Lagrange's Theorem and Integrality for Finite Commutative Hypergroups with Applications to Strongly Regular Graphs

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We introduce a notion of integrality, or resonance, for finite commutative hypergroups and their generalizations, signed hypergroups. Lagrange's theorem for subhypergroups is established using a condition of integrality of weights for the dual signed hypergroup. Order three hypergroups are studied and resonant ones whose duals have integral weights are classified. Applications are given to the theory of strongly regular graphs.


INTRODUCTION

In this paper we investigate a version of Lagrange's theorem and some related notions of integrality for finite commutative hypergroups, with particular emphasis on hypergroups of order three.

Hypergroups are generalizations of groups where we allow the product of two elements to be a probability distribution instead of a single element. Often hypergroups that appear in applications are renormalizations of similar algebraic objects in which a measure of "integrality" is present. It seems reasonable to try to encode this integrality more formally. We introduce a concept called resonance, and show that the class hypergroup and character hypergroup of a finite group \( G \) are resonant. Resonance is examined in some detail for hypergroups with three elements, which arise naturally in the theory of strongly regular graphs.

We hope that this study will help open the way for the sizable arsenal of insights and techniques from finite group theory to be brought to bear on the study of finite commutative hypergroups.
The theory of locally compact hypergroups was initiated by Dunkl [7], Jewett [10], and Spector [18]. An early survey can be found in Ross [14]; for later surveys see Heyer [8] and Wildberger [19]. Algebraic objects related to finite hypergroups have a long history, naturally due to the close connection with character tables of finite groups and other combinatorial objects. See, for example, Kawada [11], Levitan [24], Brauer [5], Bose and Mesner [4], McMullen [12], Bannai and Ito [2], Arad and Blau [1], Ross and Xu [15, 16], and Berezansky and Kalyushnyi [3].

In order to discuss both the objects and their duals consistently, we organize our discussion around the notions of generalized hypergroup and signed hypergroup. This allows a duality theory to be established.

In Section 1 we introduce the basic concepts such as the weight of an element in a generalized hypergroup and summarize the main facts of harmonic analysis which we will need. For more detail and examples the reader may consult Obata and Wildberger [13] and Wildberger [20]. We consider a class of generalized hypergroups called assemblies obtained from signed hypergroups by a renormalization involving the weight. The possibility of making such a modification is a pleasant feature of the finite theory.

In Section 2 we discuss subhypergroups and quotient hypergroups and introduce an explicit realization of any quotient hypergroup. A form of Lagrange's theorem for finite commutative hypergroups is established:

**Theorem 2.3.** Let $\mathcal{H}$ be a finite commutative hypergroup, $\mathcal{H}^\wedge$ its dual signed hypergroup, and $\mathcal{D} \subseteq \mathcal{H}$ a subhypergroup. If both $\mathcal{H}$ and $\mathcal{H}^\wedge$ have integral weights, then $\omega(\mathcal{D}) | \omega(\mathcal{H})$.

In Section 3 we introduce the notion of resonance, and show that the two hypergroups canonically associated to a finite group are both resonant. In particular, we point out that the classification of simple strongly resonant strong commutative hypergroups (this means both $\mathcal{H}$ and its dual $\mathcal{H}^\wedge$ are resonant hypergroups containing no proper subhypergroups) contains within it the classification of finite simple groups.

In Section 4 we develop some aspects of the theory of order-three hypergroups. This culminates in Theorem 4.5, which is an explicit characterization of all order-three Hermitian hypergroups whose duals have integral weights. This classification contains all the hypergroups associated to strongly regular graphs.

In Section 5 we show how Theorem 4.5 can be restated in a way which treats the hypergroup $\mathcal{H}$ and its dual $\mathcal{H}^\wedge$ symmetrically. This allows us to classify all strongly resonant order three hypergroups in terms of positive
integer solutions of the Diophantine equation

\[
\frac{1 + X}{X} \frac{1 + V}{V} = \frac{1 + X'}{X'} \frac{1 + V'}{V'}.
\]

Applications to the theory of strongly regular graphs are described in Section 6. We interpret the Integrality Condition and Krein Condition of that theory in hypergroup theoretic terms.

In Section 7 we consider the strongly regular graph of the Grassmanian of 2-planes in \( F_q^n \). Utilizing the results of the previous sections, we calculate explicitly the structure constants of the associated hypergroup and its dual.

In the Appendix we relate our work to the family of strongly regular graphs associated to the rank 3 actions of sporadic simple groups found in Hubaut [9] by explicitly describing the structure equations of each hypergroup and its dual. We find that the constants \( X, V, X', V' \) mentioned above provide a very efficient means of compressing the algebraic structure and allow us to easily read off which hypergroups are strongly resonant.

1. BASIC DEFINITIONS AND RESULTS

Definition 1.1. A generalized hypergroup is a pair \((H, A)\) where \( H = \{c_0, c_1, \ldots, c_n\} \) and \( A \) is an associative algebra with unit \( c_0 \) over \( \mathbb{C} \) and involution \( * \) satisfying the conditions

(A1) \( H \) is a basis of \( A \).

(A2) \( H^* = H \) so that we may write \( c_i^* = c_{a(i)} \).

(A3) The structure constants \( n_{ij}^{k} \in \mathbb{C} \) defined by

\[
c_j c_i = \sum_k n_{ij}^{k} c_k
\]

satisfy the conditions

\[
c_i^* = c_j \iff n_{ij}^{0} > 0
\]

\[
c_i^* \neq c_j \iff n_{ij}^{0} = 0.
\]

Unless otherwise indicated, indices will range over the set \( \{0, 1, \ldots, n\} \).

We will usually simply say that \( A \) is a generalized hypergroup \( H \) will be called commutative if \( A \) is commutative, Hermitian if \( c_i^* = c_i \ \forall i \), real if \( n_{ij}^{k} \in \mathbb{R} \ \forall i, j, k \), and positive if \( n_{ij}^{k} \geq 0 \ \forall i, j, k \). It will be called normalized
if the following additional axiom is satisfied.

\[(A\, 4) \quad \sum_{k} n^k_{ij} = 1 \quad \forall i, j.\]

A generalized hypergroup which is both real and normalized will be called a \textit{signed hypergroup}. A generalized hypergroup which is both positive and normalized will be called a \textit{hypergroup}.

Define the weight of \(c_i\) to be

\[\omega(c_i) = \left( n^0_{\sigma(i)i} \right)^{-1} > 0 \quad (1.1)\]

and define the weight of \(\mathcal{A}\) to be

\[\omega(\mathcal{A}) = \sum_{i} \omega(c_i). \quad (1.2)\]

We will also have occasion to consider the following alternative to \((A\, 4)\):

\[(A\, 4') \quad \omega(c_i)^{-1} \omega(c_j)^{-1} = \sum_{k} n^k_{ij} \omega(c_k)^{-1}.\]

A generalized hypergroup which is real and satisfies \((A\, 4')\) will be called an \textit{assembly}.

The reason for considering assemblies is the following. In many applications involving hypergroups, it is not the hypergroups themselves which appear but certain renormalizations of them. In particular, let \(\mathcal{A} = \{c_0, c_1, \ldots, c_n\} \subseteq \mathcal{A}\) be a signed hypergroup as above. Set \(C_i = \omega(c_i)c_i \in \mathcal{A}\). Then \(\mathcal{F} = \{C_0, C_1, \ldots, C_n\} \subseteq \mathcal{A}\) is a generalized hypergroup with structure equations

\[C_iC_j = \sum_{k} N^k_{ij} C_k, \quad (1.3)\]

where \(N^k_{ij} = \left( \omega(c_i)\omega(c_j) / \omega(c_k) \right)n^k_{ij}\) and such that \(\omega(C_i) = \omega(c_i)^{-1}\). One then immediately checks that \(\mathcal{F}\) is an assembly. Conversely, if \(\mathcal{F} = \{C_0, C_1, \ldots, C_n\} \subseteq \mathcal{A}\) is an assembly with structure equations \((1.3)\) and we set \(c_i = \omega(C_i)c_i\) then \(\mathcal{A} = \{c_0, c_1, \ldots, c_n\} \subseteq \mathcal{A}\) is a signed hypergroup and this procedure establishes a \(1:1\) correspondence between signed hypergroups and assemblies.

Since \((A\, 4')\) is a bit unwieldy, we make the following additional definition. For \(\mathcal{F} = \{C_0, C_1, \ldots, C_n\}\) an assembly with structure equations \((1.3)\) we declare the \textit{mass} of \(C_i\) to be \(m(C_i) = \omega(c_i)^{-1}\). Thus the mass of an element in an assembly is equal to the weight of the corresponding
element in the associated signed hypergroup. Then (A 4') becomes

\[(A 4') \quad m(C_i) m(C_j) = \sum_k N_{i,j}^k m(C_k),\]

which we call the axiom of conservation of mass for assemblies.

Finally we declare an assembly to be integral if all of the structure constants \(N_{i,j}^k\) are integers.

Let \( \mathcal{H} \subset \mathcal{A} \) be a commutative signed hypergroup as above. We now develop some harmonic analysis on \( \mathcal{H} \).

**Proposition 1.2.** The \(*\)-algebra \( \mathcal{A} \) is semisimple.

**Proof.** If \( a \in \mathcal{A} \) is non-zero then by (A 3) the coefficient of \( c_0 \) in \( aa^* \) is also non-zero so that \( aa^* \neq 0 \). Thus \( 0 \neq (aa^*)^* = a^2(a^2)^* \) so \( a^2 \neq 0 \). Thus \( \mathcal{A} \) has no non-zero nilpotent elements and so is semisimple. \( \square \)

For \( a \in \mathcal{A} \), let \( \text{ad}(a) \in \text{End}(\mathcal{A}) \) denote the operator of multiplication by \( a \).

**Lemma 1.3.** The set \( \{ \text{ad}(c_i) \} \) is linearly independent in \( \text{End}(\mathcal{A}) \).

**Proof.** If \( \sum_i r_i \text{ad}(c_i) = 0 \) then multiplying by \( c^*_j \) and considering the coefficient of \( c_0 \) gives \( r_j = 0 \). \( \square \)

The algebra \( \text{ad}(\mathcal{A}) \subset \text{End}(\mathcal{A}) \) has dimension \( n + 1 \) and is both commutative and semisimple. It is thus isomorphic to the algebra of diagonal operators in \( M(n+1, \mathbb{C}) \). That means one can find a basis \( \{ e_0, e_1, \ldots, e_n \} \) of \( \mathcal{A} \) in which the operators \( \text{ad}(c_i) \) are diagonal, that is,

\[ c_i e_j = \chi(c_i) e_j \quad \forall i, j \quad (1.4) \]

for some functions \( \chi_i \) and such that

\[ e_j e_k = \delta_{jk} e_j. \quad (1.5) \]

If \( \mathcal{F}(\mathcal{H}) \) denotes the space of all complex valued functions on \( \mathcal{H} \), then the set of functions \( \{ \chi_i \} \) is linearly independent in \( \mathcal{F}(\mathcal{H}) \).

