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DEFORMATIONS OF REAL SINGULARITIES

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IN THE theory of singularities of functions (or hypersurfaces, or complete intersections), it is well known that equivalent conditions are that the singularity be isolated, that it be finitely determined (with respect to an appropriate equivalence relation) or that it possess a finite dimensional versal unfolding. Moreover, the algorithm for writing down such an unfolding is well known.

A problem of particular interest is to determine what other singularities, or constellations of singularities, occur in the unfolding of a given one. It is familiar to workers in this area that the explicit formula mentioned above is not well adapted to the solution of this problem, except to decide the existence (or otherwise) of deformations $X \to Y$ where the singularities X and Y are similar. Alternative, more particular constructions of the miniversal unfolding have been devised to deal with this; in the case of simple singularities by Brieskorn [3, 4], for simple-elliptic singularities by Looijenga [15], and again by him for cusp singularities [16]. From these constructions, the deformations can be read off.

These constructions have all been made in the context of isolated singularities of complex surfaces. It is the object of this paper to investigate what these lead to in the real case. We retain the general background of the earlier work, and view a real singularity (or surface) as determined by a complex one together with an involution describing complex conjugation; i.e. by a Galois descent. This approach indeed applies over other fields (particularly in characteristic zero), and we make some remarks to this effect. However, though work has been done on (for example) del Pezzo surfaces over number fields, it does not seem to be of particular interest to explore deformation theory in this context, and we abstain from a detailed discussion.

The plan of this paper is as follows. There are three main parts: on simple singularities, on simple-elliptic singularities and on cusp singularities. Each of these is divided into two chapters: the complex case, then the real case. In each chapter we first describe the classification, then the geometry associated to the construction of the deformation space. I had originally intended to include also algorithms to list the singularities occurring in the deformations: however, for reasons of space these are deferred to a future publication and only general remarks are given here.

§1. DU VAL SINGULARITIES (COMPLEX CASE)

Although the case of Du Val singularities (alias rational double points, rational Gorenstein singularities, simple singularities) is well known, we recapitulate the facts briefly as we need to refer to them in the sequel, and in any case I know no reference for a discussion of some of the finer details in the real case.

Classification

The list is well known: we have $A_n (n \geq 1)$, $D_n (n \geq 4)$, $E_n (n = 6, 7, 8)$. Normal forms are also well known, as is the fact (which is not important here) that these are the quotients of $\mathbb{C}^2$ by freely acting finite subgroups of SU_2. The exceptional set in a minimal resolution consists of copies of $P^1\mathbb{C}$, each with normal degree (-2), and intersections given by the dual graph well known by the same name $\Delta(A_n, D_n,$ or $E_n)$. We refer to Bourbaki [2] for standard facts about these diagrams and the corresponding root systems.

The deformation space

The usual algorithm for hypersurface singularities permits us to write explicit equations for a miniversal deformation $\pi'\colon \mathscr{Y} \to U$ of the given singularity $Y_0 = \pi'^{-1}(0)$. An intrinsic model for this was described by Brieskorn [4] and in more detail by Slodowy [21]: here $\mathscr{Y}$ is a slice transverse to a subregular uni- (nil-) potent orbit of the complex Lie group G (of type Δ), either in G itself or in its Lie algebra $\mathfrak{g}$; π' is induced by the basic invariants for the adjoint action.

As follows from this, but was also [3, 22] shown earlier by direct construction, there is a ramified covering $q\colon \mathbb{C}^n \to U$ such that the pullback $\pi\colon \mathscr{X} \to \mathbb{C}^n$ of π' by q admits a simultaneous resolution $p\colon \tilde{\mathscr{X}} \to \mathscr{X}$ in the strong sense that (i) it is obtained by a finite sequence of blowings-up of smooth subvarieties and (ii) it induces a minimal resolution on each fibre of π. Indeed, this possibility characterises this class of singularities. The restriction of this covering to the complement of the discriminant $D \subset U$ can be identified with the monodromy covering corresponding to the action of $\pi_1(U - D, u)$, on the homology $H = H_2(Y_u)$ of the "Milnor fibre" Y_u.

Location of singularities in the deformation

This is where the advantage of the covering q becomes apparent, for we can identify $\mathbb{C}^n$ with the space on which the Weyl group W (of type Δ) acts as a reflexion group. For each $z \in \mathbb{C}^n$, the stabiliser W_z is again a reflexion group, and so can be expressed uniquely as a product ΠW_α of irreducible reflexion groups. Then if W_α has type Δ_α, the singularities of the surface X_z are Du Val singularities of types Δ_α, and correspond bijectively to the factors.

It is now easy to describe which constellations $\{\Delta_\alpha\}$ occur in this way on fibres. For z lies in some closed Weyl chamber C. The walls of C correspond bijectively to the vertices of the diagram Δ. If $\{v_i : i \in I\}$ are the vertices corresponding to the walls containing z, W_z is generated by the corresponding reflexions $\{\sigma_i : i \in I\}$; and its Dynkin diagram Δ_z consists of the vertices $\{v_i : i \in I\}$ of and those edges of (if any) whose ends both lie among these vertices.

Thus if, for example, Δ has type E_6, the edges of C have stabilisers (obtained by omitting one vertex) of type A_5, D_5, A_4A_1 or $A_2^2A_1$.

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We can add further precision here. The choice of a strong simultaneous resolution corresponds to choosing a particular Weyl chamber, or equivalently to the selection of a system R_+ of positive roots of W, where the root lattice is identified with $H = H_2(X_z)$. For $z \in \mathbb{C}^n$, the exceptional set E_z in the resolution $\tilde{X}_z$ of the fibre X_z consists of (-2)-curves, and the homology classes in H of these are the fundamental roots for W_z with respect to the system $R_+ \cap R(W_z)$ of positive roots (or equivalently, the chamber of W_z containing the chosen Weyl chamber of W).

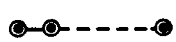
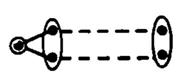
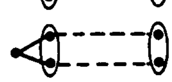
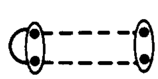
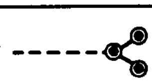
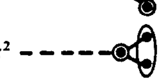
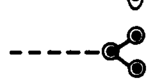
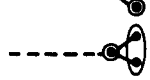
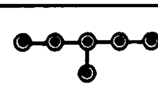
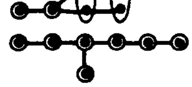
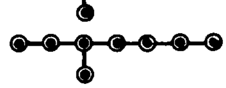
§2. DU VAL SINGULARITIES (REAL CASE)

Classification

The list may be found in several references (with slight variations, according as one is classifying functions of two or three or more variables, or the corresponding varieties): our version is taken from Slodowy [21] (see Table 1).

In the table, $G_{\mathbb{R}}$ is the real Lie group associated with the singularity by Slodowy; the Tits symbol is that for $G_{\mathbb{R}}$. It also describes the real form of the minimal resolution: ◉ denotes a curve with real points (hence isomorphic to $P^1(\mathbb{R})$); • a curve defined over $\mathbb{R}$ but with no real points, and  a pair of conjugate complex curves. This determines the real part $E_{\mathbb{R}}$ of the exceptional set up to isomorphism; $\chi(E_{\mathbb{R}})$ is its Euler characteristic. Finally as $X_{\mathbb{R}}$ is topologically a real surface with an isolated singular point P, the link of P is a closed 1-

Table 1. Real forms of du Val singularities

Notation	Normal form	Tits symbol	$G_{\mathbb{R}}$	Real roots	r	$\chi(E_{\mathbb{R}})$
A_1a	$x^2+y^2-z^2$		$SL_2\mathbb{R}=SU_{1,1}$	A_1	2	0
A_1c	$x^2+y^2+z^2$		SU_2	—	0	0
$A_{2n-1}a$	$x^{2n}+y^2-z^2$		$SL_{2n}\mathbb{R}$	A_{2n-1}	2	$2-2n$
$A_{2n-1}b$	$x^{2n}-y^2-z^2$		$SU_{n,n}$	C_n	2	0
$A_{2n-1}c$	$x^{2n}+y^2+z^2$		$SU_{n+1,n-1}$	BC_{n-1}	0	0
$A_{2n}a$	$x^{2n+1}+y^2-z^2$		$SL_{2n+1}\mathbb{R}$	A_{2n}	1	$1-2n$
$A_{2n}b$	$x^{2n+1}+y^2+z^2$		$SU_{n+1,n}$	BC_n	1	1
$D_{2n}a$	$x^{2n-1}-xy^2\pm z^2$		$SO_{2n,2n}\mathbb{R}$	D_{2n}	3	$1-2n$
$D_{2n}b$	$x^{2n-1}+xy^2\pm z^2$		$SO_{2n+1,2n-1}\mathbb{R}$	B_{2n-1}	1	$3-2n$
$D_{2n+1}a$	$x^{2n}\pm xy^2-z^2$		$SO_{2n+1,2n+1}\mathbb{R}$	D_{2n+1}	2	$-2n$
$D_{2n+1}b$	$x^{2n}\pm xy^2+z^2$		$SO_{2n+2,2n}\mathbb{R}$	B_{2n}	2	$2-2n$
E_6a	$x^4+y^3-z^2$		$E_6^{(6)}$	E_6	1	-5
E_6b	$x^4+y^3+z^2$		$E_6^{(2)}$	F_4	1	-1
E_7a	$x^3y+y^3+z^2$		$E_7^{(7)}$	E_7	2	-6
E_8a	$x^5+y^3+z^2$		$E_8^{(8)}$	E_8	1	-7

manifold, hence a union of some number r of simple closed curves. The number r is easily determined (cf. [5, 6]) and is also tabulated.

