{Z}^n$. There exist exp-polynomial functions $f_1(\beta), \ldots, f_d(\beta)$ in ns variables β such that a linear combination*:

$$\sum_{\beta = (\beta_1, \ldots, \beta_s) \in \mathbb{Z}^{ns}} b_\beta\, g_{k_1}^{(-i_1)}(\beta_1) g_{k_2}^{(-i_2)}(\beta_2) \cdots g_{k_s}^{(-i_s)}(\beta_s) v_k(\alpha - \beta_1 - \cdots - \beta_s) \tag{2.7}$$

belongs to $\widetilde{M}^{\rm rad}$ *if and only if the set* $\{b_\beta\}_{\beta\in\mathbb{Z}^{ns}}$ (*with finitely many non-zero elements*) *satisfies a finite system of linear equations*:

$$\sum_{\beta\in\mathbb{Z}^{ns}} b_\beta f_p(\beta) = 0, \quad \text{for } p = 1,\ldots,d.$$

Proof of Claim 1. Denote the sum in (2.7) by x. We have that $x\in\widetilde{M}^{\rm rad}$ if and only if

$$g_{m_1}^{(j_1)}(\gamma_1)\ldots g_{m_r}^{(j_r)}(\gamma_r)x = 0$$

for all $j_1,\ldots,j_r\in\mathbb{N}$ with $j_1+\cdots+j_r = i_1+\cdots+i_s$, $m_1\in K_{j_1},\ldots,m_r\in K_{j_r}$, $\gamma_1,\ldots,\gamma_r\in\mathbb{Z}^n$.

From the Poincaré–Birkhoff–Witt argument and the fact that the Lie algebra G and the module V are exp-polynomial, it follows that

$$g_{m_1}^{(j_1)}(\gamma_1)\ldots g_{m_r}^{(j_r)}(\gamma_r)x = \sum_{\beta\in\mathbb{Z}^{ns}} b_\beta \sum_{\ell\in J} h_\ell(\beta,\gamma) v_\ell(\alpha+\gamma_1+\cdots+\gamma_r).$$

The functions h_ℓ are exp-polynomial in $\beta=(\beta_1,\ldots,\beta_s)$ and $\gamma=(\gamma_1,\ldots,\gamma_r)$, and depend on $j_1,\ldots,j_r; m_1,\ldots,m_r$. We get that $x\in\widetilde{M}^{\rm rad}$ if and only if $\{b_\beta\}$ satisfies the system of equations:

$$\sum_{\beta\in\mathbb{Z}^{ns}} b_\beta \sum_{\ell\in J} h_\ell(\beta,\gamma) = 0 \tag{2.8}$$

for all $\gamma\in\mathbb{Z}^{nr}$, $\ell\in J$, $j_1,\ldots,j_r\in\mathbb{N}$ with $j_1+\cdots+j_r = i_1+\cdots+i_s$, $m_1\in K_{j_1},\ldots,m_r\in K_{j_r}$. Since $j_1,\ldots,j_r\in\mathbb{N}$ are bounded by the condition $j_1+\cdots+j_r = i_1+\cdots+i_s$ and $m_1,\ldots,m_r$ belong to finite sets, we conclude that only finitely many functions $h_\ell(\beta,\gamma)$ appear in system (2.8). Nonetheless system (2.8) has infinitely many equations because γ has an infinite range. Our goal is to reduce (2.8) to a system with finitely many equations.

A exp-polynomial function $h_\ell(\beta,\gamma)$ could be expanded in γ:

$$h_\ell(\beta,\gamma) = \sum_p f_p(\beta) a_p^\gamma \gamma^{\delta_p},$$

where $a_p\in\mathbb{C}^{nr}$, $\delta_p\in\mathbb{Z}^{nr}$ and the summation in p is finite. By Corollary 2.4, system (2.8) is equivalent to a finite system of linear equations:

$$\sum_{\beta\in\mathbb{Z}^{ns}} b_\beta f_p(\beta) = 0$$

with a finite number (say, d) of exp-polynomial functions $f_p(\beta)$. This establishes Claim 1.