Define a character of \( \mathcal{H} \) to be any \( \chi \in \mathcal{F}(\mathcal{H}) \) that satisfies

\[ \chi(c_i) \chi(c_j) = \sum_k n_{i,j}^k \chi(c_k) \quad \forall i, j. \quad (1.6) \]

If the linear extension of \( \chi \) to \( \mathcal{A} \) is also denoted by \( \chi \), then we have the equivalent formulation

\[ \chi(c_i) \chi(c_j) = \chi(c_i c_j) \quad \forall i, j. \quad (1.7) \]

The set of all characters of \( \mathcal{H} \) will be denoted by \( \mathcal{H}^\wedge \).
**Proposition 1.4.** \( \mathcal{X} = \{ \chi_0, \chi_1, \ldots, \chi_n \} \), and

\[
\chi_i(e_j^\ast) = \overline{\chi_j(e_i)} \quad \forall i, j.
\]

**Proof.** That each \( \chi_i \) is a character follows from (1.4). Since \( \mathcal{X} \) is isomorphic to the \(*\)-algebra \( C_n^{\ast+1} \), it has exactly \( n + 1 \) characters which all satisfy the stated condition. \( \blacksquare \)

Clearly the function identically 1 is a character of \( \mathcal{X} \) by (A.4). We will use \( \chi_0 \) to specify it. The corresponding idempotent \( e_0 \) plays a special role in the harmonic analysis of \( \mathcal{X} \); it is the Haar measure in the sense that \( c_i e_0 = e_0 \forall i. \)

For \( f, g \in \mathcal{F}(\mathcal{X}) \), define \( f^\ast(e_i) = f(c_i^\ast) \) and introduce the inner product

\[
\langle f, g \rangle = \frac{1}{\omega(\mathcal{X})} \sum_i \omega(c_i) f(c_i) \overline{g(c_i)}. \tag{1.8}
\]

Since both \( \mathcal{X} \) and \( \{ e_0, e_1, \ldots, e_n \} \) are bases of \( \mathcal{X} \), we may find constants \( \alpha_i^k \in \mathbb{C} \) such that

\[
e_j = \sum_k \alpha_j^k c_k \quad \forall j. \tag{1.9}
\]

Multiplying both sides by \( c_i^\ast \) and comparing coefficients of \( c_0 \) we find that

\[
\alpha_i^j = \omega(c_i) \chi_j(e_i^\ast) \alpha_j^0 \tag{1.10}
\]

so that

\[
e_j = \alpha_j^0 \sum_k \omega(c_k) \chi_j(e_k^\ast) c_k. \tag{1.11}
\]

Combining this with (1.5) and again comparing coefficients of \( c_0 \) we obtain

\[
\delta_{ij} = \alpha_j^0 \sum_k \omega(c_k) \chi_i(e_k) \chi_j(e_k^\ast) = \alpha_j^0 \omega(\mathcal{X}) \langle \chi_i, \chi_j \rangle. \tag{1.12}
\]

Thus

**Lemma 1.5.** \( \mathcal{X} = \{ \chi_0, \chi_1, \ldots, \chi_n \} \) is an orthogonal basis of \( \mathcal{F}(\mathcal{X}) \) with respect to the inner product \( \langle \cdot , \cdot \rangle \).
Since $\mathcal{H}(\mathcal{A})$ is a $\ast$-algebra with unit $\chi_0$ under point-wise multiplication and complex conjugation, we may write
\[ x_i x_j = \sum_k m_{ij}^k x_k \quad \text{with} \quad m_{ij}^k \in \mathbb{C} \quad (1.13) \]
and if $\chi_i^* = \chi_j$, we say the weight of $\chi_i$ is $\omega(\chi_i) = (m_{ij}^0)^{-1}$. Then
\[ \langle \chi_i, \chi_j \rangle = \langle \chi_i, \overline{\chi_j}, \chi_0 \rangle = \langle \chi_i, \chi_j^*, \chi_0 \rangle \\
= \omega(\chi_i)^{-1} \langle \chi_0, \chi_0 \rangle = \omega(\chi_i)^{-1}. \quad (1.14) \]
We conclude that $\omega(\chi_i) \in \mathbb{R}$ and $\omega(\chi_i) > 0$. Also using (1.12) we now have

**Proposition 1.6.** $e_i = (\omega(\chi_i)/\omega(\mathcal{A}))\sum_k \omega(c_k)\chi_i^*(c_k)c_k \, \forall i$.

**Corollary 1.7.** $e_0 = (1/\omega(\mathcal{A}))\sum_k \omega(c_k)c_k$.

**Theorem 1.8.** If $\mathcal{H}$ is a signed hypergroup, so is $\mathcal{H}^\wedge$. Furthermore $\omega(\mathcal{H}) = \omega(\mathcal{H}^\wedge)$.

**Proof.** Conditions (A1) and (A2) are straightforward. To show (A3), note that
\[ \overline{x_i(c_i)} x_j(c_l) = \sum_k m_{ij}^k \overline{x_k(c_i)} \quad \forall i, j, l \quad (1.15) \]
and since $x_i(c_i)/0$ for at least one $j$, $x_i(c_i) = 1$. Thus $1 = x_i(c_i)^\wedge \times x_j(c_j) = \sum_k m_{ij}^k x_k(c_i) \sum_k m_{ij}^k$ which shows that $\mathcal{H}^\wedge$ is normalized. A proof of the fact that $\omega(\mathcal{H}) = \omega(\mathcal{H}^\wedge)$ may be found in Wildberger [20].

It is clear that the definition of the weight of $\chi_i$ given above coincides with the general definition. Thus
\[ \omega(\mathcal{H}^\wedge) = \sum_j \omega(\chi_j). \quad (1.18) \]
For each $c_i \in \mathcal{H}$, the function $c_i^T \in F(\mathcal{H})$ defined by
\begin{equation}
    c_i^T(x_j) = x_j(c_i)
\end{equation}
is a character of $\mathcal{H}$ and $(c_i^T)^* = (c_i^*)^T$.

One then obtains the following duality result.

**Theorem 1.9.** Let $\mathcal{H} = \{c_0, c_1, \ldots, c_n\}$ be a signed hypergroup. Then the map $c_i \to c_i^T$ gives an isomorphism of the signed hypergroups $\mathcal{H}$ and $(\mathcal{H}^\wedge)^\wedge$.

In particular any signed hypergroup can be realized as a signed hypergroup of functions with point-wise multiplication and involution $*$ given by complex conjugation.

Two interesting hypergroups are associated to any finite group $G$. Let the conjugacy classes of $G$, say $C_0, C_1, \ldots, C_n$, be identified with their characteristic functions in the group algebra so that $C_i = \sum_{g \in C_i} g$. We may write
\begin{equation}
    C_iC_j = \sum_k N_{ij}^k C_k,
\end{equation}
where the $N_{ij}^k$ are non-negative integers. Then $\mathcal{H} = \{C_0, C_1, \ldots, C_n\} \subset \mathcal{H}$ is an assembly, with $\mathcal{H}$ the center of the group algebra. The corresponding signed hypergroup (in this case a hypergroup) is obtained by setting $c_i = C_i/|C_i|$ and $n_{ij}^k = N_{ij}^k |C_k|/|C_i||C_j|$. Note that $C_i^* = \sum_{g \in C_i} g^{-1}$. The hypergroup $\{c_0, c_1, \ldots, c_n\}$ will be called the **class hypergroup** of $G$ and denoted by $\mathcal{H}(G)$.

Consider also $(\psi_0, \psi_1, \ldots, \psi_n)$ the set of irreducible characters of $G$ with $\psi_0$ the constant function one. Let $\psi_i(e) = d_i$ and suppose that
\begin{equation}
    \psi_i\psi_j = \sum_k M_{ij}^k \psi_k.
\end{equation}
The $M_{ij}^k$ are non-negative integers satisfying
\begin{equation}
    d_id_j = \sum_k M_{ij}^k d_k.
\end{equation}

If we set $\chi_j = \psi_j/d_j$ and $m_{ij}^k = M_{ij}^k d_k/d_id_j$ then
\begin{equation}
    \chi_i\chi_j = \sum_k m_{ij}^k \chi_k, \quad \chi_i^* = \overline{\chi_i}
\end{equation}
defines a hypergroup $\mathcal{H}(G^\wedge) = \{\chi_0, \chi_1, \ldots, \chi_n\}$ which we call the **character hypergroup** of $G$. It is important to notice here that the weight of $\chi_i$ is $d_i^2$, not $d_i$, and that the associated assembly $\mathcal{H}(G^\wedge)$ consists not of the $\psi_i$'s but of the functions
\begin{equation}
    X_i = d_i^2 \chi_i = d_i \psi_i.
\end{equation}
Thus \( \mathcal{F}(G^n) \) has structure equations

\[
X_i X_j = \sum_k \frac{d_i d_j}{d_k} M_{ij}^k X_k,
\]

and in general these coefficients are not integers, so that the assembly \( \mathcal{F}(G^n) \) is not integral.

2. SUBHYPERGROUPS AND LAGRANGE'S THEOREM

Let \( \mathcal{A} = \{c_0, c_1, \ldots, c_n\} \subseteq \mathcal{A} \) be a finite commutative hypergroup and \( \mathcal{D} \subseteq \mathcal{A} \) a subhypergroup. For \( c_i \in \mathcal{A} \), \( c_i \mathcal{D} = \{c_k | n_{ij}^k \neq 0 \text{ for some } c_j \in \mathcal{D}\} \) is called a coset of \( \mathcal{D} \). Two cosets are either equal or disjoint (see Jewett [10, 10.3]). We may thus partition \( 0, 1, \ldots, n \) into subsets \( I_0, I_1, \ldots, I_p \) so that \( \mathcal{D}_l = \{c_i | i \in I_l\} \) is a distinct coset of \( \mathcal{D} \) for \( l = 0, \ldots, p \) and \( \mathcal{D}_0 = \mathcal{D} \).

Define

\[
q_l = \sum_{i \in I_l} \omega(c_i)
\]

and

\[
f_l = \frac{1}{q_l} \sum_{i \in I_l} \omega(c_i) c_i \in \mathcal{A}
\]

for all \( l = 0, \ldots, p \).

Lemma 2.1. (i) \( f_0^2 = f_0 \).
(ii) For any \( i \in I_l \), \( c_i f_0 = f_l \).