Complex conjugation acts trivially on the diagram Δ in the cases labelled a and A_1c; nontrivially in cases b and $A_{2n-1}c\,(n \geq 2)$.

The deformation space

The usual algorithm gives a miniversal deformation parametrised by a real form $U_{\mathbb{R}}$ of U. This pulls back to $q^{-1}(U_{\mathbb{R}}) \subset \mathbb{C}^n$. However, this is not a real subspace of $\mathbb{C}^n$. We can describe the situation as follows (after [14]).

W acts as a reflexion group on $\mathbb{R}^n$ and on its complexification $\mathbb{C}^n$. The action of complex conjugation on the diagram Δ defines an element of N/W, where N is the normaliser of W in $O_n(\mathbb{R})$. In fact this element can be represented by the identity (cases a) and by minus the identity in all other cases except $D_{2n}b$. Choose a representative u_0, and extend the action of u_0 to $\mathbb{C}^n$ by tensoring with complex conjugation. As this normalises W, the group $\tilde{W} = \langle W, u_0 \rangle$ acts on $\mathbb{C}^n$ and contains W as subgroup of index 2. Now we have the following.

LEMMA. *$q^{-1}(U_{\mathbb{R}})$ is the union of the fixed subspaces V_u of elements u of $\tilde{W} - W$ with $u^2 = 1$.*

As each such u is an antiholomorphic involution, each V_u is a real form of $\mathbb{C}^n$.

Proof (from [14]). If $q(z) = x \in U_{\mathbb{R}}$, then conjugation takes x into itself, hence preserving the W-orbit of z. Thus for any $c \in \tilde{W} - W$, $cz \in Wz$, and so we can choose c to leave z fixed.

Such a c normalises the stabiliser W_z. hence permutes its Weyl chambers. As W_z is transitive on these, we can find $w \in W_z$ such that $u = cw$ leaves a given chamber invariant. Then $u^2 \in W_z$ also fixes this chamber, hence is the identity.

Looijenga also shows that each such subspace V_u maps to the closure of one component of the complement of $U_{\mathbb{R}} - D_{\mathbb{R}}$ inducing a bijection between W-conjugacy classses of elements u and such components.

Note. Our u is the negative of Looijenga's, but this does not affect the description of the geometry.

We can add further precision. Suppose we have fixed a chamber C for W and the corresponding strong simultaneous resolution. This is preserved by conjugation only if u preserves C. However, for any particular fibre X_z, the minimal resolution $\tilde{X}_z$ is unique so complex conjugation extends to it, and its effect on $H_2(\tilde{X}_z)$ determines uniquely an element u_z of $\tilde{W}_z$, which will satisfy $u^2 = 1$. The element u_z is characterised as that element of $\tilde{W}_z - W_z$ which leaves invariant the chamber of W_z which contains C.

This element u_z also has the property (which does not, however, characterise it) of having maximal trace.

LEMMA. *The trace of u_z on the root lattice of W_z is maximal for involutions in $\tilde{W}_z - W_z$. It is a sum over factors of W_z of contributions as follows:*

Two factors interchanged bu u_z	0
A_na, D_na, E_na	n
$A_{2n-1}b$, $A_{2n-1}c$	0
$A_{2n}b$	-1
D_nb	$n-2$
E_6b	2.

Proof. We can write the root lattice as a sum of irreducible pieces. For two such parts interchanged by u_z (hence by each element of $\tilde{W}_z - W_z$), the trace is 0 for each element of this coset. For a factor of type a, u_z acts as the identity and clearly has maximal trace.

For cases $A_{n-1}b$ (or $A_{2n-1}c$), τ is the negative of some $\tau' \in S(n)$; its trace is maximal when that of τ' is minimal, i.e. τ' fixes the minimal possible number of elements. This is indeed achieved (among elements of order 2) by the reversal corresponding to u_z:

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ n & n-1 & n-2 & \dots & 1 \end{pmatrix}.$$

For case D_n, N is the wreath product of $S(n)$ with $\{\pm 1\}$; for the nontrivial coset for $D_n b$, we have an odd number of minus signs. An element of order 2 is a product of substitutions

$$e_i \to e_i \quad (\text{trace } 1) \qquad e_i \to -e_i \ (\text{trace } -1).$$
$$e_i \to \pm e_j \ (\text{trace } 0)$$

Maximal trace for involutions in the coset is thus $(n-2)$, and is achieved by reflexion in e_1, which interchanges the first two fundamental roots and hence agrees with u_z.

Finally for $E_6 b$ we again have $\tau = -\tau'$, where τ' is one of five conjugacy classes of involutions in $W(E_6)$ (see e.g. [27, §2]), with respective traces 6, 4, 2, 0 and -2. The maximum trace is thus $+2$, and is indeed achieved by u_z, which permutes the basic roots as indicated in Table 1.

It seems appropriate at this point to discuss real resolutions. We first note that the given strong simultaneous resolution induces one of V_u if and only if $u = u_0$ is the element of $\tilde{W} - W$ that preserves C. However, for any $z \in q^{-1}(U_{\mathbb{R}})$ the resolution of X_z admits a conjugation as above, and the real resolved surface satisfies

$$\chi(\tilde{X}_{z\mathbb{R}}) - \chi(X_{\mathbb{R}}) = \Sigma(\chi(E_p) - 1)$$

extended over singular points E_p of $X_{\mathbb{R}}$. The $\chi(E_p)$ are listed in Table 1; we have $\chi - 1 = -\text{trace } \tau$ in each case (as follows from the Lefschetz fixed point theorem).

A more natural way to desingularise $X_{\mathbb{R}}$ from the topological viewpoint is to take the "real normalisation" $X_{\mathbb{R}}^n$: remove small open conical neighbourhoods of the singular points, and close off each boundary circle with a disc (intrinsically, take the Freudenthal compactification of $X_{\mathbb{R}} - \mathrm{Sing}(X_{\mathbb{R}})$). Here we have

$$\chi(X_{\mathbb{R}}^n) - \chi(X_{\mathbb{R}}) = \Sigma(r_P - 1)$$

where r_P is the number of components of the link of P: these too are listed in Table 1.

Location of singularities in the deformation

For $z \in \mathbb{C}^n$, we consider the stabiliser $\tilde{W}_z$. The intersection W_z with W determines—as above—the singularities in the complex fibre X_z. Moreover, as $q(z) \in U_{\mathbb{R}}$, $\tilde{W}_z$ contains (by the above) an involution $u \in \tilde{W} - W$, which thus normalises W_z. Write—as before—$W_z = \Pi W_\alpha$. Then u normalises this product, so will interchange some pairs of factors and, for the others, determine an element of $N(W_\alpha)/W_\alpha$. The singular points $P_\alpha \in X_z$ corresponding to W_α are permuted accordingly by complex conjugation: we have some complex conjugate pairs. For the rest, the class of u in $N(W_\alpha)/W_\alpha$ determines the real form except in cases A_{2m-1} where it does not discriminate between types b and c (or, if $m = 1$, between a and c). This final point is subtler, and we will not go into it in this paper.

We observe that if $u \in \tilde{W} - W$ normalises W_z and satisfies $u^2 = 1$, then there is indeed a point $z' \in \mathbb{C}^n$ with stabiliser $\tilde{W}_{z'} = W_z \cup u W_z$. For if L is the fixed set of W_z, L is a complex linear subspace of $\mathbb{C}^n$, and all $z' \in L$, except those lying on a union M of certain proper linear subspaces, have the same stabiliser. Now u leaves L invariant, and defines a real form $L_{\mathbb{R}}$ of it: thus any $z' \in L_{\mathbb{R}}$, $z' \notin M$ will do. However, in the ambiguous cases above, the connected components of $L_{\mathbb{R}} - M$ may correspond to different types $A_{2m-1}b$ and $A_{2m-1}c$.

§3. SIMPLE-ELLIPTIC SINGULARITIES (COMPLEX CASE)

Classification

Following Saito [20], a normal surface singularity is called simple-elliptic if the exceptional set of its minimal resolution consists of an elliptic curve E. To classify such singularities, it suffices to know the isomorphism class (or j-invariant) of E, and the degree $-D$ of its normal bundle. The cases $D = 1, 2, 3$ yield hypersurface singularities (Arnold's [1] parabolic singularities); $D = 4$ a complete intersection (of two quadric cones). Although our primary interest is in these, the arguments below cover the cases $D \leq 6$ and, with minor changes, cases $D \leq 9$.