Claim 2. *In the notation of Claim* 1, *the set*

$$\left\{g_{k_1}^{(-i_1)}(\beta_1)\ldots g_{k_s}^{(-i_s)}(\beta_s) v_k(\alpha-\beta_1-\cdots-\beta_s)\right\}_{\beta\in\mathbb{Z}^{ns}}$$

spans a subspace of dimension less or equal to d *in* $M_\alpha^{(-i_1-\cdots-i_s)}(V)$.

Proof of Claim 2. To prove this claim we need to show that any $d+1$ vectors in this set are linearly dependent. Let B be a subset in $\mathbb{Z}^{ns}$ of size $d+1$. The homogeneous system of d linear equations

$$\sum_{\beta \in B} b_\beta f_p(\beta) = 0$$

in $d+1$ variables $\{b_\beta\}_{\beta \in B}$ has a non-trivial solution. By Claim 1, the set

$$\{g_{k_1}^{(-i_1)}(\beta_1) \dots g_{k_s}^{(-i_s)}(\beta_s) v_k(\alpha - \beta_1 - \dots - \beta_s)\}_{\beta \in B}$$

is linearly dependent modulo $\widetilde{M}^{\mathrm{rad}}$. Claim 2 is now proved.

Theorem 1.5 now follows immediately from Claim 2 since the space $M_\alpha^{(-i)}$ is spanned by the elements

$$\{g_{k_1}^{(-i_1)}(\beta_1) \dots g_{k_s}^{(-i_s)}(\beta_s) v_k(\alpha - \beta_1 - \dots - \beta_s)\}$$

with $i_1 + \dots + i_s = i$; $i_1, \dots, i_s \in \mathbb{N}$. Thus there are finitely many possibilities for $(i_1, \dots, i_s)$, and the indices $k_1, \dots, k_s$ and k run over finite sets $K_{-i_1}, \dots, K_{-i_s}$ and J. □

Proof of Theorem 1.7. The proof of Theorem 1.7 may be derived from the proof of Theorem 1.5 by forgetting the $\mathbb{Z}^n$-grading on the module V and replacing expressions $v_k(\alpha - \beta_1 - \dots - \beta_s)$ etc., simply with v_k. □

3. Weight modules over generalized Virasoro algebras

The *Virasoro algebra* Vir := Vir[$\mathbb{Z}$] over $\mathbb{C}$ is the Lie algebra with the $\mathbb{C}$-basis $\{c, d_i \mid i \in \mathbb{Z}\}$ and subject to the following commutator relations:

$$[c, d_i] = 0,$$

$$[d_i, d_j] = (j - i) d_{i+j} + \delta_{i,-j} \frac{i^3 - i}{12} c \quad \text{for } i, j \in \mathbb{Z}.$$

The structure and representation theory for the Virasoro algebra has been well developed. For details, we refer the readers to [16], the book [14], and the references therein.

Generalized Virasoro algebras Vir[M] in characteristic 0 were introduced by Patera and Zassenhaus in [20], which are Lie algebras obtained from the above Virasoro algebra definition by replacing the index group $\mathbb{Z}$ with an arbitrary non-zero subgroup M of the base field $\mathbb{C}$, which are also the one-dimensional universal central extensions of the generalized Witt algebras [6]. When $M \simeq \mathbb{Z}^n$ and $n > 1$, the algebras Vir[M] are called *higher rank Virasoro algebras*. Representations for Generalized Virasoro algebras Vir[M] have been studied in several references. In [23], all weight modules over any generalized Virasoro algebra with weight multiplicity 1 were determined, which are so-called intermediate series modules. In [18], it was proved that all irreducible weight modules with finite dimensional weight spaces over the generalized

Virasoro algebra Vir[$\mathbb{Q}$] over $\mathbb{C}$ are intermediate series modules (where $\mathbb{Q}$ is the rational number field). In [13], the irreducibility of Verma modules over the generalized Virasoro algebra Vir[M] over any characteristic 0 field was completely determined. In [22], irreducible weight modules over higher rank Virasoro algebras with finite dimensional weight spaces were divided into two subclasses. One of these subclasses is called finitely dense modules, and from Definition 2.10 in [22], these finitely dense modules seem to have very complicated structure, and [22] lacks any classification of such modules (actually in this section you will see that those modules have very nice structures). Fortunately, this classification problem, i.e., the classification of irreducible weight modules with finite dimensional weight spaces over higher rank Virasoro algebras has been completely solved in [16] by using the following construction of such modules. The classification of irreducible weight modules with finite dimensional weight spaces over the classical Virasoro algebra was solved by Mathieu in [17] by using an entirely different method.