Proof. (i) This follows from the fact that \( f_0 \) is the Haar measure for \( \mathcal{D} \).
(ii) For \( i \in I_l \), we may write \( c_i f_0 = \sum_{j \in I_l} \alpha_j c_j \) with \( \alpha_j > 0 \) for all \( j \in I_l \) and \( \sum_{j \in I_l} \alpha_j = 1 \). From (i), it follows that

\[
c_i f_0 = \sum_{j \in I_l} \alpha_j c_j f_0.
\]
identical. Now from Corollary 1.7

$$\frac{1}{\omega(\mathcal{R})} \sum_i \omega(c_i) c_i = e_0 f_0$$

$$= \frac{1}{\omega(\mathcal{R})} \sum_i \omega(c_i) c_i f_0.$$  \hfill (2.4)

Comparing the components of both sides supported in $\mathcal{D}_i$ and combining it with the above observation gives the result. \hfill \Box

**Theorem 2.2.** There exist non-negative constants $r_{ij}^l, 0 \leq i, j, l \leq p$ such that

$$f_i f_j = \sum_{l=0}^{p} r_{ij}^l f_l \quad \text{for all } 0 \leq i, j \leq p,$$

and such that $\mathcal{F} = \{ f_0, f_1, \ldots, f_p \}$ becomes a finite commutative hypergroup with the above product and involution $\ast$. Furthermore we have the equations

(i) \hspace{1em} $r_{ij}^l = \sum_{k \in I_i} n_{ts}^{k}$ \hspace{1em} for any $t \in I_i, s \in I_j$

(ii) \hspace{1em} $r_{ij}^l = (q_i / q_s q_l) (1 / \omega(c_k)) \sum_{t \in I_i, s \in I_j} \omega(c_i) \omega(c_j) n_{ts}^{k}$ \hspace{1em} for any $k \in I_i$

(iii) \hspace{1em} $\omega(f_t) = \left( \sum_{k \in I_o} n_{ts}^{k} \right)^{-1} = q_t / q_0$ \hspace{1em} for any $t \in I_i, s \in I_{p(l)}$

**Proof.** (i) For any $0 \leq i, j \leq p$, take $t \in I_i$ and $s \in I_j$. Then by Lemma 2.1,

$$f_i f_j = (c_i f_0) (c_j f_0) = (c_i c_j) f_0$$

$$= \sum_k n_{ts}^{k} c_k f_0$$

$$= \sum_{l=0}^{p} \left( \sum_{k \in I_i} n_{ts}^{k} \right) f_l = \sum_{l=0}^{p} r_{ij}^l f_l,$$  \hfill (2.5)

where $r_{ij}^l = \sum_{k \in I_i} n_{ts}^{k}$.

Since the $f_t$'s have disjoint supports, these coefficients are unique. Verification that $\mathcal{F}$ forms a hypergroup is routine.
(ii) We use (2.2) to obtain
\[
    f_i f_j = \frac{1}{q_i d_j} \left( \sum_{r \in I_i} \omega(c_r) c_r \right) \left( \sum_{s \in I_j} \omega(c_s) c_s \right) \\
    = \frac{1}{q_i d_j} \sum_{r \in I_i} \omega(c_r) \omega(c_s) \sum_{l=0}^{p} \sum_{k \in I_l} n_i^k c_k \\
    = \frac{1}{q_i d_j} \sum_{l=0}^{p} \sum_{k \in I_l} \left( \sum_{r \in I_i} \omega(c_r) \omega(c_s) n_i^k \right) c_k.
\]
(2.6)

Equating this with \( \sum_{l=0}^{p} r_i^l f_i = \sum_{l=0}^{p} r_j^l (1/q_l) \sum_{k \in I_l} \omega(c_k) c_k \) gives the result.

(iii) Suppose that \( f_i^* = f_j \). Then setting \( l = 0 \) in (i) we get
\[
    \omega(f_i) = (r_{ij})^{-1} = \left( \sum_{k \in I_0} n_i^k \right)^{-1} \quad \text{for any } t \in I_i, s \in I_j
\]
(2.7)
while setting \( l = 0 \) and \( k = 0 \) in (ii) gives
\[
    \omega(f_i) = (r_{ij}^0)^{-1} = \frac{q_i q_j}{q_0} \left( \sum_{r \in I_i} \omega(c_r) \omega(c_s) n_i^0 \right)^{-1} \\
    = \frac{q_i}{q_0} \left( \sum_{t \in I_i} \omega(c_t)^2 \omega(c_s)^{-1} \right)^{-1}
\]
(2.8)

The hypergroup \( \mathcal{F} = \{f_0, f_1, \ldots, f_p\} \) provides an explicit realization of the quotient hypergroup \( \mathcal{H}/\mathcal{D} \). Note that from (2.1),
\[
    \omega(\mathcal{F}) = \sum_{l=0}^{p} \omega(f_l) = \sum_{l=0}^{p} \frac{q_l}{q_0} = \frac{\omega(\mathcal{H})}{\omega(\mathcal{D})}.
\]
(2.9)

If \( \mathcal{H} \) has integral weights, then both \( \omega(\mathcal{H}) \) and \( \omega(\mathcal{F}) \) are positive integers. It is easy to give examples however where \( \omega(\mathcal{D}) \) does not divide \( \omega(\mathcal{F}) \).
Theorem 2.3 (Lagrange's Theorem for Finite Commutative Hypergroups). Let \( \mathcal{H} \) be a finite commutative hypergroup, \( \mathcal{H}^\wedge \) its dual, and \( \mathcal{D} \subseteq \mathcal{H} \) a subhypergroup. If both \( \mathcal{H} \) and \( \mathcal{H}^\wedge \) have integral weights, then \( \omega(\mathcal{D}) | \omega(\mathcal{H}) \).

Proof. Let \( \mathcal{F} = \mathcal{H}/\mathcal{D} \) be as above. The map \( \phi: \mathcal{H} \to \mathcal{F} \) given by \( \phi(c_i) = c_i f_\mathcal{D} \) is a hypergroup homomorphism and so there is an injection of \( \mathcal{F}^\wedge \) into \( \mathcal{H}^\wedge \). But then \( \omega(\mathcal{F}^\wedge) \) is an integer. But by Theorem 1.8 and (2.9) we have
\[
\omega(\mathcal{F}^\wedge) = \frac{\omega(\mathcal{F})}{\omega(\mathcal{D})}.
\]

3. Integrality and Resonance

One of our main aims is to establish some plausible conditions under which a signed hypergroup \( \mathcal{H} \) exhibits "integrality." The obvious condition—that the associated assembly \( \mathcal{A} \) be integral—has a serious drawback which we have already noticed; it can fail for the character hypergroups of finite groups. We therefore propose a variant of this condition.

Let \( \mathcal{A} = \{c_0, c_1, \ldots, c_n\} \) be a signed hypergroup with structure equations \( c_i c_j = \sum k n_{ij}^k \) whose associated assembly \( \mathcal{A}_C = \{C_0, C_1, \ldots, C_n\} \) has structure equations \( C_i C_j = \sum_k N_{ij}^k C_k \).

Definition 3.1. The signed hypergroup \( \mathcal{H} \) is resonant if

1. \( \omega(c_i) \omega(c_j) n_{ij}^k \in \mathbb{Z} \forall i, j, k \)
2. \( \gcd(\omega(c_i), \omega(c_j))|\omega(c_i)\omega(c_j)n_{ij}^k \forall i, j, k. \)

Definition 3.2. The assembly \( \mathcal{A}_C \) is resonant if

1. \( N_{ij}^k m(C_k) \in \mathbb{Z} \forall i, j, k \)
2. \( \gcd(m(C_i), m(C_j))|N_{ij}^k m(C_k) \forall i, j, k. \)

Note that in both situations, condition (1) implies that the weights or masses are necessarily integral so that condition (2) is well-defined. A quick check shows that a signed hypergroup \( \mathcal{H} \) is resonant if and only if the associated assembly \( \mathcal{A}_C \) is resonant.

We now wish to show that for any finite group \( G \), both \( \mathcal{A}(G) \) and \( \mathcal{A}^\wedge(G) \) are resonant. For the first, we will prove a more general result. Let \( H \) be a finite group and \( G \) a subgroup of \( \text{Aut}(H) \). Let \( \{C_0, \ldots, C_n\} = \mathcal{A}_C \) be the set of orbits of \( G \) on \( H \) with \( C_0 = \{e\} \). By identifying \( C_i \) with its characteristic function \( \sum h \in C_i h \in \text{the group algebra of } H \), \( \mathcal{A}_C \) becomes a (generally non-commutative) assembly with structure equations \( C_i C_j = \sum k N_{ij}^k C_k \) for some non-negative integers \( N_{ij}^k \). Clearly \( m(C_i) = |C_i| \).
Theorem 3.3. With the above notation, \( m(C_i) \) divides \( N^k_{ij} m(C_k) \) \( \forall i, j, k \).

Proof. Fix \((x, y) \in C_i \times C_j\) such that \(xy \in C_k\). Let the orbits of \(G\) on \(C_i \times C_j\) be \(\Omega_1, \ldots, \Omega_n\) with \(G(x, y) = \Omega_1\). Thus \(\Omega_1 = \{(g \cdot x, g \cdot y) \mid g \in G\} \). If \(G_{(x, y)}\) denotes the stabilizer subgroup of \((x, y)\) and \(G_x, G_y\) the stabilizer subgroups of \(x\) and \(y\) respectively, then \(G_{(x, y)} = G_x \cap G_y\), \(|C_i| = |G|/|G_x|\), and \(|\Omega_1| = |G|/|G_{(x, y)}|\). Thus

\[
\frac{|\Omega_1|}{|C_i|} = \frac{|G_x|}{|G_{(x, y)}|}
\]

and both fractions are integers. Under the multiplication map \(M: C_i \times C_j \rightarrow H\) the orbit \(\Omega_1\) maps onto \(C_k\) and so by \(G\)-invariance \(|\Omega_1| = q|C_k|\) for some positive integer \(q\). Note that this does not mean that \(N^k_{ij}\) is equal to \(q\), since other orbits \(\Omega_i\) might also map to \(C_k\) under \(M\). However,

\[
\frac{|G_x|}{|G_{(x, y)}|} |C_i| = q|C_k|
\]

so \(|C_i|\) divides \(q|C_k|\). Since this is true for all orbits \(\Omega_i\), mapping to \(C_k\), we see that \(|C_i|\) divides \(N^k_{ij} |C_k|\).

Corollary 3.4. For any finite group \(G\), the class hypergroup \(\mathcal{H}(G)\) is resonant.

Proof. This is an immediate consequence of the above in the case when \(G\) acts on itself by conjugation.

Proposition 3.5. Let \(G\) be a finite group and \(\mathcal{H}(G^\wedge) = \{\chi_0, \chi_1, \ldots, \chi_n\}\) its character hypergroup. Then \(\mathcal{H}(G^\wedge)\) is resonant.

Proof. We use the notation at the end of Section 1 with

\[
X_i X_j = \sum_k \frac{d_i d_j}{d_k} M^k_{ij} X_k
\]

the structure equations of the assembly \(\mathcal{H}(G^\wedge)\). The quantity \(N^k_{ij} m(X_k)\) is then the integer \(M^k_{ij} d_i d_j d_k\) which is divisible by \(\gcd(m(X_i), m(X_j)) = \gcd(d_i^2, d_j^2)\).

Now define a signed hypergroup \(\mathcal{S}\) to be strongly resonant if both \(\mathcal{S}\) and \(\mathcal{S}^\wedge\) are resonant. Recall that a hypergroup is called strong if its dual is also a hypergroup. Corollary 3.4 and Proposition 3.5 show that class hypergroups of finite groups are strongly resonant, as well as strong.