The deformation space

For $D = 3$, the standard equation is a cubic form in three variables. A versal deformation is obtained by varying the modulus and adding lower order terms, thus giving a cubic surface. It is convenient to complete this to the corresponding projective surface X (the curve E then reappears as the intersection with the plane at infinity), and consider deformations likewise as cubic surfaces in $P^3(\mathbb{C})$.

There is a similar construction for other values of $D \leq 9$, but it is convenient to reformulate this somewhat. First we construct the surfaces X.

Take the projective plane P^2 and blow up, in succession, $n = 9 - D$ points, to give a surface $\tilde{X}_n$. We will suppose that the points are in "almost general position" (see [7] for details). Write κ for the canonical class of $\tilde{X}_n$. Then the projective image of $\tilde{X}_n$ defined by $(-r\kappa)$ is birational if $r + 5 \geq n$, and gives a surface X_n. The basic properties of this construction are given in Manin [17] and Demazure [7]. The case $r = 1$ ($n \leq 6$), with X smooth, gives the classical del Pezzo surfaces of degree D in P^D. An updated version of this characterisation was given in Hidaka and Watanabe [11]: if X is a normal Gorenstein surface with ample anticanonical divisor, then either X is a cone over an elliptic curve, or X arises as above, or $D = 8$ and X is either $P^1 \times P^1$ or the Hirzebruch surface F_2 with the (-2)-curve collapsed to a point.

The Picard group P of $\tilde{X}$ is a free abelian group with basis ε_0, the preimage of a line in P^2, and the classes ε_i $(1 \leq i \leq n)$ of the exceptional curves. We have

$$\varepsilon_0^2 = 1, \qquad \varepsilon_i^2 = -1 (1 \leq i \leq n), \qquad \varepsilon_i \varepsilon_j = 0 \ (i \neq j).$$

The canonical class is

$$\kappa = -3\varepsilon_0 + \sum^n \varepsilon_i$$

so $\kappa^2 = 9 - n = D > 0$. The orthogonal complement Q of κ is negative definite: write R for the set of roots

$$R = \{\xi \in P: \kappa \cdot \xi = 0, \xi^2 = -2\}.$$

Then R spans Q provided $n \geq 3$. The cases $\tilde{X} = P^1 \times P^1$ or F_2 are exceptional: here P has basis η_1, η_2 with $\eta^2 = 0, \eta_1 \cdot \eta_2 = 1$ and $\kappa = 2(\eta_1 + \eta_2)$ so $R = \{\pm(\eta_1 - \eta_2)\}$ spans Q in this case. In general, R is a root system: the reflexions s_r corresponding to $r \in R$ generate a reflexion group W. If $n \geq 3$, a fundamental system of roots is given by

$$\rho_1 = \varepsilon_0 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 \qquad \rho_r = \varepsilon_{r-1} - \varepsilon_r \ (2 \leq r \leq n)$$

so the type of the root system is given by

n	3	4	5	6	7	8
Δ	$A_1 \times A_2$	A_4	D_5	E_6	E_7	E_8

We are interested in the pair (X, E), where E is a smooth hyperplane section (more precisely, the support of an anticanonical divisor). Then E is an elliptic curve. We consider the restriction

$$\tilde{\varphi}: \operatorname{Pic} \tilde{X} \to \operatorname{Pic} E.$$

Since E represents $-\kappa$, for any $\xi \in P$, $\tilde{\varphi}(\xi)$ has degree $-\xi \cdot \kappa$. In particular, $\tilde{\varphi}(\varepsilon_i)$ has degree 1, so is a point P_i of E. Clearly P_i is the intersection of E with the exceptional curve of the ith blowing up.

We can reconstruct $\tilde{X}$ as follows. Embed E in P^2 by $\tilde{\varphi}(\varepsilon_0)$: now blow up P^2 successively at the points P_i. Observe that this gives a simultaneous resolution of the family of surfaces X. For if U denotes the family of homomorphisms $\tilde{\varphi}$, we blow up $P^2(\mathbb{C}) \times U$ successively along the submanifolds which are the sections (over U) defined by the points $\tilde{\varphi}(\varepsilon_i)$.

We can even construct X projectively: embed P^2 in P^9 as the Veronese surface and now project successively from the image of P_r to P^{9-r}. Thus $\tilde{\varphi}$ determines $\tilde{X}$ and X. Indeed, it is sufficient to have the restriction to Q, $\varphi: Q \to \operatorname{Pic}_0 E = \operatorname{Jac} E$. For if $\tilde{\varphi}'$, $\tilde{\varphi}$ both induce φ, they differ only by a translation of E.

We now ask to what extent (X, E) determines φ. Here a little more care is necessary. Observe that $R_\varphi = \{r \in R: \varphi(r) = 0\}$ is also a root system: write W_φ for the group generated by the corresponding reflexions.

PROPOSITION. *Let $\tilde{X}$, $\tilde{X}'$ be surfaces as above, with smooth curves E, E' as above and a commutative diagram*

$$\begin{array}{ccc} \operatorname{Pic} \tilde{X} & \xrightarrow{\tilde{\varphi}} & \operatorname{Pic} E \\ \downarrow \alpha & & \downarrow \operatorname{Pic} \beta \\ \operatorname{Pic} \tilde{X}' & \xrightarrow{\tilde{\varphi}'} & \operatorname{Pic} E' \end{array}$$

with α an isomorphism, $\beta: E \to E'$ an isomorphism. Then there is a uniquely determined $w \in W_\varphi$ and a unique isomorphism $\sigma: \tilde{X} \to \tilde{X}'$ taking E to E' by β and inducing $\alpha \circ w$.

This result is proved in [19, Theorem 3.7]. Observe that if $w \in W_\varphi$, then $\tilde{\varphi} \circ w = \tilde{\varphi}$ (the converse is not always true).

Let us identify $\operatorname{Pic} \tilde{X}$ with the standard lattice $\langle \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n \rangle$. Since $\deg \tilde{\varphi}\ (\xi) = -\xi \cdot \kappa$, the essential information contained in $\tilde{\varphi}$ is given by its restriction to Q, $\varphi: Q \to \operatorname{Jac} E$. We consider $\operatorname{Hom}(Q, \operatorname{Jac} E)$ as our model for the deformation space of the simple-elliptic singularity.

In the cases $D \leq 3$, the dimension of this space is 2 less than the Milnor number of the corresponding hypersurface singularity. One of these two dimensions corresponds to deformations of the curve E. The other corresponds to the fact that, the singularity being weighted homogeneous so that U admits a $\mathbb{C}^*$-action, we have taken the quotient.

As in the case of Du Val singularities, our space is still a branched covering of this. Its universal cover $\operatorname{Hom}(Q, \mathbb{C})$ (where $\mathbb{C}$ is the universal cover of $\operatorname{Jac} E$) corresponds to the monodromy covering; thus the branching behaviour is essentially the same in the two cases.

Location of singularities in the deformation

A homomorphism φ: $Q \to \operatorname{Jac} E$ determines $\tilde{\varphi}$ as the pushout in the diagram

$$\begin{array}{ccccccc} 0 \to & Q & \to & P & \to & \mathbb{Z} & \to 0 \\ & \downarrow \varphi & & \downarrow \tilde{\varphi} & & \| & \\ 0 \to & \operatorname{Jac} E & \to & \operatorname{Pic} E & \to & \mathbb{Z} & \to 0; \end{array}$$

and this in turn determines a (del Pezzo) surface X_φ. We have already defined $R_\varphi = R \cap \operatorname{Ker} \varphi$ and the corresponding group W_φ. Now, just as in §1, W_φ determines the list of singularities which appear on X_φ. A formal proof of this is given in [19].

One can also determine which reflexion groups can occur as W_φ. For W_φ is the reflexion subgroup of the stabiliser in W of $\varphi \in U = \operatorname{Hom}(Q, \operatorname{Jac} E)$. The action of W on this lifts to an action on the universal cover of U by the semidirect product $\tilde{W}$ of W with the fundamental group $Q^\# \times Q^\#$ of U, where $Q^\#$ denotes the dual lattice $\operatorname{Hom}(Q, \mathbb{Z})$; and stabilisers in $\tilde{W}$ are isomorphic to those in W.

Consider first a single factor of $\operatorname{Jac} E \cong S^1$. The corresponding group $W.Q^\#$ acts on $\operatorname{Hom}(Q, \mathbb{R})$ as the affine reflexion group formed by extending W. The stabilisers are found as in §1, but now using a chamber which is a euclidean simplex (or product of such): in particular, no point can belong to all the walls. Thus we must delete at least one vertex from (each component of) the corresponding extended Dynkin diagram.