In this section we assume that $M = \mathbb{Z} \oplus M_0 \subset \mathbb{C}$ where M_0 is a non-zero subgroup of $\mathbb{C}$. We simply denote $L = \mathrm{Vir}[M]/\mathbb{C}c$, and for any $i \in \mathbb{Z}$ we denote

$$L_i = \bigoplus_{a\in M_0} \mathbb{C}d_{i+a},$$

$$L_+ = \bigoplus_{i\in\mathbb{Z}_+} L_i,\ \ L_- = \bigoplus_{i<0} L_i.$$

Then $L_0 \simeq \mathrm{Vir}[M_0]/\mathbb{C}c$.

For any $\alpha, \beta \in \mathbb{C}$, we have the L_0 module $V(\alpha,\beta,M_0) = \bigoplus_{a\in M_0} \mathbb{C}v_a$ subject to the actions

$$d_a v_b = (b + \alpha + a\beta)v_{a+b} \quad \text{for } a,b \in M_0. \tag{3.1}$$

We extend the L_0 module structure on $V(\alpha,\beta,M_0)$ to an $L_+ + L_0$ module structure by defining

$$L_+ V(\alpha,\beta,M_0) = 0. \tag{3.2}$$

Then we obtain the induced L module

$$\widetilde{V} = \widetilde{V}(\alpha,\beta,M_0) = \mathrm{Ind}^L_{L_+ + L_0} V(\alpha,\beta,M_0) = U(L) \bigotimes_{U(L_+ + L_0)} V(\alpha,\beta,M_0),$$

where $U(L)$ is the universal enveloping algebra of the Lie algebra L. It is clear that, as vector spaces, $\widetilde{V} \simeq U(L_-) \otimes_{\mathbb{C}} V(\alpha,\beta)$. The module $\widetilde{V}$ has a unique maximal proper submodule J trivially intersecting with $V(\alpha,\beta,M_0)$. Then we obtain the quotient module

$$\overline{V} = \overline{V}(\alpha,\beta,M_0) = \widetilde{V}/J. \tag{3.3}$$

It is clear that $\overline{V}$ is uniquely determined by the constants α, β, and $\overline{V} = \bigoplus_{i\in\mathbb{Z}_+} \overline{V}_{-i+\alpha+M_0}$ where

$$\overline{V}_{-i+\alpha+M_0} = \bigoplus_{a\in M_0} \overline{V}_{-i+\alpha+a},\ \ \overline{V}_{-i+\alpha+a} = \{v \in \overline{V} | d_0 v = (-i + \alpha + a)v\}. \tag{3.4}$$

We can similarly define $\widetilde{V}_{i+\alpha+M_0}$ and $\widetilde{V}_{-i+\alpha+a}$. It is easy to see that

$$\dim \widetilde{V}_{-i+\alpha+a} = \infty, \quad \text{for } i \in \mathbb{N},\ a \in M_0.$$

It will be different for $\overline{V}_{-i+\alpha+a}$.

Note that if $M_0 \simeq \mathbb{Z}^n$, then L is a $\mathbb{Z}^n$-extragraded Lie algebra, $V(\alpha,\beta,M_0)$ is a L module with polynomial action, and $\overline{V}(\alpha,\beta,M_0)=M(V(\alpha,\beta,M_0))$ as defined in Section 2. In this case, from Theorem 1.5, we know that $\overline{V}(\alpha,\beta,M_0)$ has finite dimensional weight spaces, i.e., $\dim \overline{V}_{-i+\alpha+a} < \infty$ for all $i \in \mathbb{N}$, $a \in M_0$. More generally (without the restriction $M_0 \simeq \mathbb{Z}^n$), we have

Theorem 3.1. *The module $\overline{V}(\alpha,\beta,M_0)$ defined in (3.3) over* Vir$[M]$ *has finite dimensional weight spaces, more precisely,* $\dim \overline{V}_{-i+\alpha+a} \leqslant 1\cdot 3\cdot \cdots \cdot (2i+1)$ *for all $i \in \mathbb{N}$, $a \in M_0$.*