The classification of all simple commutative strongly resonant strong hypergroups seems to be an interesting yet demanding task which awaits
the efforts of algebraists. Since finite simple groups are determined by their class hypergroups this classification would include within it (in some sense) that of the finite simple groups.

4. RESONANT HYPERGROUPS OF ORDER THREE

In this section we investigate all resonant order three hypergroups $\mathcal{H}$ whose duals $\mathcal{H}^\perp$ have integral weights. This can be viewed as a preliminary to classifying all order three strongly resonant hypergroups, but it also has independent interest due to the connection with the theory of strongly regular graphs.

We distinguish between two cases: $\mathcal{H}$ is not Hermitian or $\mathcal{H}$ is Hermitian. The first situation is simple and the second considerably less so.

Suppose first that $\mathcal{H} = \{c_0, c_1, c_2\}$ is not Hermitian, so that $c_1^2 = c_2$ and $\omega(c_1) = \omega(c_2) = \omega$. It is easy to see that the structure equations for $\mathcal{H}$ are determined by $\omega$ and are

$$
c_1 c_2 = \frac{1}{\omega} c_0 + \frac{\omega - 1}{2 \omega} c_1 + \frac{\omega - 1}{2 \omega} c_2
$$

$$
c_1^2 = \frac{\omega - 1}{2 \omega} c_1 + \frac{\omega + 1}{2 \omega} c_2
$$

$$
c_2^2 = \frac{\omega + 1}{2 \omega} c_1 + \frac{\omega - 1}{2 \omega} c_2.
$$

This implies that the dual $\mathcal{H}^\perp$ is isomorphic to $\mathcal{H}$. The equations of the associated assembly are

$$
C_1 C_2 = \omega C_0 + \frac{\omega - 1}{2} C_1 + \frac{\omega - 1}{2} C_2
$$

$$
C_1^2 = \frac{\omega - 1}{2} C_1 + \frac{\omega + 1}{2} C_2
$$

$$
C_2^2 = \frac{\omega + 1}{2} C_1 + \frac{\omega - 1}{2} C_2.
$$

If $\mathcal{H}$ is resonant, then $\omega$ must be a positive integer and Condition (1) of Definition 3.1 is automatic, while Condition (2) is satisfied only if $\omega$ is an odd integer. By duality, $\mathcal{H}$ is resonant implies $\mathcal{H}$ is strongly resonant.
Now consider the Hermitian case. Let us write the structure equations of the Hermitian hypergroup $\mathcal{H} = \{c_0, c_1, c_2\}$ as

\[
c_1^2 = \frac{1}{\omega_1} c_0 + a_1 c_1 + b_1 c_2
\]
\[
c_2^2 = \frac{1}{\omega_2} c_0 + b_2 c_1 + a_2 c_2
\]
\[
c_1 c_2 = e_1 c_1 + e_2 c_2
\]

and set

\[
W = \omega(\mathcal{H}) = 1 + \omega_1 + \omega_2. \tag{4.4}
\]

Then (A4) and associativity give the relations

\[
e_2 = 1 - e_1
\]
\[
b_1 = \frac{e_1 \omega_2}{\omega_1} \quad b_2 = \frac{e_2 \omega_1}{\omega_2} \tag{4.5}
\]
\[
a_1 = 1 - \frac{1 + e_1 \omega_2}{\omega_1} \quad a_2 = 1 - \frac{1 + e_2 \omega_1}{\omega_2}
\]

so that all of the structure constants are determined by $e_1$, $\omega_1$, and $\omega_2$.

Manipulating (4.3) shows that the values of the characters on $c_1$ and $c_2$ are the roots of the equations

\[
0 = (t - 1) \left( t^2 + t(e_1 - a_1) - \frac{e_2}{\omega_1} \right) = (t - 1)(t - x)(t - y)
\]
\[
0 = (t - 1) \left( t^2 + t(e_2 - a_2) - \frac{e_1}{\omega_2} \right) = (t - 1)(t - z)(t - v), \tag{4.6}
\]

where $x$, $y$, $z$, $v$ are the real numbers

\[
x = \frac{a_1 - e_1}{2} + \frac{\omega_2}{2} D \geq 0
\]
\[
y = \frac{a_1 - e_1}{2} - \frac{\omega_2}{2} D \leq 0
\]
\[
z = \frac{a_2 - e_2}{2} - \frac{\omega_1}{2} D \leq 0 \tag{4.7}
\]
\[
v = \frac{a_2 - e_2}{2} + \frac{\omega_1}{2} D \geq 0
\]
and

\[ D = \sqrt{(1 + e_1 \omega_2 - e_2 \omega_1)^2 + 4e_2 \omega_1 / \omega_1 \omega_2} \]

\[ = zy - vx > 0. \tag{4.8} \]

The character table for \( \mathcal{R} \) can thus be written

\[
\begin{array}{c|ccc}
\chi_0 & \chi_1 & \chi_2 \\
c_0 & 1 & 1 & 1 \\
c_1 & 1 & x & y \\
c_2 & 1 & z & v \\
\end{array}
\tag{4.9}
\]

We write the structure equations for \( \mathcal{R}^\sim = \{ \chi_0, \chi_1, \chi_2 \} \) as

\[ \chi_1^2 = \frac{1}{\mu_1} \chi_0 + d_1 \chi_1 + f_1 \chi_2 \]

\[ \chi_2^2 = \frac{1}{\mu_2} \chi_0 + f_2 \chi_1 + d_2 \chi_2 \tag{4.10} \]

\[ \chi_1 \chi_2 = g_1 \chi_1 + g_2 \chi_2. \]

Then

\[ W = \omega(\mathcal{R}^\sim) = 1 + \mu_1 + \mu_2 \tag{4.11} \]

and

\[ g_2 = 1 - g_1 \]

\[ f_1 = \frac{g_1 \mu_2}{\mu_1} \]

\[ f_2 = \frac{g_2 \mu_1}{\mu_2} \tag{4.12} \]

\[ d_1 = 1 - \left( \frac{1 + g_1 \mu_2}{\mu_1} \right) \]

\[ d_2 = 1 - \left( \frac{1 + g_2 \mu_1}{\mu_2} \right). \]

The orthogonality relations of the character table are

\[ 1 + \omega_1 x + \omega_2 z = 0 \]

\[ 1 + \omega_1 y + \omega_2 v = 0 \]

\[ 1 + \omega_1 xy + \omega_2 zv = 0 \tag{4.13} \]

and

\[ 1 + \mu_1 x + \mu_2 y = 0 \]

\[ 1 + \mu_1 z + \mu_2 v = 0 \]

\[ 1 + \mu_1 xz + \mu_2 yv = 0. \tag{4.14} \]
From (4.8) and the orthogonality relations we may deduce that

\[
\begin{align*}
\omega_1 &= \frac{v - z}{D} \quad \omega_2 = \frac{x - y}{D} \\
\mu_1 &= \frac{v - y}{D} \quad \mu_2 = \frac{x - z}{D}
\end{align*}
\]

and the somewhat remarkable formula

\[
D = \sqrt{\frac{W}{\omega_1 \omega_2 \mu_1 \mu_2}}.
\]

In fact it is not hard using the orthogonality relations to check that

\[
\begin{align*}
x &= \frac{-1 + \omega_2 \mu_2 D}{W - 1} \quad y = \frac{-1 - \omega_2 \mu_1 D}{W - 1} \\
z &= \frac{-1 - \omega_1 \mu_2 D}{W - 1} \quad v = \frac{-1 + \omega_1 \mu_1 D}{W - 1}
\end{align*}
\]

From the quadratic equations (4.6) and the corresponding equations

\[
0 = (t - 1) \left(t^2 + t(g_1 - d_1) - \frac{g_2}{\mu_1}\right) = (t - 1)(t - x)(t - z)
\]

\[
0 = (t - 1) \left(t^2 + t(g_2 - d_2) - \frac{g_1}{\mu_2}\right) = (t - 1)(t - y)(t - v)
\]

for the values of the characters, we may deduce the relations

\[
\begin{align*}
xy &= -\frac{c_2}{\omega_1} \quad x + y = a_1 - e_1 \\
zv &= -\frac{c_1}{\omega_2} \quad z + v = a_2 - e_2 \\
xz &= -\frac{g_2}{\mu_2} \quad x + z = d_1 - g_1 \\
yw &= -\frac{g_1}{\mu_2} \quad y + v = d_2 - g_2.
\end{align*}
\]
Combining these with (4.17), we get

\[
\begin{align*}
e_1 &= \frac{\omega_1 W - \omega_2 + \omega_1 \omega_2 (\mu_2 - \mu_1) D}{(W - 1)^2} \\
e_2 &= \frac{\omega_2 W - \omega_1 + \omega_1 \omega_2 (\mu_2 - \mu_1) D}{(W - 1)^2} \\
g_1 &= \frac{\mu_1 W - \mu_2 + \mu_1 \mu_2 (\omega_1 - \omega_2) D}{(W - 1)^2} \\
g_2 &= \frac{\mu_2 W - \mu_1 + \mu_1 \mu_2 (\omega_2 - \omega_1) D}{(W - 1)^2}.
\end{align*}
\] (4.20)

We now turn to the normalized equations for the associated assemblies obtained by setting \( C_s = c_i \) and \( X_s = \mu_i x_i, i = 1, 2, 3 \). We obtain

\[
\begin{align*}
C_1 &= \omega_1 C_0 + a_1 \omega_1 C_1 + e_1 \omega_1 C_2 = \omega_1 C_0 + A_1 C_1 + B_1 C_2 \\
C_2 &= \omega_2 C_0 + e_2 \omega_2 C_1 + a_2 \omega_2 C_2 = \omega_2 C_0 + B_2 C_1 + A_2 C_2 \quad (4.21) \\
C_1 C_2 &= e_1 \omega_2 C_1 + e_2 \omega_2 C_2 = E_1 C_1 + E_2 C_2
\end{align*}
\]

and

\[
\begin{align*}
X_1^2 &= \mu_1 X_0 + d_1 \mu_1 X_1 + g_1 \mu_1 X_2 = \mu_1 X_0 + D_1 X_1 + F_1 X_2 \\
X_2^2 &= \mu_2 X_0 + g_2 \mu_2 X_1 + d_2 \mu_2 X_2 = \mu_2 X_0 + F_2 X_1 + D_2 X_2 \quad (4.22) \\
X_1 X_2 &= g_1 \mu_2 X_1 + g_2 \mu_2 X_2 = G_1 X_1 + G_2 X_2.
\end{align*}
\]

**Proposition 4.1.** The following are equivalent

(i) \( \mathcal{A} \) is resonant.

(ii) All the coefficients in (4.21) are integers.

(iii) \( \omega_1, \omega_2, E_1, E_2 \in \mathbb{Z} \).