For the full result, since there were two factors S^1, we repeat the operation (of taking the extended diagram, then a proper subset) once more. For further details of this, see [19].

§4. SIMPLE-ELLIPTIC SINGULARITIES (REAL CASE)

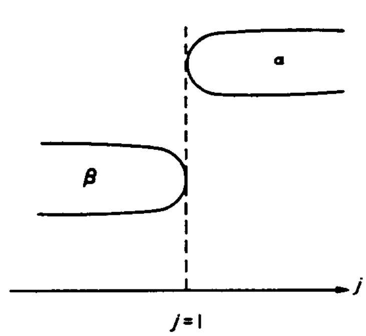
Classification

The abelian variety $J = \operatorname{Jac} E$ of dimension 1 has a real form if and only if the j-invariant $j(J)$ is real. There are two types.

(α) If $j(J) \geq 1$, J is the quotient of $\mathbb{C}$ by a rectangular lattice L, which we can take to be generated by $1, i\tau\, (\tau > 0)$. Real forms are defined by the conjugations $z \to \pm \bar{z}$: for $\tau \neq 1$ (i.e. $j \neq 1$) these are inequivalent. The group $J_\mathbb{R}$ of real points has (in each case) two components.

(β) If $j(J) \leq 1$, J is the quotient of $\mathbb{C}$ by a rhombic lattice, which we may suppose generated by $1 \pm i\tau$. Again, there are just two real forms $z \to \pm \bar{z}$, inequivalent for $\tau \neq 1$ (i.e. $j \neq 1$). The group $J_\mathbb{R}$ has just one component.

The set of all real forms is a smooth one-dimensional family with these two components. We can represent it graphically:



Any real form of E determines one of the jacobian, Jac E. Conversely it is easy to show that if $J_{\mathbb{R}}$ has type β, then E has a real point (so is isomorphic to J over $\mathbb{R}$); if $J_{\mathbb{R}}$ has type α, there are two corresponding real forms for $E_{\mathbb{R}}$: one isomorphic to $J_{\mathbb{R}}$, the other with no real points (for the above example, we can take conjugation as $z \to \bar{\zeta} + \frac{1}{2}(\text{mod } L)$.

The real forms of the simple-elliptic singularities were determined in [18]. Since the singularity is determined by a curve E and a bundle over it, a real form is determined by a curve $E_{\mathbb{R}}$ and a real class of real divisors D. The theory of real divisors is expounded in [10]. We obtain the following list:

D odd Then $E_{\mathbb{R}}$ must have a real point. There are just two strata: β_1 and α_3.

D even If $E_{\mathbb{R}}$ has 0 or 1 component, it determines a unique singularity. The case where $E_{\mathbb{R}}$ has 2 components splits into two, according to whether D has even degree on each component or odd degree on each. There are thus four strata:

$$\beta_2, \alpha_0, \alpha_2 \text{ and } \alpha_4.$$

In each case $E_{\mathbb{R}}$ (or $J_{\mathbb{R}}$) determines the singularity up to (real) isomorphism.

The suffix in this notation is the number (r) of components of the real link. To determine this, we observe that a component of $E_{\mathbb{R}}$ is one-sided or two-sided in the resolution according to whether D has odd or even degree on it.

As this classification is not so well known, I now give the relation to equations in normal form.

$$D = 1 \quad (\tilde{E}_8) \quad z^2 - 4y^3 + g_2x^4y + g_3x^6,$$
$$D = 3 \quad (\tilde{E}_6) \quad z^2x - 4y^3 + g_2x^2y + g_3x^3.$$

I have left in both parameters g_2 and g_3 since either may vanish, and not only the ratio $g_2^3 : g_3^2$ but also the sign of g_3 is needed for the classification.

$$\beta_1 : g_2^3 < 27g_3^2. \qquad \alpha_3 : g_2^3 > 27g_3^2.$$

$D = 2$ ($\tilde{E}_7$). Write

$$f_\lambda = (x^4 + (4\lambda - 2)x^2y^2 + y^4) \quad (\lambda \in R; \lambda \neq 0, 1);$$
$$g_\mu = x^4 + 2\mu x^2y^2 - y^4 \qquad (\mu \in \mathbb{R});$$

then $j \geq 1$ for f_λ, $j \leq 1$ for g_μ. We have strict normal forms

$$\begin{array}{llll} \beta_2 & g_\mu + z^2 & (\sim g_{-\mu} - z^2) & \mu \in \mathbb{R} \\ \alpha_0 & f_\lambda + z^2 & (\sim f_{\lambda^{-1}} + z^2) & 0 < \lambda < 1 \\ \alpha_2 & f_\lambda - z^2 & (\sim f_{\lambda^{-1}} - z^2) & 0 < \lambda < 1 \\ \alpha_4 & f_\lambda + z^2 & (\sim f_{\lambda^{-1}} - z^2) & \lambda < 0. \end{array}$$

$D = 4$ ($\tilde{D}_5$). We have a general pencil $\lambda_1 f_1 + \lambda_2 f_2$ of homogeneous quadratics in 4 variables. Eigenvalues are the 4 points $(\lambda_1 : \lambda_2) \in P^1\mathbb{C}$ corresponding to quadratics of rank less than 4.

β_2: 2 eigenvalues real, 2 complex conjugate.
α_2: 2 pairs of complex conjugate eigenvalues.
α_0: All eigenvalues real; some form $\lambda_1 f_1 + \lambda_2 f_2$ positive definite.
α_4: All eigenvalues real, otherwise.

Normal forms can again be given, but none is particularly convenient.

The deformation space

Our identification of the deformation space in the complex case derived from the proposition in §3. The proposition remains true if $\beta: E \to E'$ is a semialgebraic isomorphism: the resulting isomorphism σ is then also semialgebraic (interpreted here in the sense of complex conjugation). Indeed, the proof requires no essential change.

Now given a real del Pezzo surface $X_{\mathbb{R}}$ and real hyperplane section $E_{\mathbb{R}}$ we can complexify, and define $\tilde{\varphi}$ as before; however, now complex conjugation acts on the Picard groups and $\tilde{\varphi}$ is an equivariant homomorphism. Conversely, given a $\mathbb{Z}_2$-equivariant homomorphism

$$\tilde{\varphi}: \operatorname{Pic} X \to \operatorname{Pic} E,$$

where the $\mathbb{Z}_2$-action on $\operatorname{Pic} X$ leaves invariant κ and the intersection numbers, and the action on E is induced by choice of a real form of E, the proposition guarantees us a corresponding surface X over $\mathbb{C}$, and a semialgebraic involution, thus defining a real structure on X. However, the induced automorphism τ on Pic X only agrees with the given one up to an element of W_φ. The geometric reason for this is clear: τ has to permute the irreducible components of the exceptional set for the resolution $\tilde{X} \to X$, and as in §2, the fact that we have a simultaneous resolution picks out a particular class. It follows from the earlier discussion that the preferred τ is that which leaves invariant the system of positive roots of W_φ consisting of the elements of R_φ which are non-negative linear combinations of $\rho_1, \ldots, \rho_n$.

We consider the space $\operatorname{Hom}_\tau(Q, \operatorname{Jac} E)$ of equivariant homomorphisms as our moduli space. An element φ of this group determines $\tilde{\varphi}$ by using the pushout diagram of Galois modules

$$\begin{array}{ccccccccc} 0 \to & Q & \to & P & \to & \mathbb{Z} & \to 0 \\ & \downarrow\varphi & & \downarrow\tilde{\varphi} & & \| & \\ 0 \to & \operatorname{Jac} E & \to & \operatorname{Pic} E & \to & \mathbb{Z} & \to 0. \end{array}$$

There are many components of our moduli space. We have already commented on the possible actions of τ on $\operatorname{Jac} E$. The action on Q preserves the roots, and extends uniquely to an action on P preserving κ. It follows from [7] that the action of τ on Q is induced by some element $w \in W$ of order 2 in the Weyl group. The classification of these up to conjugacy was discussed in my paper [27], where we also described how the choice of w and the integer $n = 9 - D$ determines the real surface $\tilde{X}_{\mathbb{R}}$ up to homeomorphism in the cases when $X \cong X$ is smooth.