Proof. Since L_- is generated by L_{-1}, and L_+ is generated by L_1, we deduce that

$$L_{-1}\overline{V}_{-i+\alpha+M_0} = \overline{V}_{-i-1+\alpha+M_0}, \quad \text{for } i \in \mathbb{Z}_+,$$

and, if $v \in \overline{V}_{-i+\alpha+M_0}$ where $i \in \mathbb{N}$, satisfies $L_1 v = 0$ then $v=0$. We also know that, for any $n \in \mathbb{N}$, $a \in M_0$,

$$\overline{V}_{-n+\alpha+a} = \text{span}\{d_{-1+a_n} d_{-1+a_{n-1}} \cdots d_{-1+a_1} v_{a_0} | a_i \in M_0,$$
$$\text{with } a_0 + a_1 + \cdots + a_n = a\}.$$

Claim 1. *For any $n \in \mathbb{Z}_+$, and $b_i \in \mathbb{C}$, $\alpha_i \in M_0$, (only finitely many b_i are not zero):*

(a) *if*

$$\sum_{i\in\mathbb{Z}} \alpha_i^k b_i = 0, \quad for\ 0 \leqslant k \leqslant 2n+2, \tag{3.5}$$

then

$$\sum_{i\in\mathbb{Z}} b_i d_{-1+\alpha_i} d_{-1+a_n} \cdots d_{-1+a_1} v_{a_0-\alpha_i} = 0; \tag{3.6}$$

(b) *if*

$$\sum_{i\in\mathbb{Z}} \alpha_i^k b_i = 0, \quad for\ 0 \leqslant k \leqslant 2n+1, \tag{3.5$'$}$$

then

$$\sum_{i\in\mathbb{Z}} b_i d_{\alpha_i} d_{-1+a_n} \cdots d_{-1+a_1} v_{a_0-\alpha_i} = 0 \tag{3.7}$$

for all $a_0, a_1, \ldots, a_n \in M_0$, where all the sums are finite.

Now we first consider $n=0$. Suppose (3.5$'$) holds for $n=0$. We have

$$\sum_{i\in\mathbb{Z}} b_i d_{\alpha_i} v_{a_0-\alpha_i} = \sum_{i\in\mathbb{Z}} b_i (a_0 - \alpha_i + \alpha + \beta\alpha_i) v_{a_0} = 0,$$

which is (3.7) for $n=0$. Now suppose (3.5) holds for $n=0$. For any $\gamma\in M_0$, we deduce

$$d_{1+\gamma}\left(\sum_{i\in\mathbb{Z}} b_i d_{-1+\alpha_i} v_{a_0-\alpha_i}\right)=\sum_{i\in\mathbb{Z}} b_i(-2+\alpha_i-\gamma)d_{\gamma+\alpha_i}v_{a_0-\alpha_i}$$

$$=\sum_{i\in\mathbb{Z}} b_i(-2+\alpha_i-\gamma)(a_0-\alpha_i+\alpha+\beta(\gamma+\alpha_i))v_{a_0+\gamma}=0,$$

to yield $\sum_{i\in\mathbb{Z}} b_i d_{-1+\alpha_i}v_{a_0-\alpha_i}=0$, which is (3.6) for $n=0$.

Suppose our claim holds for any $n\leqslant m$ for a fixed $m\in\mathbb{Z}_+$. Now we consider the claim for $n=m+1$. To get (3.7) for $n=m+1$, we assume (3.5′) for $n=m+1$. For any $\gamma\in M_0$, we compute

$$\begin{aligned}&d_{1+\gamma}\left(\sum_{i\in\mathbb{Z}} b_i d_{\alpha_i} d_{-1+a_{m+1}}\cdots d_{-1+a_1}v_{a_0-\alpha_i}\right)\\&=\sum_{i\in\mathbb{Z}} b_i(-1-\gamma+\alpha_i)d_{1+\gamma+\alpha_i}d_{-1+a_{m+1}}\cdots d_{-1+a_1}v_{a_0-\alpha_i}\\&\quad+\sum_{i\in\mathbb{Z}} b_i d_{\alpha_i}d_{1+\gamma}d_{-1+a_{m+1}}\cdots d_{-1+a_1}v_{a_0-\alpha_i}.\end{aligned}\tag{3.8}$$