**Proof.** (i) \( \Rightarrow \) (iii). Suppose \( \mathcal{A} \) is resonant. Then \( \omega_1, \omega_2 \) are positive integers, \( \omega_1 \) divides \( e_1 \omega_1 \omega_2 \), and \( \omega_2 \) divides \( e_2 \omega_1 \omega_1 \). This implies that \( e_1 \omega_2 = E_1 \) and \( e_2 \omega_1 = E_2 \) are integers. (iii) \( \Rightarrow \) (ii). Suppose \( \omega_1, \omega_2, E_1, E_2 \in \mathbb{Z} \). Applying conservation of mass to (4.21) we obtain \( \omega_2 = 1 + a_1 \omega_1 + e_1 \omega_2 = 1 + A_1 + E_1 \) and \( \omega_1 = 1 + e_2 \omega_1 + a_2 \omega_2 = 1 + E_2 + A_2 \). Thus \( A_1, A_2 \in \mathbb{Z} \). Since \( B_1 = e_1 \omega_1 = \omega_1 - E_2 \) and \( B_2 = e_2 \omega_2 = \omega_2 - E_1 \), all coefficients of (4.21) are integers. (ii) \( \Rightarrow \) (i). If all the coefficients of (4.21) are integers, then to show \( \mathcal{A} \) resonant we need only prove that \( \omega_1 \mid B_1 \omega_2 \) and \( \omega_2 \mid B_2 \omega_1 \). But \( B_1 \omega_2 = e_1 \omega_1 \omega_2 = E_1 \omega_1 \) and \( B_2 \omega_1 = e_2 \omega_1 \omega_1 = E_2 \omega_2 \) so the result follows. \( \square \)
Similarly, we have

**Proposition 4.2.** The following are equivalent

(i) $\mathcal{H}$ is resonant.

(ii) All the coefficients in (4.22) are integers.

(iii) $\mu_1, \mu_2, G_1, G_2 \in \mathbb{Z}$.

**Corollary 4.3.** If $\mathcal{H}$ is resonant and $\mathcal{H}$ has integral weights then either

\[ \mu_1 = \mu_2 \text{ or } D \in \mathbb{Q} \] (or both). Similarly if $\mathcal{H}$ is resonant, then either

\[ \omega_1 = \omega_2 \text{ or } D \in \mathbb{Q} \] (or both).

**Proof.** If $\mathcal{H}$ is resonant, then $e_1 = E_1/\omega_1 \in \mathbb{Q}$. But from (4.20)

\[ e_1 = \frac{\omega_1 W - \omega_2 + \omega_2 \omega_1 (\mu_1 - \mu_2) D}{(W - 1)^2}. \tag{4.23} \]

Since $\omega_1, \omega_2 \in \mathbb{Z}^+$ are non-zero and $\mu_1, \mu_2 \in \mathbb{Z}^+$ the claim follows. The case of $\mathcal{H}$ is identical.

**Proposition 4.4.** Suppose $\mathcal{H}$ is resonant and $\mu_1 = \mu_2$. Then $\omega_1 = \omega_2$ and the structure equations for $\mathcal{H}$ must be of the form

\[ \begin{align*}
  c_1^2 &= \frac{1}{2k} c_0 + \left( \frac{1}{2} - \frac{1}{2k} \right) c_1 + \frac{1}{2} c_2 \\
  c_2^2 &= \frac{1}{2k} c_0 + \frac{1}{2} c + \left( \frac{1}{2} - \frac{1}{2k} \right) c_2 \\
  c_1 c_2 &= \frac{1}{2} c_1 + \frac{1}{2} c_2
\end{align*} \]

for some $k \in \mathbb{Z}^+$.

**Proof.** We assume $\mathcal{H}$ resonant. The condition $\mu_1 = \mu_2$ is by (4.15) equivalent to the condition $x + y = z + v$ which by (4.19) is equivalent to $a_1 - c_1 = a_2 - c_2$. Using (4.5) and (4.21) this becomes $\omega_2 - \omega_1 = (E_1 - E_2)(\omega_1 + \omega_2)$. Since $\omega_1$ and $\omega_2$ are positive integers and $E_1, E_2$ are integers, we may deduce that $\omega_1 = \omega_2$ and $E_1 = E_2$.

From (4.20) we further see that

\[ E_1 = E_2 = e_1 \omega_2 = \frac{2 \omega_1^3}{4 \omega_1^2} = \frac{\omega_1}{2} \tag{4.24} \]

so that $\omega_1 = \omega_2 = 2k$ for some $k \in \mathbb{Z}^+$. The structure equations for $\mathcal{H}$ follow and are as given. Conversely for any $k \in \mathbb{Z}^+$ the above structure equations agree with (4.5) so define a hypergroup.
Now consider the case when $K$ is resonant, $K^\wedge$ has integral weights, and $D \in \mathbb{Q}$. Note that from (4.16) we immediately get a non-trivial number theoretic condition on $\omega_1, \omega_2, \mu_1,$ and $\mu_2$. We wish to classify the possibilities. As a first consequence, we note that (4.7) implies that $x, y, z, v \in \mathbb{Q}$. Set 

\[ X = \omega_1 x, \quad Y = \omega_1 y, \quad Z = \omega_2 z, \quad \text{and} \quad V = \omega_2 v. \quad (4.25) \]

Then $X, Y$ and $Z, V$ are the roots of

\[ 0 = t^2 + t(1 + E_1 - E_2) - E_2 \quad (4.26) \]
\[ 0 = t^2 + t(1 + E_2 - E_1) - E_1, \quad (4.27) \]

respectively. It follows that $X, Y, Z, V$ are algebraic integers and thus rational integers. From (4.8),

\[ \Delta^2 = (1 + E_1 - E_2)^2 + 4E_2 = (1 + E_2 - E_1)^2 + 4E_1 \]
\[ = 1 + E_2^2 + E_1^2 + 2E_1 + 2E_2 - 2E_1E_2 \quad (4.28) \]

is the common discriminant of the quadratic equations.

Note that from the orthogonality relations (4.13) we have $Z = -(1 + X)$ and $Y = -(1 + V)$, so that

\[ E_1 = -VZ = V(X + 1) \]
\[ E_2 = -XY = X(V + 1) \quad (4.29) \]

and

\[ \Delta^2 = (1 + X + V)^2. \quad (4.30) \]

Although $X$ and $V$ thus determine $Y, Z, E_1,$ and $E_2$, they do not determine $\omega_1$ and $\omega_2$ which are subject to the additional equation (from conservation of mass) $\omega_1 \omega_2 = E_1 \omega_1 + E_2 \omega_2$. The solutions of this equation are given by $\omega_1 = E_2 + B_2, \omega_2 = E_1 + B_1,$ where $B_1, B_2$ are integers (possibly negative) satisfying $B_1B_2 = E_1E_2$. If we wish the corresponding parameters to define a hypergroup, then we must have $B_1, B_2 \geq 0$ and $A_i = \omega_i - 1 - E_i \geq 0$, $i = 1, 2$. Suppose henceforth that $E_1 \geq E_2$ (or equivalently $V \geq X$). Then $A_2 \geq 0$ holds automatically and the condition $A_1 \geq 0$ is

\[ B_2 \geq 1 + E_1 - E_2 = 1 + V - X. \quad (4.31) \]
One more condition remains to be considered—the integrality of \( \mu_1 \) and \( \mu_2 \). Since \( \mu_2 \) is integral if \( \mu_1 \) is, we need only consider \( \mu_1 \). Now

\[
D = zy - xv = \frac{ZY - XV}{\omega_1 \omega_2} = \frac{1 + X + V}{\omega_1 \omega_2}
\] (4.32)

and so we find

\[
\mu_1 = \frac{v - y}{D} = \frac{\omega_1 Y - \omega_2 Y}{1 + X + V} = V + \frac{V(B_1 + B_2 + 2VX + X) + B_1}{1 + X + V}.
\] (4.33)

We summarize our results.

**Theorem 4.5.** Let \( X, V \) be non-negative integers with \( V \geq X \). Set \( Z = -1 \), \( Y = -(1 + V) \), \( E_1 = V(X + 1) \), \( E_2 = X(V + 1) \). Find non-negative integers \( B_1, B_2 \) such that

1. \( B_1 B_2 = E_1 E_2 = XV(X + 1)(V + 1) \)
2. \( B_2 \geq 1 + V - X \)
3. \( (1 + X + V) | V(B_1 + B_2 + 2VX + X) + B_1 \).

Set \( \omega_1 = E_1 + B_1 \) and \( \omega_2 = E_1 + B_1 \). Then there exists a unique hypergroup \( \mathcal{K} \) with associated assembly \( \mathcal{K} = \{C_0, C_1, C_2\} \) given by

\[
C_1^2 = \omega_1 C_0 + (\omega_1 - 1 - E_1)C_1 + B_2 C_2 \\
C_2^2 = \omega_2 C_0 + B_1 C + (\omega_2 - 1 - E_2)C_2 \\
C_1 C_2 = E_1 C_1 + E_2 C_2.
\]

This hypergroup is resonant and its dual \( \mathcal{K}^\vee \) has integral weights. Furthermore any resonant Hermitian hypergroup of order 3 whose dual \( \mathcal{K}^\vee \) has integral weights is either one of the family given in Proposition 4.4 or obtained as above for a suitable choice of \( X, V, B_1, \) and \( B_2 \) satisfying conditions (1), (2), and (3).

We will refer to \( X \) and \( V \) as the seeds of the assembly \( \mathcal{K} \) (or of the hypergroup \( \mathcal{K} \)).

There are some obvious candidates for factors \( B_1, B_2 \) of \( E_1 E_2 = XV \times (X + 1)(V + 1) \), and they result in particularly simple families of examples.

1. If \( B_1 = XV \) and \( B_2 = (X + 1)(V + 1) \), then \( \omega_1 = (2X + 1) \times (V + 1) \), \( \omega_2 = V(2X + 1) \) and \( \mu_1 = 2V + 2VX(2V + 1)/(1 + X + V) \) so the only condition required is (3) \((1 + X + V) | 2VX(2V + 1)\).
(2) If \( B_1 = X(1 + X) \) and \( B_2 = V(1 + V) \) then \( \omega_1 = (X + V) \times (V + 1) \), \( \omega_2(X + V)(X + 1) \) and \( \mu_1 = \omega_1 \). In this case, conditions (1), (2), and (3) are automatic.

(3) If \( B_1 = X(V + 1) \) and \( B_2 = V(X + 1) \) then \( \omega_1 = 2XV + X + V = \omega_2 \) and \( \mu_1 = (2XV + X + V)(2V + 1)/(1 + X + V) \) so the only condition is (3) \((1 + X + V)/(2XV - 1)(2V + 1)\).