Once the actions of τ on Q and on $J = \operatorname{Jac} E$ are fixed, there are two main subcases, depending whether $J_{\mathbb{R}}$ has 1 or 2 components. In the former case, J is topologically equivariantly isomorphic to $(S^1 \times S^1, s)$ where s is the swap $s(x, y) = (y, x)$. Thus if $\varphi = (\varphi_1, \varphi_2)$, $\varphi_2 = \varphi_2 \circ \tau$, and $\operatorname{Hom}_\tau(Q, J) \cong \operatorname{Hom}(Q, S^1)$ is connected. In the latter, we can write J as a product $S^1 \times S^1 = J_+ \times J_-$ of two circles, where τ acts trivially on the first component and by inversion on the second. We define $c: S^1 \times S^1 \to S^1 \times S^1$ by $c(x, y) = (x, \bar{y})$. Correspondingly $\varphi = \varphi_+ \times \varphi_-$. The homomorphism φ_+ can take arbitrary values on elements of Q invariant by τ, but those with $\tau(\xi) = -\xi$ are sent to $\{\pm 1\}$. To state this invariantly, we use Galois cohomology: as we only consider the real field here, this refers to cohomology of the group $\mathbb{Z}_2$ generated by τ. For any module M, the Tate cohomology groups $\hat{H}^i(\mathbb{Z}_2; M)$ are periodic with period 2. We will simply write $H^0(M)$, $H^1(M)$.

Then φ_+ induces a homomorphism

$$\varepsilon^1: H^1(Q) \to H^1(J_+) = H^1(J) \cong \{\pm 1\}$$

and connected components of the family of possible φ_+ correspond bijectively to homomorphisms ε^1. A similar discussion holds for φ_-, with

$$\varepsilon^0\colon H^0(Q) \to \{\pm 1\}.$$

We now describe how φ discriminates between real forms of the singularity. The type of involution on J (1 or 2 fixed components) already determines whether we have an α-stratum or a β-stratum. The only case when we need look further is when D is even and $J_{\mathbb{R}}$ has 2 components, so that ε^0, ε^1 are defined. Now the canonical class κ is an invariant element of P of degree $D = 2m$. For any $\lambda \in P$ of degree m, $\kappa - \lambda - \tau(\lambda)$ is an invariant element of Q, and its class $\kappa_0 \in H^0(Q)$ clearly does not depend on λ. Similarly, for $\mu \in P$ of degree 1, $\mu - \tau(\mu)$ is an anti-invariant element of Q whose class $\kappa_1 \in H^1(Q)$ does not depend on the choice of μ.

LEMMA. *In this situation, we have the α_0 stratum if $\varepsilon^1(\kappa_1) = -1$, while if $\varepsilon^1(\kappa_1) = +1$, we have α_2 or α_4 depending whether $\varepsilon^0(\kappa_0) = -1$ or $+1$.*

Proof. In the pushout diagram

$$\begin{array}{ccccccc} 0 \to & Q & \to & P & \to & \mathbb{Z} & \to 0 \\ & \downarrow\varphi & & \downarrow\varphi & & \| & \\ 0 \to & J & \to & \operatorname{Pic} E & \to & \mathbb{Z} & \to 0 \end{array}$$

both sequences split additively; the obstructions to equivariant splitting are classes in $H^1(Q)$, $H^1(J)$ and φ maps the former (represented by $\mu - \tau\mu$, hence equal to κ_1) to the latter. But the lower sequence splits if and only if $\operatorname{Pic}_1 E$ is nonempty.

If it does split, we have the α_2 or α_4 stratum depending whether κ is not or is in that invariant component of $\operatorname{Pic}_D E$ consisting of doubles of invariant elements of $\operatorname{Pic}_m E$. But this is determined by the value of $\varepsilon^0(\kappa_0)$.

§5. CUSP SINGULARITIES (COMPLEX CASE)

Classification

Following Hirzebruch [12], a singularity is called a cusp singularity if its minimal resolution consists of a cycle of rational curves. It is determined up to isomorphism by the normal degrees $-b_i$ of these curves. These satisfy $b_i \geq 2$ and $\sum_{i=1}^{D^*} (b_i - 2) = D > 0$. The case when the number (D^*) of curves is ≤ 2 is different in some respects, but we adopt the convention that the infinite cyclic cover consists of smooth rational curves E_i' $(i \in \mathbb{Z})$, with E_i', E_j' disjoint if $|i-j| \geq 2$, and meeting transversely in a single point (only) if $|i-j| = 1$: then the normal degree of E_i' is b_i, where this is defined to be periodic in i with period D^*.

There is a natural duality in the class of cusp singularities. If the cycle $\mathbf{b} = (b_i)$ is arranged as

$$\underbrace{2, \ldots, 2}_{k_1^* - 1}, k_1 + 2, \underbrace{2, \ldots, 2}_{k_2^* - 1}, k_2 + 2, \ldots, k_g + 2;$$

with each $k_i, k_i^* \geq 1$, then the dual cycle is

$$k_1^* + \underbrace{2, \ldots, 2}_{k_1 - 1}, k_2^* + 2, \underbrace{2, \ldots, 2}_{k_2 - 1}, \ldots, \underbrace{2, \ldots, 2}_{k_g - 1}.$$

We have $D = \Sigma k_i$, $D^* = \Sigma k_1^*$, so these are exchanged by the duality.

The singularity is a hypersurface singularity if and only if $D \le 3$, a complete intersection if $D = 4$, a Pfaffian singularity in $\mathbb{C}^5$ if $D = 5$. Equations can readily be given in terms of the b_i^* for a normal form (see e.g. [9, 3.2]). In these cases the Milnor number is given by

$$\mu = 11 + D^* - D.$$

We will often associate to the cusp singularity the dual graph Δ of the resolution: a polygon with D^* vertices labelled by the integers b_i. Sometimes we use also the graph Δ^* of the dual.

The deformation space

The construction is due to Looïjenga [16]: we attempt only a condensed statement of his main results. The two dual cusp singularities can be "glued" to give an Inoue–Hirzebruch surface X (which is closed), with the two singular points x_0, x_1.

The deformation space of X maps isomorphically to the product of those for these two singularities. We will however keep x_1 fixed and deform x_0: moreover, we consider only the (dense, Zariski-open) subset of the deformation space where x_0 is deformed into (at worst) a set of Du Val singularities. The corresponding surfaces X_t are rational, the cycle of curves resolving x_1 is an anticanonical divisor $E = \Sigma[E_i]$.

We can parametrise the family of such surfaces in the same manner as in §3. Indeed, following Looïjenga [16], we define here also a strong simultaneous resolution and a root system. First, by [16, Theorem 1.1], the surface $\tilde{X}_t$ can be constructed as follows.

Start with the smooth rational surface $\bar{Y}$ and the anticanonical cycle $\bar{E} = \Sigma \bar{E}_i$ defined by:

if $D = 1$, $\bar{Y} = P^2(\mathbb{C})$ and $\bar{E}$ is a nodal cubic curve;

if $D = 2$, $\bar{Y} = P^1(\mathbb{C}) \times P^1(\mathbb{C})$ is a quadric surface and $\bar{E}_1$, $\bar{E}_2$ are conics (smooth plane sections) meeting transversely;

if $D = 3$, $\bar{Y} = P^2(\mathbb{C})$ and $\bar{E}_i (1 \le i \le 3)$ are lines forming a triangle;

if $D = 4$, $\bar{Y} = P^1(\mathbb{C}) \times P^1(\mathbb{C})$ and $\bar{E}_1 \cup \bar{E}_3 = P^1(\mathbb{C}) \times \{0, \infty\}$, $\bar{E}_2 \cup \bar{E}_4 = \{0, \infty\} \times P^1(\mathbb{C})$;

if $D = 5$, $\bar{Y}$ is the nonsingular del Pezzo surface of degree 5 and the $\bar{E}_i$ are lines (exceptional curves) forming a pentagon—for example, those corresponding to the classes $\varepsilon_1, \varepsilon_0 - \varepsilon_1 - \varepsilon_2, \varepsilon_2, \varepsilon_0 - \varepsilon_2 - \varepsilon_3, \varepsilon_0 - \varepsilon_1 - \varepsilon_4$ in Pic $\bar{Y}$. Then $\bar{E}_i$ has selfintersection number $4 - D$ (except if $D = 1$, when we have 7). Now blow up a sequence of $b_i^* + 4 - D$ (or $b^* + 7$, if $D = 1$) points P_i^j on $\bar{E}_i$ (but not the intersection points with other $\bar{E}_j$).

This gives the construction and (using the ordering of the P_i^j, which need not be distinct) the strong simultaneous resolution. The rank of the Picard group $P = \text{Pic } Y$ of a resulting surface Y is obtained from that of $\bar{Y}$ (1, 2, 1, 2, 5 in the five cases) by adding

$$\sum_{i \bmod D} (b_i^* + 4 - D) = \Sigma(b_i^* - 2) + D(6 - D)$$
$$= D^* + D(6 - D)$$

(or $b^* + 7 = D^* + 9$ when $D = 1$); thus we have rank $P = D^* + 10$ in all cases, agreeing (as it must) with $10 - \kappa^2 = 10 - E^2$.