Since $d_{1+\gamma}d_{-1+a_{m+1}}\cdots d_{-1+a_1}v_{a_0-\alpha_i}\in(L_{-1})^m v_{a_0-\alpha_i}$ can be expressed as a combination of elements of the form $d_{-1+a'_{m+1}}\ldots d_{-1+a'_1}v_{a'_0-\alpha_i}$ with the coefficients which are degree at most one polynomials in α_i, by the inductive hypothesis and (3.5′), we see that the second term on the right hand side of (3.8) is 0. So

$$\begin{aligned}&d_{1+\gamma}\left(\sum_{i\in\mathbb{Z}} b_i d_{\alpha_i} d_{-1+a_{m+1}}\cdots d_{-1+a_1}v_{a_0-\alpha_i}\right)\\&=\sum_{i\in\mathbb{Z}} b_i(-1-\gamma+\alpha_i)d_{1+\gamma+\alpha_i}d_{-1+a_{m+1}}\cdots d_{-1+a_1}v_{a_0-\alpha_i}\\&=\sum_{i\in\mathbb{Z}} b_i(-1-\gamma+\alpha_i)(-2-\gamma-\alpha_i+a_{m+1})d_{\gamma+\alpha_i+a_{m+1}}d_{-1+a_m}\cdots d_{-1+a_1}v_{a_0-\alpha_i}\\&\quad+\cdots+d_{-1+a_{m+1}}\cdots d_{-1+a_2}\\&\quad\times\sum_{i\in\mathbb{Z}} b_i(-1-\gamma+\alpha_i)(-2-\gamma-\alpha_i+a_1)d_{\gamma+\alpha_i+a_1}v_{a_0-\alpha_i}.\end{aligned}\tag{3.9}$$

From the inductive hypothesis and (3.5′), we know that each sum on the right-hand side of (3.9) is 0. Thus, for all $\gamma\in M_0$,

$$d_{1+\gamma}\left(\sum_{i\in\mathbb{Z}} b_i d_{\alpha_i} d_{-1+a_{m+1}}\cdots d_{-1+a_1}v_{a_0-\alpha_i}\right)=0,$$

which gives

$$\sum_{i\in\mathbb{Z}} b_i d_{\alpha_i} d_{-1+a_{m+1}} \cdots d_{-1+a_1} v_{a_0-\alpha_i} = 0.$$

So (3.7) holds for $n=m+1$.

To verify (3.6) for $n=m+1$, we suppose (3.5) holds for $n=m+1$. By using (3.7) for $n=m+1$, for any $\gamma\in M_0$ we deduce that

$$\begin{aligned} d_{1+\gamma}&\sum_{i\in\mathbb{Z}} b_i d_{-1+\alpha_i} d_{-1+a_{m+1}} \cdots d_{-1+a_1} v_{a_0-\alpha_i} \\ &=\sum_{i\in\mathbb{Z}} b_i[(-2+\alpha_i-\gamma)d_{\gamma+\alpha_i} d_{-1+a_{m+1}} \cdots d_{-1+a_1} v_{a_0-\alpha_i} + \cdots \\ &\quad + (a_0-\alpha_i+\alpha+\beta(\gamma+1))d_{-1+\alpha_i} d_{-1+a_{m+1}} \cdots d_{-1+a_1} v_{1+\gamma+a_0-\alpha_i}] = 0, \end{aligned}$$

which implies (3.6) for $n=m+1$. Thus Claim 1 holds for $n=m+1$. By the inductive principle, Claim 1 follows.

Fix $p\in M_0\setminus\{0\}$. Let $P_i=\{kp|0\leqslant k\leqslant 2i\}$ for any $i\in\mathbb{N}$. For any $n\in\mathbb{N}$, $a\in M_0$, $\alpha_1,\ldots,\alpha_n\in\mathbb{Z}p$ with $\alpha_i\in P_i$, let

$$\begin{aligned} &W(a,\alpha_1,\alpha_2,\ldots,\alpha_n) \\ &\quad=\operatorname{span}\{d_{-1+\alpha_{n+1}} d_{-1+\alpha_n} \cdots d_{-1+\alpha_1} v_{a-\alpha_1-\alpha_2-\cdots-\alpha_{n+1}} | \alpha_{n+1}\in P_{n+1}\}. \end{aligned}$$