### 5. STRONGLY RESONANT STRONG HYPERGROUPS

We will now provide an alternate formulation of the result contained in Theorem 4.5 which will be more convenient when we wish to emphasize the symmetry between \( S \) and \( S^\wedge \). In parallel with (4.25), set

\[
X' = \mu_1 x, \quad Y' = \mu_2 y, \quad Z' = \mu_1 z, \quad \text{and} \quad V' = \mu_2 v. \tag{5.1}
\]

Then

\[
Y' = -(X' + 1) \quad \text{and} \quad Z' = -(V' + 1). \tag{5.2}
\]

We now will attempt to express all the other coefficients in (3.19) and (3.20) solely in terms of the quantities \( X, V, X', \) and \( V' \). Eliminating \( x, y, z, \) and \( v \) from (4.25) and (5.1) gives

\[
\begin{align*}
\mu_1 X &= \omega_1 X' \\
\mu_2 (1 + V) &= \omega_2 (1 + X') \\
\omega_2 (1 + V') &= \mu_1 (1 + X) \\
\mu_2 V &= \omega_2 V'.
\end{align*}
\tag{5.3}
\]

Solving for \( \mu_1 / \mu_2 \) in two different ways, we find

\[
(1 + X)(1 + V)X'V' = (1 + X')(1 + V')XV. \tag{5.4}
\]

This is a necessary condition on \( X, V, X', V' \). It will play a crucial role in the following analysis. We henceforth assume that \( X, V, X', V' \) satisfy (5.4). Then we get the ratios

\[
\begin{align*}
[ \omega_1; \omega_2; \mu_1; \mu_2 ] &= K[(1 + X)(1 + V)V'; (1 + X)(1 + X)V; \\
& \quad (1 + V')(1 + X')V; (1 + X')(1 + X) V] \tag{5.5}
\end{align*}
\]
for some constant $K$ not specified by (5.3). Invoking the orthogonality relation (4.13)

$$1 + \frac{XY}{\omega_1} + \frac{ZV}{\omega_2} = 0$$

(5.6)

together with (5.2), (5.4), and (5.5) yields

$$K = \frac{X + XX' + V' + XV''}{(1 + X)(1 + X')V'} = \frac{X' + XX' + V + VX'}{(1 + X)(1 + X')V'}.$$ 

(5.7)

Then from (5.5) we get

$$\omega_1 = X(1 + V') + \frac{V'(1 + X)(1 + V)}{1 + X'}$$
$$\omega_2 = X' + XX' + V + VX'$$
$$\mu_1 = X'(1 + V) + \frac{V(1 + X')(1 + V')}{1 + X}$$
$$\mu_2 = X + XX' + V' + XV'.$$

(5.8)

Now using Theorem 4.5 for both $\mathcal{F}$ and $\mathcal{F}^\wedge$ gives (recall the definitions in (4.21) and (4.22))

$$B_1 = X' + XX' + VX' - VX$$
$$B_2 = V'(1 + X)(1 + V) = \frac{XY(1 + V')}{1 + X'}$$
$$F_1 = X + XX' + V'X - V'X'$$
$$F_2 = V'(1 + X')(1 + V') = \frac{X'V'(1 + V)}{X}.$$ 

(5.9)

In order for $\mathcal{F}$ to be a hypergroup we need $B_1$ and $B_2$ to satisfy $B_1 \geq 0$ and $B_2 \geq 1 + V - X$. Both of these conditions are easily shown to be consequences of (5.4) and the inequalities $0 \leq X' \leq V'$ and $0 \leq X \leq V$. Since we may argue similarly for $\mathcal{F}^\wedge$, we have proven the following:

**Theorem 5.1.** Let $X, V, X', V'$ be non-negative integers satisfying

1. $0 \leq X \leq V$ and $0 \leq X' \leq V'$ and
2. $(1 + X)(1 + V)V' = (1 + X')(1 + V')XV$. 

Then there exists a unique Hermitian strongly resonant strong hypergroup \( \mathcal{H} \) of order three whose seeds are \( X, V \) and whose dual \( \mathcal{H}^\dagger \) has seeds \( X', V' \). The various values of \( \omega_1, \omega_2, \mu_1, \mu_2, B_1, B_2, F_1, \) and \( F_2 \) are given by (5.8) and (5.9), while the other values \( E_1, E_2, A_1, A_2, G_1, G_2, C_1, \) and \( C_2 \) are given as in Theorem 4.5. Conversely every Hermitian strongly resonant strong hypergroup \( \mathcal{H} \) of order three occurs in this fashion through the choice of non-negative integers \( X, V, X', V' \) satisfying (1) and (2).

Note that if one of \( X, X' \) is zero, the other is also and \( V, V' \) can be chosen arbitrarily. Such solutions we may term degenerate, and if we exclude them our analysis reduces to the solution in positive integers of

\[
\frac{1 + X}{V} \frac{1 + V}{X'} = \frac{1 + X'}{V'} \frac{1 + V'}{X}. \tag{5.10}
\]

This equation has an infinite number of solutions and certain families of solutions can be obtained with little effort. The proper classification of all its solutions is a problem the author offers to those with an interest in number theory. We remark that Fermat had an interest in a class of questions closely connected with the above equation—the determination of how many ways a number of the form \((1 + X)/X\) could be expressed as a product of similar numbers (with the number of terms in the product being unspecified).

6. APPLICATIONS TO STRONGLY REGULAR GRAPHS

A graph \( X \) of order \( p \) is strongly regular if there exist integers \( k, \lambda, \mu \) such that

(i) \( X \) is regular of valency \( k \)

(ii) given any two distinct vertices \( v \) and \( w \), the number of vertices adjacent to both \( v \) and \( w \) is \( \lambda \) if \( v \) and \( w \) are adjacent, and \( \mu \) otherwise.

References for the theory of strongly regular graphs are Cameron [6] and Hubaut [9].

Fix a point \( c_0 \) of \( X \), let \( c_1 \) denote the set of points adjacent to \( c_0 \), and \( c_2 \) the set of points of distance two from \( c_0 \). Then \( X \) is the union of these three sets, and there is naturally defined a convolution of these sets entirely analogous to the convolution of \( K \) orbits on a symmetric space \( G/K \) (see Wildberger [22, 23] for more information about this approach). That is, we consider \( c_1 \) and \( c_2 \) to be the spheres of radius 1 and 2, respectively, about \( c_0 \). To convolve \( c_1 \) and \( c_j \), take an arbitrary point \( P_i \) of
and consider a random point \( Q \) of distance \( j \) from \( P \). Let \( n^k_{ij} \) be the probability that \( Q \) lies in \( c_k \). Then \( c_i c_j = \sum_k n^k_{ij} c_k \) is a well-defined hypergroup given specifically by

\[
\begin{align*}
    c_1^2 &= \frac{1}{k} c_0 + \frac{\lambda}{k} c_1 + \frac{k-\lambda-1}{k} c_2 \\
    c_2^2 &= \frac{1}{l} c_0 + \frac{k-\mu}{l} c_1 + \frac{l+\mu-k-1}{l} c_2 \\
    c_1 c_2 &= \frac{\mu}{k} c_1 + \frac{k-\mu}{k} c_2,
\end{align*}
\]

where we have introduced \( l = p - k - 1 \).

Equations (6.1) define a Hermitian hypergroup \( \mathcal{H} = \{c_1, c_1, c_2\} \) of order three, and we may immediately deduce from (4.5) that a necessary relationship between the parameters \( p, k, \lambda, \mu, \) and \( l \) is

\[
k(k - \lambda - 1) = l \mu.
\]

This condition is well known (see Cameron [6]). We call \( \mathcal{H} = \mathcal{H}(X) \) the hypergroup of the strongly regular graph \( X \). One of the simplest strongly regular graphs is the pentagon with hypergroup

\[
\begin{align*}
    c_1^2 &= \frac{1}{2} c_0 + \frac{1}{2} c_2 \\
    c_2^2 &= \frac{1}{2} c_0 + \frac{1}{2} c_1 \\
    c_1 c_2 &= \frac{1}{2} c_1 + \frac{1}{2} c_2,
\end{align*}
\]

This hypergroup has particularly pleasant properties; it deserves to be called the Golden hypergroup.

The study of the algebraic system (6.1) has clearly been of importance in the theory of strongly regular graphs, although perhaps the explicit role of the hypergroup being advocated here is new. We now have the theory of the previous sections within which to work. We note first that \( k \) and \( l \) are the weights of \( c_1, c_2 \) and that the associated assembly \( \mathcal{H}(X) = \{C_0, C_1, C_2\} \) is given by

\[
\begin{align*}
    C_1^2 &= kC_0 + \lambda C_1 + \mu C_2 \\
    C_2^2 &= lC_0 + (p - 2k + \lambda)C_1 + (l + \mu - k - 1)C_2 \\
    C_1 C_2 &= (k - \lambda - 1)C_1 + (k - \mu)C_2.
\end{align*}
\]
Not every hypergroup of the form (6.1) comes from a strongly regular graph. It is known that there are two important necessary conditions on the parameters. The first, known as the **Integrality Condition**, asserts that the quantities

\[
\frac{1}{2} \left( k + l + \frac{(k + l)(\mu - \lambda) - 2k}{\sqrt{(\mu - \lambda)^2 + 4(k - \mu)}} \right)
\]

and

\[
\frac{1}{2} \left( k + l - \frac{(k + l)(\mu - \lambda) - 2k}{\sqrt{(\mu - \lambda)^2 + 4(k - \mu)}} \right)
\]

are non-negative integers. These numbers are the multiplicities of the eigenvalues of the graph (see Schwenk and Wilson [17]). Can we interpret these rather mysterious quantities in hypergroup-theoretic terms? Using the notation of the previous sections, and using \(\omega_1 = k, \omega_2 = l\), etc., (4.8) gives

\[
D = \sqrt{(\mu - \lambda)^2 + 4(k - \mu)}/kl
\]

and (4.15) together with (4.7) shows the two quantities in (6.5) are \(\mu_1\) and \(\mu_2\).

Thus the Integrality Condition is nothing but the requirement that the dual \(\mathcal{R}(X)^{\sim}\) have integral weights.

The second condition, known as the **Krein Condition**, asserts that if \(X\) is a strongly regular graph such that both \(X\) and its complement \(\overline{X}\) are connected (\(\overline{X}\) is also strongly regular), and if \(k, r,\) and \(s\) are the eigenvalues of \(X\), then

\[
(r + 1)(k + r + 2rs) \leq (k + r)(s + 1)^2
\]

and

\[
(s + 1)(k + s + 2sr) \leq (k + s)(r + 1)^2.
\]

Again we may reasonably ask if this somewhat mysterious condition has a formulation in hypergroup theoretic terms, and again the answer is affirmative. It turns out that (6.7) is exactly equivalent to the condition that \(\mathcal{R}(X)^{\sim}\) is itself a hypergroup (as opposed to simply being a signed hypergroup, which is automatic by Theorem 1.7).

Let us summarize our observations.
Theorem 6.1. Let \( X \) be a strongly regular graph. Then \( \mathcal{H}(X) \) is a resonant Hermitian hypergroup whose dual \( \mathcal{H}(X)^\wedge \) has integral weights. If further both \( X \) and \( X \) are connected, then \( \mathcal{H}(X)^\wedge \) is a hypergroup.

Proof. It requires only to show the resonance of \( \mathcal{H}(X) \), which follows from the form of Eq. (6.4) and Proposition 5.1.