Define L to be the sublattice of P generated by the E_i (hence of rank D), and Q its orthogonal complement. A root system is defined in Q by listing a system of fundamental roots. If ε_i^j denotes the proper transform of P_i^j, we start by choosing the $\varepsilon_i^j - \varepsilon_i^{j+1}$: there are $D^* + D(5 - D)$ (or, if $D = 1$, $D^* + 8$) of these, while Q has rank $D^* + 10 - D$, thus we need 1, 2, 1, 2, or 5 further roots for $D = 1, 2, 3, 4$ or 5. If $D = 1$ resp. 3, and η is the class obtained from a general line in P^2, we take $\eta - \varepsilon^1 - \varepsilon^2 - \varepsilon^3$ resp. $\eta - \varepsilon_1^1 - \varepsilon_2^1 - \varepsilon_3^1$ as our final root. If

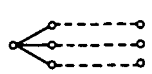
$D = 2$ resp. 4 and η_1, η_2 are the classes obtained from the two rulings of $P^1(\mathbb{C}) \times P^1(\mathbb{C})$ we take as further roots

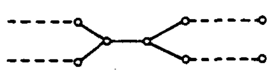
$$\eta_1 - \varepsilon_1^1 - \varepsilon_2^1, \eta_2 - \eta_1 \text{ resp. } \eta_1 - \varepsilon_1^1 - \varepsilon_3^1, \eta_2 - \varepsilon_2^1 - \varepsilon_4^1.$$

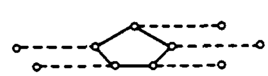
If $D = 5$, and η_i is the class of the total transform of E_i we take as further roots $\eta_i - \varepsilon_i^1$ (i mod 5). We then find that the root system has type described by graphs as follows

$$\begin{array}{ll} D=1 & T_{2,3,b_1^*+4} \\ D=2 & T_{2,b_1^*+2,b_2^*+2} \\ D=3 & T_{b_1^*+1,b_2^*+1,b_3^*+1} \\ D=4 & \Pi_{b_1^*,b_2^*,b_3^*,b_4^*} \\ D=5 & \Omega_{b_1^*-1,b_2^*-1,b_3^*-1,b_4^*-1,b_5^*-1} \end{array}$$

where (as in [9]) the symbols T, Π, Ω denote graphs of the respective shapes

T 

Π 

Ω 

and the suffixes are the lengths of the arms, increased by 1.

As in §3, we parametrise our family using the restriction homomorphism

$$\tilde{\varphi}: P = \operatorname{Pic} \tilde{X}_t \to \operatorname{Pic} E.$$

A divisor on E is equivalent to one whose support avoids the singular set of E. It then has a degree on each component E_i; thus we have a natural surjection of Pic E onto the dual lattice $L^\#$ of L. Its kernel (which can be calculated using the coefficient exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_E \to \mathcal{O}_E^* \to 0$ on E) is isomorphic to the multiplicative group $\mathbb{C}^\times$. Explicitly, let z_i be a coordinate on E_i taking the values 0 on $E_i \cap E_{i-1}$, ∞ on $E_i \cap E_{i+1}$ (more precisely, we should do this on components E_i' of the universal cover): then given a divisor D we multiply the coordinates of its points with the indicated multiplicities to obtain an element of $\mathbb{C}^\times$.

The homomorphism $\tilde{\varphi}$ fits into a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & Q \arrow[r] \arrow[d, "\varphi"] & P \arrow[r] \arrow[d, "\tilde{\varphi}"] & L^\# \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{C}^\times \arrow[r] & \operatorname{Pic} E \arrow[r] & L^\# \arrow[r] & 0 \end{tikzcd}$$

so is essentially determined by its restriction φ. Given φ we define R_φ to consist of the roots in $\operatorname{Ker} \varphi$ and W_φ to be the group generated by the corresponding reflexions.

The positive cone C^+ is that component of $\{x \in P_{\mathbb{R}}: x.x > 0\}$ which contains ample divisor classes. The choice of this component is equivalent to orienting a positive definite summand of P, or equivalently (since L is negative definite) of Q.

We can now state the result describing how φ determines the surface $\tilde{X}_t$ (denoted X_φ from now on).

THEOREM. *Let X, X' be rational surfaces with anticanonical cycles E, E' and other notations as above. If α: $\operatorname{Pic} X \to \operatorname{Pic} X'$ is an isomorphism such that*

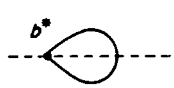
(i) *α is an isometry*
(ii) $\alpha[E_i] = [E_i']$
(iii) $\alpha R = R'$
(iv) $\alpha C^+ = C'^+$
(v) $\tilde{\varphi}' \circ \alpha = \tilde{\varphi}$

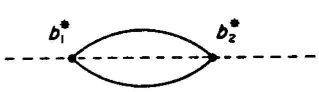
then there exist a uniquely determined $w \in W_\varphi$ and a unique isomorphism σ: $X \to X'$ with $\sigma E_i = E_i'$, inducing $\alpha \circ w$, and preserving the orientation of the cycle.

Proof [16, I.5.3].

We seek to reformulate this result in terms of φ. Note first that the orientation of the cycle can be identified with one of the graph Δ. For $D \geq 3$ this is already determined by the cyclic order of the curves E_i. Our description of the identification of $\operatorname{Pic} E$ shows that a change of this orientation will replace the isomorphism ζ: Ker $(\operatorname{Pic} E + L^\#) \to \mathbb{C}^\times$ by its inverse $\zeta'(D) = (\zeta(D))^{-1}$.

Now suppose X, X' as above, and that we know the cycles E, E' determine isomorphic graphs Δ. Let $\bar{\alpha}$: $Q \to Q'$ be an isomorphism satisfying (iii) $\bar{\alpha} R = R'$, (iv) $\bar{\alpha}(C^+) = C'^+$ and (v) $\varphi' \circ \bar{\alpha} = \varphi$. We seek to extend $\bar{\alpha}$ to α satisfying (i) and (ii). Since $\bar{\alpha}$ satisfies (iii), it is an isometry. As the two situations are abstractly isomorphic, we can take $Q = Q'$. Consider the induced automorphism of the discriminant group, $\hat{\alpha}$: $Q^\#/Q \to Q^\#/Q$; this is an isometry for the induced forms [23]. Now any element of the Weyl group W of R induces the identity on $Q^\#/Q$. The group N of automorphisms of Q preserving R and C^+ normalises W, and N/W is isomorphic to the group G of automorphisms of the diagram (T, Π or Ω) describing a fundamental chamber of the root system. For $D = 3, 4, 5$ G is isomorphic to the group Aut Δ of automorphisms of the diagram Δ. For $D = 1, 2$, Aut Δ is the direct product of G and the group generated by the obvious reflexion:





We have (cf. [24]) a natural isomorphism of $Q^\#/Q$ on $L^\#/L$, since Q and L are mutual orthogonal complements in the unimodular lattice P. Given automorphisms of Q and of L combine to give an automorphism of P if and only if the induced automorphisms of the discriminant group coincide. But Aut Δ acts on L (by permuting the E_i) and

LEMMA. *If $\rho \in \operatorname{Aut} \Delta$ acts trivially on $L^\#/L$, then either $\rho = 1$ or $D \leq 2$ and ρ is the obvious reflexion (which acts trivially on L).*

Proof [26].

Thus in all cases G acts faithfully on $L^\#/L$, or equivalently on $Q^\#/Q$.

From the above discussion we deduce the following.

PROPOSITION. *Let $\bar{\alpha}$: $Q \to Q'$ be an isomorphism with $\bar{\alpha}(R) = R'$, $\bar{\alpha}(C^+) = C'^+$. Then there is a unique isometry α: $P \to P'$ which extends $\bar{\alpha}$ and satisfies $\alpha[E_i] = [E'_{k+\varepsilon i}]$ for some fixed k and $\varepsilon = \pm 1$.*

We can thus reformulate the theorem.

THEOREM. *Let X, X' be as above; α: $Q \to Q'$ satisfy*

$$\alpha(R) = R', \quad \alpha(C^+) = C'^+, \quad \varphi' \circ \bar{\alpha} = \varphi.$$

Then if $D \leq 2$, for a unique $w \in W_\varphi$ we have an isomorphism $\sigma: X \to X'$ inducing $\alpha \circ w$. If $3 \leq D \leq 5$, α determines $\varepsilon = \pm 1$, and only if $\varepsilon = +1$ do we have σ.

Indeed, in the latter case, an isomorphism σ induces α satisfying $\varphi' \circ \alpha = \varepsilon \varphi$, so except in the case $\varphi(Q) \subset \{\pm 1\}$, if $\varepsilon = -1$ the above does not lead to a surface.

As in §3, we consider $\mathrm{Hom}(Q, \mathbb{C}^\times)$ as our model for the deformation space. Observe that L has rank D; Q has rank $D^* - D + 10$ which coincides with the Turina number of the singularity. Again, the universal cover $\mathrm{Hom}(Q, \mathbb{C})$ is essentially the branched cover given by the monodromy of that part of the deformation space corresponding to Du Val singularities.