Claim 2. *For any $\beta\in M_0$, we have*

$$d_{-1+\beta} d_{-1+a_n} \cdots d_{-1+a_1} v_{a-\alpha_1-\cdots-\alpha_n-\beta} \in W(a,\alpha_1,\alpha_2,\cdots,\alpha_n).$$

It is clear that we can find nontrivial $b_{\alpha_{n+1}}\in\mathbb{C}$ for $\alpha_{n+1}\in P_{n+1}$ (consider the following as a linear system of $2n+3$ unknowns with $2n+3$ equations) such that

$$\beta^k + \sum_{\alpha_{n+1}\in P_{n+1}} \alpha_{n+1}^k b_{\alpha_{n+1}} = 0, \quad \text{for } 0\leqslant k\leqslant 2n+2,$$

which are equalities of the form (3.5), where $\beta^0=1$. Then applying (3.6), we see that Claim 2 follows.

From Claim 2 we know that

$$\begin{aligned} &\dim \overline{V}_{-n+\alpha+a} \\ &\quad=\{d_{-1+\alpha_n} d_{-1+a_{n-1}} \cdots d_{-1+a_1} v_{a-\alpha_1-\alpha_2-\cdots-\alpha_n} : \alpha_i\in\{kp|0\leqslant k\leqslant 2i\}\} \\ &\quad\leqslant 1\cdot 3\cdot\cdots\cdot(2n+1), \quad \text{for } n\in\mathbb{Z}_+, a\in M_0. \end{aligned}$$

Note that $\dim\overline{V}_{\alpha+a}=1$ for all $a\in M_0$. Thus our theorem follows. □

It is clear that $\overline{V}(\alpha,\beta,M_0)$ is an irreducible Vir[M] module if and only if $V(\alpha,\beta,M_0)$ defined by (3.4) is an irreducible Vir[M_0] module. Thus, if we start with the irreducible Vir[M_0] module $V'(\alpha,\beta,M_0)$ (which is the irreducible subquotient of $V(\alpha,\beta,M_0)$ and which may not be an exp-polynomial module), instead of $V(\alpha,\beta,M_0)$, we get an irreducible Vir[M] module $\overline{V}'(\alpha,\beta,M_0)$, which can be also realized by taking the irreducible subquotient of $\overline{V}(\alpha,\beta,M_0)$ for all $\alpha,\beta\in\mathbb{C}$.

The module $\overline{V}(\alpha,\beta,M_0)$ contains highest weight Vir[$\mathbb{Z}$] (the classical Virasoro algebra) modules $U(\mathrm{Vir}[\mathbb{Z}])v_a$ for any $a\in M_0$. Thus not all weight multiplicities of $\overline{V}(\alpha,\beta,M_0)$ are 1, which indicates the modules $\overline{V}(\alpha,\beta,M_0)$ are not modules of intermediate series.

It is natural to ask the following questions: Are the Vir[$\mathbb{Z}$] submodules $U(\mathrm{Vir}[\mathbb{Z}])v_a$ irreducible? Is it true that $U(\mathrm{Vir}[\mathbb{Z}])v_a=\bigoplus_{i\in\mathbb{Z}_+}\overline{V}_{i+a}$?

It is important to calculate the character formula for the modules $\overline{V}(\alpha,\beta,M_0)$. We know the following about $\dim\overline{V}_{-i+\alpha+a}$ for $i\in\mathbb{Z}_+$, $a\in M_0$.

Corollary 3.2. *For any* $i\in\mathbb{Z}_+$, $a_1,a_2\in M_0$, *if* $(-i+a_1)(-i+a_2)\neq 0$, *we have*

$$\dim\overline{V}_{-i+\alpha+a_1}=\dim\overline{V}_{-i+\alpha+a_2}. \tag{3.10}$$

Proof. Suppose $a_1\neq a_2$. Let $a=a_1-a_2$, $G=\mathrm{Vir}[\mathbb{Z}a]=\bigoplus_{i\in\mathbb{Z}}\mathbb{C}d_{ia}$. Then G is isomorphic to the classical centerless Virasoro algebra. Consider the G module $W=\bigoplus_{x\in\mathbb{Z}a}\overline{V}_{-i+\alpha+a_2+x}$. From Theorem 3.1, we know that W is a uniformly bounded G module. By using Theorem 4.6 in [23], we see that all the dimensions of $\overline{V}_{-i+\alpha+a_2+x}$ are equal except for $\overline{V}_0$. The corollary follows. □

References

[1] S. Berman, Y. Billig, Irreducible representations for toroidal Lie algebras, J. Algebra 221 (1) (1999) 188–231.