7. A Hypergroup of Grassmanians

We will illustrate the results of the last few sections by investigating in some detail a particular hypergroup associated to the family of 2-planes in \( F_q^n \), or lines in \( PG(n-1, q) \). We declare two lines to be adjacent if they intersect. From the action of \( PSL_n(q) \), we can see that this will give us a strongly regular graph and one can compute that the parameters will be

\[
\begin{align*}
    p &= \frac{(q^{n+1} - 1)(q^n - 1)}{(q + 1)(q - 1)^2} \\
    k &= \frac{q(q + 1)(q^{n-1} - 1)}{q - 1} \\
    \lambda &= \frac{q^n - 1}{q - 1} + q^2 - 2 \\
    \mu &= (q + 1)^2 \\
    l &= \frac{q^4(q^{n-1} - 1)(q^{n-2} - 1)}{(q + 1)(q - 1)^2}.
\end{align*}
\]

To find the seeds \( X, V \), we must solve the quadratic equations (4.26) and (4.27). This is simplified by the observation that

\[
\Delta = \frac{q^n - 2q^2 + 1}{q - 1}
\]

from which we quickly obtain

\[
\begin{align*}
    X &= \frac{q^n - q^2 - q + 1}{q - 1} \\
    V &= q.
\end{align*}
\]
From the remark following (4.28) we have
\[ Y = -(q + 1) \]
\[ Z = -\frac{q^n - q^2}{q - 1} \]  \hspace{1cm} (7.4)

and then from (4.25) we deduce that the character values are
\[ x = \frac{q^n - q^2 - q + 1}{q(q + 1)(q^{n-1} - 1)} \]
\[ y = -\frac{(q - q)^2}{q(q + 1)(q^{n-1} - 1)} \]
\[ z = -\frac{(q - 1)(q + 1)}{q^2(q^{n-1} - 1)} \]
\[ w = \frac{(q + 1)(q - 1)^2}{q^3(q^{n-1} - 1)(q^{n-2} - 1)}. \]  \hspace{1cm} (7.5)

Now using (4.8) and (4.15), we may compute that
\[ \mu_1 = \frac{q(q^n - 1)}{q - 1} \]
\[ \mu_2 = \frac{q^2(q^{n+1} - 1)(q^{n-2} - 1)}{(q + 1)(q - 1)^2}. \]  \hspace{1cm} (7.6)

Notice that these are both integers. To compute the seeds \( X', V' \) of the dual \( \hat{\mathcal{R}} \), we use (5.1) to get
\[ X' = \frac{(q^n - 1)(q^n - q^2 - q + 1)}{(q - 1)(q + 1)(q^{n-1} - 1)} \]
\[ V' = \frac{q^{n+1} - 1}{q(q^{n-1} - 1)}. \]  \hspace{1cm} (7.7)

These are not generally integers, so that \( \hat{\mathcal{R}} \) is not generally resonant. To complete the picture, we calculate the structure constants of the assemblies associated to \( \mathcal{R} \) and its dual as given by (4.21) and (4.22). These follow immediately from (5.9) and algebraic manipulation. The results are
\[ A_1 = q^2 - 2 + \frac{q^n - 1}{q - 1} \]
\[ A_2 = \frac{q(q^{2n} - 2q^{n+2} - 2q^n + q^{n+1} + q^n + q^{n-1} + 3q^3 - 2q^2 - 2q + 1)}{(q + 1)(q - 1)^2}. \]
\[ B_1 = (q + 1)^2 \]
\[ B_2 = \frac{q^3(q^{2n-2} - 2q^n - 2q^{n-1} + q^{n-2} + q^{n-1} - q^{n-3} + q^2 + q - 1)}{(q + 1)(q - 1)^2} \]
\[ E_1 = \frac{q^3(q^{n-2} - 1)}{q - 1} \]
\[ E_2 = \frac{(q + 1)(q^n - q^2 - q + 1)}{q - 1} \]
\[ D_1 = \frac{(q^n - 1)(q^{2n-1} + q^{n+2} - 2q^{n+1} - 4q^n + 2q^{n-1} + q^{n-2} + 2q^2 + q - 2)}{(q - 1)(q + 1)(q^{n-1} - 1)^2} \]
\[ D_2 = \frac{(q^n - 1)(q^{n+1} - 2q^{2n+1} - 2q^{2n} + q^{2n-1} + q^{n+3} - q^n + 3q^{n+2} - 3q^3 + q^2 + 2q - 1)}{q(q + 1)(q - 1)^2(q^{n-1} - 1)^2} \]
\[ G_1 = \frac{(q^{n+1} - 1)^2(q^{n-2} - 1)}{(q + 1)(q^{n-1} - 1)^2(q - 1)} \]
\[ G_2 = \frac{(q^n - q^2 - q + 1)(q^n - 1)^2}{q(q - 1)(q^{n-1} - 1)^2} \]
\[ F_1 = \frac{(q^{n+1} - 1)(q^n - 1)}{q(q^{n-1} - 1)^2} \]
\[ F_2 = \frac{(q^n - q^2 - q + 1)(q^n - 1)(q^{n+1} - 1)}{(q^{n-1} - 1)^2(q + 1)(q - 1)^2} \]

To be very specific, we give both sets of assembly equations in the case \( q = 5, n = 3 \).

\[ C_1^2 = 180C_0 + 54C_1 + 36C_2 \]
\[ C_2^2 = 625C_0 + 500C_1 + 480C_2 \] (7.9)
\[ C_1C_2 = 125C_1 + 144C_2 \]
\[ X_1^2 = 155X_0 + \frac{124}{3}X_1 + \frac{403}{15}X_2 \]
\[ X_2^2 = 650X_0 + \frac{1612}{3}X_1 + \frac{7813}{15}X_2 \]
\[ X_1X_2 = \frac{338}{3}X_1 + \frac{1922}{15}X_2. \]

The seeds are
\[ X = 24, \quad V = 5, \quad X' = \frac{62}{3}, \quad \text{and} \quad V' = \frac{26}{5} \]
and the character table of the associated hypergroup \( \mathcal{H} = \{ c_1, c_2, c_3 \} \) is
\[
\begin{array}{c|ccc}
 & X_0 & X_1 & X_2 \\
\hline
 c_0 & 1 & 1 & 1 \\
 c_1 & 1 & \frac{2}{15} & -\frac{1}{30} \\
 c_2 & 1 & -\frac{1}{25} & \frac{1}{125} \\
\end{array}
\]

In the Appendix we construct the explicit equations of the hypergroups and their duals associated to the strongly regular graphs related to sporadic groups given by the list (S) of Hubaut [9]. For convenience we write down the associated assemblies. The seeds \( X, V, X', V' \) are given for each and one can see how efficiently these numbers encode the structure of the hypergroups. The strongly resonant hypergroups in this list are thus seen to be \( S_3, S_9, S_{10}, S_{11}, S_{12}, S_{13} \) and \( S_{19} \).

**APPENDIX**

S1. \( PSL_3(4) \) acting on an orbit of 56 complete conics of \( PG(2, 4) \).
\[
C_1^2 = 10C_0 + 0C_1 + 2C_2 \quad X_1^2 = 35X_0 + \frac{70}{3}X_1 + \frac{56}{3}X_2 \\
C_2^2 = 45C_0 + 36C_1 + 36C_2 \quad X_2^2 = 20X_0 + \frac{8}{3}X_1 + \frac{28}{3}X_2 \\
C_1C_2 = 9C_1 + 8C_2 \quad X_1X_2 = \frac{32}{3}X_1 + \frac{49}{3}X_2 \\
X = 2, V = 3, X' = 7, V' = 4/3.\]
S2. $M_{22}$ acting on the 77 blocks of $S(3, 6, 22)$.

\[ C_1^2 = 16C_0 + 0C_1 + 4C_2 \quad X_1^2 = 55X_0 + \frac{1287}{32}X_1 + \frac{1155}{32}X_2 \]

\[ C_2^2 = 60C_0 + 45C_1 + 47C_2 \quad X_2^2 = 21X_0 + \frac{35}{32}X_1 + \frac{231}{32}X_2 \]

\[ C_1C_2 = 15C_1 + 12C_2 \quad X_1X_2 = \frac{441}{32}X_1 + \frac{605}{32}X_2 \]

$X = 2, V = 5, X' = 55/8, V'' = 7/4.$

S3. $PSU_3(5^2)$ acting over subsets of autoconjugate triangles in $PG(2, 5^3)$ with an hermitian conic:

\[ C_1^2 = 7C_0 + 0C_1 + 1C_2 \quad X_1^2 = 28X_0 + 18X_1 + 12X_2 \]

\[ C_2^2 = 42C_0 + 36C_1 + 35C_2 \quad X_2^2 = 21X_0 + 4X_1 + 12X_2 \]

\[ C_1C_2 = 6C_1 + 6C_2 \quad X_1X_2 = 9X_1 + 16X_2 \]

$X = 2, V = 2, X' = 8, V'' = 1.$

S4. $PSL_4(4)$ acting on the 105 flags of $PG(2, 4)$.

\[ C_1^2 = 32C_0 + 4C_1 + 12C_2 \quad X_1^2 = 84X_0 + \frac{539}{8}X_1 + \frac{525}{8}X_2 \]

\[ C_2^2 = 72C_0 + 45C_1 + 51C_2 \quad X_2^2 = 20X_0 + \frac{5}{8}X_1 + \frac{35}{8}X_2 \]

\[ C_1C_2 = 27C_1 + 20C_2 \quad X_1X_2 = \frac{125}{8}X_1 + \frac{147}{8}X_2 \]

$X = 2, V = 9, X' = 6, V'' = 5/2.$

S5. $PSL_4(4)$ acting on an orbit of 120 Baer subplanes of $PG(2, 4)$.

\[ C_1^2 = 42C_0 + 8C_1 + 18C_2 \quad X_1^2 = 99X_0 + \frac{4002}{48}X_1 + \frac{3960}{49}X_2 \]

\[ C_2^2 = 77C_0 + 44C_1 + 52C_2 \quad X_2^2 = 20X_0 + \frac{40}{49}X_1 + \frac{180}{49}X_2 \]

\[ C_1C_2 = 33C_1 + 24C_2 \quad X_1X_2 = \frac{800}{49}X_1 + \frac{891}{49}X_2 \]

$X = 2, V = 11, X' = 33/7, V'' = 20/7.$
S6. $M_{23}$ acting on the 253 blocks of $S(4, 7, 23)$.

\[
\begin{align*}
C_1^2 &= 112C_0 + 36C_1 + 60C_2 & X_1^2 &= 230X_0 + \frac{81903}{392}X_1 + \frac{82225}{392}X_2 \\
C_2^2 &= 140C_0 + 65C_1 + 87C_2 & X_2^2 &= 22X_0 + \frac{297}{392}X_1 + \frac{759}{392}X_2 \\
C_1C_2 &= 75C_1 + 52C_2 & X_1X_2 &= \frac{7865}{392}X_1 + \frac{7935}{392}X_2
\end{align*}
\]

$X = 2, V = 25, X' = 115/28, V' = 55/44.$

S7. $M_{22}$ acting on the 176 blocks of $M_{23}$ avoiding one point.