Location of singularities in the deformation

This now proceeds as before. For a given φ, the singularities on X_φ correspond to the irreducible factors of W_φ. The group W_φ stabilises φ; also a lift in $\mathrm{Hom}(Q, \mathbb{C})$. The algorithm to determine possible W_φ runs as follows (cf. [16]): take the diagram (of type T, Π, Ω) corresponding to the root system R in Q. Delete vertices till each component of the resulting graph is of finite type. Then each component, with at most one exception—say Δ_0—is of type A_n for some n. Replace the component Δ_0 by the corresponding extended diagram; then delete (at least) one vertex from it.

§6. CUSP SINGULARITIES (REAL CASE)

Classification

This was determined in my paper [25]. Each real form is determined by a conjugation map, which acts on E and hence induces an automorphism of Δ.

If this automorphism is trivial, the isomorphism classes of real forms correspond to $T/2T$, where T is the discriminant group $L^\#/L$. As T admits 2 generators, this has order 1, 2 or 4. The values of r for these real forms are

$$\begin{array}{ll} (1 \text{ form}) & r = 2 \\ (2 \text{ forms}) & r = 1, 3 \\ (4 \text{ forms}) & r = 2, 2, 2, 4. \end{array}$$

If the automorphism is a rotation of Δ of order 2, we have a unique real form. It has $r = 0$.

If the automorphism is a reflexion of Δ, it has 2 fixed points P, Q on Δ. Each is either (a) the mid-point of an edge, (b_1) a vertex with b_i odd, or (b_2) a vertex with b_i even. In cases (a), (b_1) define $n_P = 1$, $r_P = 1$; in case (b_2) define $n_P = 2$, $r_P = \{0, 2\}$. Then we have $n_P n_Q$ real forms; the corresponding values of r are given by $r_P + r_Q$.

We observe also that in deriving this result one first shows directly that the real forms correspond to $H^0(\tau; L^\#/L)$ (trivial action case) or $H^1(\tau, L^\#/L)$ (reflexion case) where the action of τ is induced by the given permutation of the E_i and is defined more precisely in [26].

The deformation space

As in §4, we now claim that the proof of the theorem goes through without essential change to give semialgebraic isomorphisms: here hypothesis (v) must be replaced by

$$\tilde{\varphi}' \circ \alpha = \tau \circ \tilde{\varphi}$$

where τ denotes complex conjugation in $\mathbb{C}^\times$, or—if α reverses the orientation of the cycle Δ—its inverse. This allows us to determine the real forms of the surfaces X_φ. Taking into account the further discussion in §4, we see that our moduli space must now be taken as the space of equivariant homomorphisms

$$\mathrm{Hom}_\tau(Q, \mathbb{C}^\times).$$

Here again, if $3 \le D \le 5$ then the variety X determines an orientation of the cycle, and the action of τ on $\mathbb{C}^\times$ is determined by whether conjugation on Q preserves or reverses this. If $D \le 2$, for each conjugation on Q we have to consider both types of action of τ on $\mathbb{C}^\times$.

The discussion of the effect of $w \in W_\varphi$ is now just the same as in [27] (§4). The possible actions of $\mathbb{Z}_2$ on Q are those which preserve R and C^+: as above, these belong to N but not necessarily to W. We can classify involutions in W using the same techniques as before, but I do not see how to extend this to N.

More interesting is the action $\mathbb{Z}_2$ on $\mathrm{Pic}\, E$. As before, this is determined as a pushout of equivariant homomorphisms

$$\begin{array}{ccccccc} 0 \to Q & \to & P & \to & L^\# & \to & 0 \\ \downarrow \varphi & & \downarrow \tilde{\varphi} & & \| & & \\ 0 \to \mathbb{C}^\times & \to & \mathrm{Pic}\, E & \to & L^\# & \to & 0 \end{array}$$

The action of $\mathbb{Z}_2$ on $L^\#$, and indeed also on $\mathbb{C}^\times$ is determined by the action on Δ.

The exact sequence for $\mathrm{Pic}\, E$ is determined by the boundary homomorphisms

$$H^0(L^\#) \to H^1(\mathbb{C}^\times), H^1(L^\#) \to H^0(\mathbb{C}^\times).$$

It is immediate from the fact that $\mathbb{Z}_2$ acts by permuting basis elements that $H^1(L) = 0 = H^1(L^\#)$. If $\mathbb{Z}_2$ acts trivially or by rotation on Δ, then it acts by conjugation on $\mathbb{C}^\times$, so $H^1(\mathbb{C}^\times) = 0$. Hence in these cases the sequence for $\mathrm{Pic}\, E$ splits.

If $\mathbb{Z}_2$ acts by reflexion on Δ, it acts on $\mathbb{C}^\times$ by $z \to \bar{\xi}^{-1}$, so $H^1(\mathbb{C}^\times) = \{\pm 1\}$. Here the sequence is determined by δ: $H^0(L^\#) \to \{\pm 1\}$ or, by duality, $\hat{\delta} \in H^0(L)$.

Identification of the real form

We are now ready to relate the homomorphism φ to the real form of the singularity. In the "rotation" cases, φ induces

$$H^0(\varphi): H^0(Q) \to H^0(\mathbb{C}^\times) = \{\pm 1\};$$

while in the reflexion cases we have

$$H^1(\varphi): H^1(Q) \to H^1(\mathbb{C}^\times) = \{\pm 1\}.$$

We recall that the real form of the singularity is determined by $\hat{\varepsilon}_0 \in H^0(L^\#/L)$ in the case of trivial action, is unique for a rotation of order 2, and is determined by $\hat{\varepsilon}_1 \in H^1(L^\#/L)$ for a reflexion. Write $\hat{\eta}_0 \in H^0(Q^\#)$ (rotation cases), $\hat{\eta}_1 \in H^1(Q^\#)$ (reflexion cases) for the classes dual to the homomorphisms $H^0(\varphi)$, $H^1(\varphi)$.

THEOREM. *The projection $q: Q^\# \to T$ has*

$$H^0(q)\hat{\eta}_0 = \hat{\varepsilon}_0 \quad (\textit{rotation cases}),$$
$$H^1(q)\hat{\eta}_1 = \hat{\varepsilon}_1 \quad (\textit{reflexion cases}).$$

Moreover, the boundary map ∂ for the exact sequence $0 \to L \to L^\# \to T \to 0$ has $\partial \hat{\varepsilon}_1 = \hat{\delta}$ (reflexion cases).

Proof. The four exact sequences making up the commutative diagram

$$\begin{tikzcd}
L \arrow[dr, "i"] \arrow[rr, bend left, "mi"] & & L^{\#} \arrow[dr, "l"] & \\
& P \arrow[ur, "m"] \arrow[dr, "p"] & & T \\
Q \arrow[ur, "j"] \arrow[rr, bend right, "pj"'] & & Q^{\#} \arrow[ur, "q"'] &
\end{tikzcd}$$

have the exact cohomology sequences which fit into the diagram

$$\begin{tikzcd}[column sep=small]
& H^1(L^{\#}) \arrow[dr, "H^1(l)"] \arrow[rr, bend left] & & H^0(Q) \arrow[dr, "H^0(j)"] \arrow[rr, bend left] & & H^0(Q^{\#}) \arrow[dr, "H^0(q)"] \arrow[rr, bend left] & & H^1(L) \arrow[dr, "H^1(i)"] & \\
H^1(P) \arrow[ur, "H^1(m)"] \arrow[dr, "H^1(p)"'] & & H^1(T) \arrow[ur, "\partial'"] \arrow[dr, "\partial"] & & H^0(P) \arrow[ur, "H^0(p)"] \arrow[dr, "H^0(m)"] & & H^0(T) \arrow[ur, "\partial"] \arrow[dr, "\partial'"] & & H^1(P) \\
& H^1(Q^{\#}) \arrow[ur, "H^1(q)"'] \arrow[rr, bend right] & & H^0(L) \arrow[ur, "H^0(i)"'] \arrow[rr, bend right] & & H^0(L^{\#}) \arrow[ur, "H^0(l)"'] \arrow[rr, bend right] & & H^1(Q) \arrow[ur, "H^1(j)"'] &
\end{tikzcd}$$

The natural dualities between Q and $Q^{\#}$, L and $L^{\#}$, P and itself induce dualities (of vector spaces over $\mathbb{F}_2$) of $H^0 Q$ and $H^0 Q^{\#}$, $H^1 Q$ and $H^1 Q^{\#}$ etc., under which the whole diagram becomes self-dual, with $H^0 T$ and $H^1 T$ dually paired.

Now commutativity in the basic pushout diagram implies that $H^1\varphi\circ(\partial' 1^*) = \delta$. Thus by duality,

$$(\partial q^*)(\hat{\eta}) = \hat{\delta}.$$

Of course, this is only of interest in the reflexion case. As ∂ is injective, and as we have not given a formal definition of ε_1, to complete the proof in this case it will suffice to show that the real form is indeed determined by the real form of Pic E, viz. δ. Here we must use the notation of [25, §3]. $H^0(L^{\#})$ is spanned by the classes of points on the invariant curves E_i. For a conjugation $\tau\rho_1\kappa$, E_0 is invariant and the point with coordinate $u_0 = z$ is mapped to that with $u_0 = c_0\bar{z}^{-1}$, so $\delta[E_0] = c_0 \in \{\pm 1\}$. But we showed (see especially [25, p. 227, penultimate paragraph]) that c_0 and the corresponding number for the other invariant cycle (if any) determined the real form.