[2] S. Berman, J. Szmigielski, Principal realization for the extended affine Lie algebra of type sl_2 with coordinates in a simple quantum torus with two generators, Recent developments in quantum Affine Algebras and Related Topics, Raleigh, NC, 1998, Contemporary Mathematics, 248, American Mathematical Society, Providence, RI, 1999, pp. 39–67.

[3] Y. Billig, Principal vertex operator representations for toroidal Lie algebras, J. Math. Phys. 39 (7) (1998) 3844–3864.

[4] Y. Billig, Energy-momentum tensor for the toroidal Lie algebras, (2002) preprint math.RT/0201313.

[5] V. Chari, Integrable representations of affine Lie-algebras, Invent. Math. 85 (2) (1986) 317–335.

[6] D.Z. Djokovic, K. Zhao, Derivations, isomorphisms, and second cohomology of generalized Witt algebras, Trans. Amer. Math. Soc. 350 (2) (1998) 643–664.

[7] D.Z. Djokovic, K. Zhao, Some infinite-dimensional simple Lie algebras in characteristic 0 related to those of Block, J. Pure Appl. Algebra 127 (2) (1998) 153–165.

[8] S. Eswara Rao, Classification of irreducible integrable modules for toroidal Lie algebras with finite dimensional weight spaces, (2002) preprint math. RT/0209060.

[9] S. Eswara Rao, Iterated loop modules and a filtration for vertex representation of toroidal Lie algebras, Pacific J. Math. 171 (2) (1995) 511–528.

[10] S. Eswara Rao, Irreducible representations of the Lie-algebra of the diffeomorphisms of a d-dimensional torus, J. Algebra 182 (2) (1996) 401–421.

[11] Y. Gao, Representations of extended affine Lie algebras coordinatized by certain quantum tori, Compositio Math. 123 (1) (2000) 1–25.

[12] M. Golenishcheva-Kutuzova, D. Lebedev, Vertex operator representation of some quantum tori Lie algebras, Comm. Math. Phys. 148 (2) (1992) 403–416.

[13] J. Hu, X. Wang, K. Zhao, Verma modules over generalized Virasoro algebras Vir[G], J. Pure Appl. Algebra 177 (1) (2003) 61–69.

[14] V.G. Kac, A.K. Raina, Bombay Lectures on Highest Weight Representations of Infinite dimensional Lie Algebras, World Science, Singapore, 1987.
[15] T.A. Larsson, Lowest-energy representations of non-centrally extended diffeomorphism algebras, Comm. Math. Phys. 201 (1999) 461–470.
[16] R. Lu, K. Zhao, Classification of irreducible weight modules over higher rank Virasoro algebras, (2003) preprint math.RT/0308133.
[17] O. Mathieu, Classification of Harish-Chandra modules over the Virasoro algebra, Invent. Math. 107 (1992) 225–234.
[18] V. Mazorchuk, Classification of simple Harish-Chandra modules over Q-Virasoro algebra, Math. Nachr. 209 (2000) 171–177.
[19] R.V. Moody, A. Pianzola, Lie Algebras with Triangular Decompositions, Wiley, New York, 1995.
[20] J. Patera, H. Zassenhaus, The higher rank Virasoro algebras, Comm. Math. Phys. 136 (1991) 1–14.
[21] G. Shen, Graded modules of graded Lie algebras of Cartan type. I. Mixed products of modules, Sci. Sinica, Ser. A 29 (6) (1986) 570–581.
[22] Y. Su, Classification of Harish-Chandra modules over the higher rank Virasoro algebras, Comm. Math. Phys. 240 (2003) 539–551.
[23] Y. Su, K. Zhao, Generalized Virasoro and super-Virasoro algebras and modules of intermediate series, J. Algebra 252 (1) (2002) 1–19.