\[
\begin{align*}
C_1^2 &= 70C_0 + 18C_1 + 34C_2 & X_1^2 &= 154X_0 + \frac{3366}{25}X_1 + \frac{3366}{25}X_2 \\
C_2^2 &= 105C_0 + 54C_1 + 68C_2 & X_2^2 &= 21X_0 + \frac{16}{25}X_1 + \frac{66}{25}X_2 \\
C_1C_2 &= 51C_1 + 36C_2 & X_1X_2 &= \frac{459}{25}X_1 + \frac{484}{25}X_2
\end{align*}
\]

$X = 2, V = 17, X' = 22/5, V' = 17/5.$

S8. $PSU_3(5^2)$ acting on the 175 edges of the Hoffman-Singleton graph (S.3).

\[
\begin{align*}
C_1^2 &= 72C_0 + 20C_1 + 36C_2 & X_1^2 &= 153X_0 + \frac{1069}{8}X_1 + \frac{1071}{8}X_2 \\
C_2^2 &= 102C_0 + 51C_1 + 65C_2 & X_2^2 &= 21X_0 + \frac{7}{8}X_1 + \frac{21}{8}X_2 \\
C_1C_2 &= 51C_1 + 36C_2 & X_1X_2 &= \frac{147}{8}X_1 + \frac{153}{8}X_2
\end{align*}
\]

$X = 2, V = 17, X' = 17/4, V' = 7/2.$

S9. Rank 3 representation of $G_2(2)$ on 36 points.

\[
\begin{align*}
C_1^2 &= 14C_0 + 4C_1 + 16C_2 & X_1^2 &= 21X_0 + 12X_1 + 12X_2 \\
C_2^2 &= 21C_0 + 12C_1 + 12C_2 & X_2^2 &= 14X_0 + 4X_1 + 6X_2 \\
C_1C_2 &= 9C_1 + 8C_2 & X_1X_2 &= 8X_1 + 9X_2
\end{align*}
\]

$X = 2, V = 3, X' = 3, V' = 2.$
S10. Rank 3 representation of $HS$ in 100 points.
\[ C_1^2 = 22C_0 + 0C_1 + 6C_2 \quad X_1^2 = 77X_0 + 60X_1 + 56X_2 \]
\[ C_2^2 = 77C_0 + 56C_1 + 60C_2 \quad X_2^2 = 22X_0 + 0X_1 + 6X_2 \]
\[ C_1C_2 = 21C_1 + 16C_2 \quad X_1X_2 = 16X_1 + 21X_2 \]
$X = 2, V = 27, X' = 7, V' = 2.$
S11. Rank 3 representation of $HI$ in 100 points.
\[ C_1^2 = 36C_0 + 14C_1 + 12C_2 \quad X_1^2 = 36X_0 + 14X_1 + 12X_2 \]
\[ C_2^2 = 63C_0 + 42C_1 + 38C_2 \quad X_2^2 = 63X_0 + 38X_1 + 42X_2 \]
\[ C_1C_2 = 21C_1 + 24C_2 \quad X_1X_2 = 21X_1 + 24X_2 \]
$X = 6, V = 3, X' = 3, V' = 6.$
S12. Rank 3 representation of $PSU(3)$ on 162 points.
\[ C_1^2 = 56C_0 + 10C_1 + 24C_2 \quad X_1^2 = 140X_0 + 121X_1 + 120X_2 \]
\[ C_2^2 = 105C_0 + 60C_1 + 72C \quad X_2^2 = 21X_0 + 0X_1 + 3X_2 \]
\[ C_1C_2 = 45C_1 + 32C_2 \quad X_1X_2 = 18X_1 + 20X_2 \]
$X = 2, V = 15, X' = 15, V' = 2.$
S13. Rank 3 representation of $McL$ on 275 points.
\[ C_1^2 = 112C_0 + 30C_1 + 56C_2 \quad X_1^2 = 252X_0 + \frac{1385}{6}X_1 + 231X_2 \]
\[ C_2^2 = 162C_0 + 81C_1 + 105C_2 \quad X_2^2 = 22X_0 + 0X_1 + \frac{11}{6}X_2 \]
\[ C_1C_2 = 81C_1 + 56C_2 \quad X_1X_2 = \frac{121}{6}X_1 + 21X_2 \]
$X = 2, V = 27, X' = 27, V' = 2.$
\[ C_1^2 = 100C_0 + 36C_1 + 20C_2 \quad X_1^2 = 65X_0 + \frac{52}{3}X_1 + \frac{26}{3}X_2 \]
\[ C_2^2 = 315C_0 + 252C_1 + 234C_2 \quad X_2^2 = 350X_0 + \frac{878}{3}X_1 + \frac{910}{3}X_2 \]
\[ C_1C_2 = 63C_1 + 80C_2 \quad X_1X_2 = \frac{140}{3}X_1 + \frac{169}{3}X_2 \]
$X = 20, V = 3, X' = 13, V' = 10/3.$
S15. Rank 3 representation of $Suz$ on 1782 vertices.

\[ C_1^2 = 416C_0 + 100C_1 + 96C_2 \quad X_1^2 = 780X_0 + \frac{711}{2}X_1 + 330X_2 \]
\[ C_2^2 = 1365C_0 + 1050C_1 + 1044C_2 \quad X_2^2 = 1001X_0 + 550X_1 + \frac{1155}{2}X_2 \]
\[ C_1C_2 = 315C_1 + 320C_2 \quad X_1X_2 = \frac{847}{2}X_1 + 450X_2 \]

$X = 20, V = 15, X' = 75/2, V' = 11.$

S16. Rank 3 representation of $Fi_{22}$ on 3510 vertices.

\[ C_1^2 = 693C_0 + 180C_1 + 126C_2 \quad X_1^2 = 429X_0 + 78X_1 + \frac{195}{4}X_2 \]
\[ C_2^2 = 2816C_0 + 2304C_1 + 2248C_2 \quad X_2^2 = 3080X_0 + \frac{10795}{4}X_1 + 2730X_2 \]
\[ C_1C_2 = 512C_1 + 567C_2 \quad X_1X_2 = 350X_1 + \frac{1521}{4}X_2 \]

$C = 63, V = 8, X' = 39, V' = 35/4.$

S17. Rank 3 representation of $Fi_{23}$ on 31671 vertices.

\[ C_1^2 = 3510C_0 + 693C_1 + 51C_2 \quad X_1^2 = 782X_0 + \frac{4301}{50}X_1 + \frac{3519}{200}X_2 \]
\[ C_2^2 = 28160C_0 + 25344C_1 + 25000C_2 \]
\[ X_2^2 = 30888X_0 + \frac{6024519}{200}X_1 + \frac{1509651}{50}X_2 \]
\[ C_1C_2 = 2816C_1 + 3159C_2 \quad X_1X_2 = \frac{34749}{50}X_1 + \frac{152881}{200}X_2 \]

$X = 351, V = 8, X' = 391/2, V' = 35/4.$

S18. Rank 3 representation of $Fi_{24}$ on 306,936 vertices.

\[ C_1^2 = 31671C_0 + 3510C_1 + 3240C_2 \]
\[ X_1^2 = 57477X_0 + 11221X_1 + \frac{21315}{2}X_2 \]
\[ C_2^2 = 275264C_0 + 247104C_1 + 246832C_2 \]
\[ X_2^2 = 249458X_0 + \frac{405275}{2}X_1 + 203203X_2 \]
\[ C_1C_2 = 28160C_1 + 28431C_2 \]
\[ X_1X_2 = 46255X_1 + \frac{93639}{2}X_2 \]

\[ X = 351, \quad V = 80, \quad X' = 637, \quad V' = 145/2. \]

**S19. Rank 3 representation of** \( 2^2 \cdot M_{24} \) **on** 2048 **vertices.**

\[ C_1^2 = 759C_0 + 310C_1 + 264C_2 \quad X_1^2 = 276X_0 + 44X_1 + 36X_2 \]
\[ C_2^2 = 1288C_0 + 840C_1 + 792C_2 \quad X_2^2 = 1771X_0 + 1530X_1 + 1540X_2 \]
\[ C_1C_2 = 448C_1 + 495C_2 \quad X_1X_2 = 231X_1 + 240X_2 \]

\[ X = 55, \quad V = 8, \quad X' = 20, \quad V' = 11. \]

**S20. Rank 3 representation of** \( M_{24} \) **over** \( 2 \cdot M_{12}. \)

\[ C_1^2 = 495C_0 + 206C_1 + 180C_2 \]
\[ X_1^2 = 252X_0 + \frac{6566}{121}X_1 + \frac{5796}{121}X_2 \]
\[ C_2^2 = 792C_0 + 504C_1 + 476C_2 \]
\[ X_2^2 = 1035X_0 + \frac{100418}{121}X_1 + \frac{101430}{121}X_2 \]
\[ C_1C_2 = 288C_1 + 315C_2 \]
\[ X_1X_2 = \frac{23805}{121}X_1 + \frac{24696}{121}X_2 \]

\[ X = 35, \quad V = 8, \quad X' = 196/11, \quad V' = 115/11. \]

**S21. Rank 3 representation of** \( Co \cdot 2 \) **over** \( 2 \cdot PSU_6(2). \)

\[ C_1^2 = 891C_0 + 378C_1 + 324C_2 \quad X_1^2 = 275X_0 + \frac{350}{9}X_1 + \frac{575}{18}X_2 \]
\[ C_2^2 = 1408C_0 + 896C_1 + 840C_2 \quad X_2^2 = 2024X_0 + \frac{32039}{18}X_1 + \frac{16100}{9}X_2 \]
\[ C_1C_2 = 512C_1 + 567C_2 \quad X_1X_2 = \frac{2116}{9}X_1 + \frac{4375}{18}X_2 \]

\[ X = 63, \quad V = 8, \quad X' = 175/9, \quad V' = 23/2. \]
S22. Rank 3 representation of $PSU_6(2)$ over $PSU_6(3)$.

\[ C_1^2 = 567C_0 + 246C_1 + 216C_2 \quad X_1^2 = 252X_0 + \frac{148}{3}X_1 + 44X_2 \]

\[ C_2^2 = 840C_0 + 520C_1 + 488C_2 \quad X_2^2 = 1155X_0 + 946X_1 + \frac{2860}{3}X_2 \]

\[ C_1C_2 = 320C_1 + 351C_2 \quad X_1X_2 = \frac{605}{3}X_1 + 208X_2 \]

$X = 39$, $V = 8$, $X' = 52/3$, $V' = 11$.

S23. Rank 3 representation of Rudvalis group on $F_4(2)$.

\[ C_1^2 = 1755C_0 + 730C_1 + 780C_2 \quad X_1^2 = 3276X_0 + \frac{10577}{4}X_1 + 2639X_2 \]

\[ C_2^2 = 2304C_0 + 1280C_1 + 1328C_2 \quad X_2^2 = 783X_0 + 145X_1 + \frac{609}{4}X_2 \]

\[ C_1C_2 = 1024C_1 + 975C_2 \quad X_1X_2 = \frac{2523}{4}X_1 + 637X_2 \]

$X = 15$, $V = 64$, $X' = 28$, $V' = 87/4$.

REFERENCES