In the case where τ acts by a nontrivial rotation of ΔL, hence also T, has trivial cohomology and the result is thus trivial. There remains the case where τ acts trivially on Δ, which is considerably more delicate.

Write N for a (conjugation-invariant) regular neighbourhood of E in X, M for its closed complement. The homology groups form the diagram of exact sequences

$$\begin{tikzcd}[column sep=small]
& H_3(M,\partial N) \arrow[dr] \arrow[rr, bend left] & & H_2(N) \arrow[dr] \arrow[rr, bend left] & & H_2(N,\partial N) \arrow[dr] \arrow[rr, bend left] & & H_1(M) \arrow[dr] & \\
H_3(X) \arrow[ur] \arrow[dr] & & H_2(\partial N) \arrow[ur] \arrow[dr] & & H_2(X) \arrow[ur] \arrow[dr] & & H_1(\partial N) \arrow[ur] \arrow[dr] & & H_1(X) \\
& H_3(N,\partial N) \arrow[ur] \arrow[rr, bend right] & & H_2(M) \arrow[ur] \arrow[rr, bend right] & & H_2(M,\partial N) \arrow[ur] \arrow[rr, bend right] & & H_1(N) \arrow[ur] &
\end{tikzcd}$$

which can be identified, subject to two qualifications, with essentially the diagram above:

$$\begin{tikzcd}
 & 0 \arrow[dr] \arrow[rr, bend left] & & L \arrow[dr] \arrow[rr, bend left] & & L^{\#} \arrow[dr] \arrow[rr, bend left] & & 0 \arrow[dr] & \\
0 \arrow[ur] \arrow[dr] & & \mathbb{Z} \arrow[ur] \arrow[dr] & & P \arrow[ur] \arrow[dr] & & \mathbb{Z}\oplus T \arrow[ur] \arrow[dr] & & 0 \\
 & \mathbb{Z} \arrow[ur] \arrow[rr, bend right] & & \mathbb{Z}\oplus Q \arrow[ur] \arrow[rr, bend right] & & \mathbb{Z}\oplus Q^{\#} \arrow[ur] \arrow[rr, bend right] & & \mathbb{Z} \arrow[ur] &
\end{tikzcd}$$

The first qualification is that the splittings are of additive groups only, and not natural. The second is that in identifying divisor classes with two-dimensional homology groups, the action of conjugation τ acquires a sign, since it reverses the orientations of complex curves.

Now $\hat{\varepsilon}_0$ can be identified with the obstruction to splitting $H_1(\partial N) \cong \mathbb{Z} \oplus T$ equivariantly: this is clear form the identification [26, §2F] of the action on $H_1(\partial N)$ of automorphisms and conjugations of the cusp. Hence this lifts to the obstruction to splitting $H_2(M, \partial N) \cong \mathbb{Z} \oplus Q^{\#}$ equivariantly, which is the same (by duality) as the obstruction to splitting $H_2(M) \cong \mathbb{Z} \oplus Q$. It remains to identify this last obstruction with $\hat{\eta}_0$.

The map $H^0(\varphi)$ can be interpreted as follows. An element x of $H^0(Q)$ can be represented by a difference $F = F' - F''$ of cycles defined over $\mathbb{R}$. We must look at the intersection with the exceptional locus $E = \cup_1^n E_i$. Now, as we are in the split case, each E_i is isomorphic over $\mathbb{R}$ to $P^1(\mathbb{R})$. We have chosen coordinates $\mathbb{Z}_i$ on E_i, which we may suppose real on $E_i^{\mathbb{R}}$. Then $H^0\varphi(x)$ is the sign of the product of coordinates z_i of points of $F' \cap E_i$, $F'' \cap E_i$. We can thus ignore pairs of complex conjugate points. Indeed, if we write

$$E^- = \cup_i \{P \in E_i^{\mathbb{R}} : z_i(P) \leq 0\}$$

we have $H^0\varphi(x) = (-1)^n$, n the mod 2 intersection number of the cycles $F_{\mathbb{R}} = F'_{\mathbb{R}} + F''_{\mathbb{R}}$ and E^- in $X_{\mathbb{R}}$.

On the other hand, the splitting obstruction for $H_2(M)$ can be computed from the surface F representing x by taking a homology B of F to a cycle avoiding E (hence, we may suppose, lying in M) and then taking the homology class of $B \cup \tau B$ in $H_3(X, M) \cong H_3(N, \partial N) \cong \mathbb{Z}$, modulo 2. To identify these two, we must be more explicit about B.

We observe that it is not necessary for the cycle F above to be chosen algebraic.

Indeed, the intersection number of $F_{\mathbb{R}}$ and E^- in $X_{\mathbb{R}}$ is determined by the mod 2 homology class of $F_{\mathbb{R}}$, and hence by the equivariant mod 2 homology class of F. But—as a short calculation shows—$H^2(X; \mathbb{Z}_2)$ is an extension by $H^0(\mathbb{Z}_2; H^2(X; \mathbb{Z}_2))$ of a subgroup $\mathbb{Z}_2$ represented by mapping a 2-sphere with the antipodal involution to a point of $X_{\mathbb{R}}$, and hence not contributing to our invariant.

We can thus suppose that F avoids singular points of E, and meets N in a collection of discs giving fibres over points of $F \cap E$ of the normal bundle of E; deformations of these points give deformations of E. First we deform pairs of complex conjugate intersections to be real. Next deform all real intersection points to $z_i = +1$ or $z_i = -1$. As we are working mod 2, each of these intersections can be taken to have multiplicity 0 or 1. Moreover as $x \in Q$, the two intersection points on E_i both have the same multiplicity (0 or 1).

It remains to identify the two invariants after these normalisations: it suffices to show that each component E_i such that $F.E_i = \{-1, 1\}$ contributes to the class of $B \cup \tau B$ in $H_3(N, \partial N; \mathbb{Z}_2)$.

We construct B by joining -1 to 1 by the semicircle $\{e^{i\theta} : 0 \leq \theta \leq \pi)$: the union of the corresponding normal discs gives the desired homology. Then τB is given by the other half of the unit circle. The cycle $B \cup \tau B$ in $(N, \partial N)$ meets the 1-cycle E^- transversely in one point ($z_i = -1$). As E^- generates $H_1(N; \mathbb{Z}_2) \cong \mathbb{Z}_2$, this completes the proof.

Since this proof is quite involved, it seems worth presenting an alternative argument which, though it gives a weaker result, gives more geometrical insight into the split case.

The real exceptional set $E_{\mathbb{R}} = \cup E_{i\mathbb{R}}$ is a connected graph. A basis for $H_1(E_{\mathbb{R}})$ is given by the $E_{i\mathbb{R}}$ together with E^-. The natural retraction of $X_{\mathbb{R}} \cap N$ on $E_{\mathbb{R}}$ has degree 2 on the boundary, so for each i there are 2 pieces mapping to E_i^- and 2 to E_i^+. As we saw in [25, p. 226], as E_i moves round the cycle these 4 components get permuted. The real form determines the class of the permutation in the symmetric group S(4): its class in the quotient S(3) by the four group is already determined by the complex class of the singularity. Types are thus as follows:

Complex type	Standard real form	Nonstandard real form
A	3, 1	—
B	2, 1, 1	4
C	1, 1, 1, 1	2, 2

Each cycle yields a boundary component of $N_{\mathbb{R}}$. We can determine the homology classes in $N_{\mathbb{R}}$: e.g. for a cycle of lenth 1, we have a sum, with E_i^+ or E_i^- for each i. It follows that a cycle of even length is homologous to a linear combination of the $E_i(\mathbb{R})$ (and hence to 0 in N): one of odd length is not.

Now if F is (as above) an invariant cycle representing an element of Q, we want to relate $F_{\mathbb{R}} \cap E^-$ to the real form. As $[F] \in Q$, $F_{\mathbb{R}} \cdot E^{\mathbb{R}} = 0$. Now $[E^-]$ is not a linear combination of the $[E_i^{\mathbb{R}}]$ in $[X_{\mathbb{R}}] \Leftrightarrow$ there is a 1-cycle L in $X_{\mathbb{R}}$ with $L.\ E^{\mathbb{R}} = 0$, $L.\ E^- = 1, \Leftrightarrow$ there is an invariant 2-cycle F, with $[F] \in Q$, $F_{\mathbb{R}}.\ E^- = 1$. For L spans a surface D in X, which may be taken to have zero intersection numbers with the E_i: take $F = D \cup \tau D$.

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