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MILGRAM'S CLASSIFYING SPACE OF A TOPOLOGICAL GROUP†

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§1. INTRODUCTION

A CLASSIFYING space for a topological group G is the base space B_G of a principal G-bundle $E_G \to B_G$ such that E_G is a contractible space. It is a universal object in the sense that any principal G-bundle over a complex K admits a bundle mapping into E_G. General properties of G-bundles and their characteristic classes are obtained by studying $E_G \to B_G$.

The first functorial construction of an $E_G \to B_G$ was given by Milnor in 1956 [5]. In 1959, Dold and Lashof [1] reformulated Milnor's construction in such a way that it applies when G is a topological monoid (i.e. an associative H-space with a two-sided unit). Recently Milgram [4] gave a different functorial construction, and proved two useful properties: first, if G is an abelian monoid, then B_G has a natural (functorial) structure as an abelian monoid; secondly, if G is a complex such that the multiplication $G \times G \to G$ is skeletal (i.e. for each q, it maps the q-skeleton into the q-skeleton), then B_G becomes a complex in a natural way so that the chain group $C(B_G)$ is isomorphic to the bar resolution of $C(G)$. Thus Milgram's construction can be regarded as the geometric analog of the algebraic bar construction.

In this paper, we present a reformulation of Milgram's construction. It has three advantages: it is well motivated, the degree of generality of the results can be precisely stated, and the relation of Milgram's construction to that of Dold and Lashof becomes apparent (it is a quotient of the latter).

We shall establish two further properties. First, the construction preserves products $E_{G\times H} = E_G \times E_H$. Secondly, if G is a group (or an abelian monoid), then E_G has a natural structure as a topological group (abelian monoid) such that G is a subgroup (submonoid), B_G is the coset space E_G/G, and, when G is abelian, B_G is a quotient group (monoid).

The fact that B_G preserves products combined with its functorial property implies immediately that it carries an abelian monoid (group) into an abelian monoid (group); one need only observe that the multiplication $G \times G \to G$, and the inverse mapping $G \to G$ are morphisms. In the category of semisimplicial monoids, the W-construction of Eilenberg–MacLane [3] has exactly these properties (see John Moore [6]). In view of this, it appears

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likely that Milgram's resolution, applied to the geometric realization of a semi-simplicial monoid, gives a space that is naturally equivalent to the geometric realization of the W-construction applied to the monoid.

§2. ENLARGING AN ACTION

We assume throughout this paper that G denotes a topological monoid (equivalently, an associative H-space with a two-sided unit e). All spaces are assumed to be compactly generated (a set that meets every compact set in a closed set is closed); and product spaces are formed in the sense of the category of these spaces [8, 4.1].

An *action* of G in a space X is a mapping $X \times G \to X$ (the image of (x, g) is written xg) satisfying the associativity law $x(gg') = (xg)g'$, and the unit condition $xe = x$. A space X with an action is called a *G-space*. A mapping $f: X \to Y$ of one G-space in another is a *G-mapping* if $f(xg) = (fx)g$ for all x, g. For any product of the form $X \times G$, the *right action* of G is defined by $(x, g)g' = (x, gg')$ for all x, g, g'. Then, if X is a G-space, the action mapping $X \times G \to X$ is a G-mapping relative to right action by virtue of the associative law.

2.1. *Definition.* Let A be closed in X, and $h : A \times G \to A$ an action. Form the adjunction space $\bar{X} = A \cup_h (X \times G)$; this is the quotient space obtained from $X \times G$ by collapsing $A \times G$ into A by h. Right action of G in $X \times G$ induces an action of G in $\bar{X}$. The resulting G-space is called the *enlargement* to X of the G-action on A.

We identify X with the image in $\bar{X}$ of $X \times e$; this identification on A preserves the action of G. In this way $\bar{X}$ is the union of the closed set A with the action h and the open set $(X - A) \times G$ with right action.

Let us recall [8, 6.2] that A is called a neighborhood deformation retract of X (briefly, (X, A) is an NDR) if there is a mapping $u: X \to I = [0, 1]$ and a homotopy $k: X \times I \to X$ such that $u^{-1}0 = A$, $k(x, 0) = x$ for all $x \in X$, $k(a, t) = a$ for all $a \in A$, $t \in I$, and $k(x, 1) \in A$ for all x such that $ux < 1$.

2.2. LEMMA. *If (X, A) is an NDR, then $(\bar{X}, A)$ is an NDR. If also (G, e) is an NDR, then $(\bar{X}, X)$ is an NDR.*

Proof. It is easily seen that $(X \times G, A \times G)$ is an NDR (see [8, 6.3]), then the first statement follows from the general proposition [8; 8.5] about adjunction spaces. By [8; 6.3]

$$(X, A) \times (G, e) = (X \times G, X \times e \cup A \times G)$$

is an NDR. Since the quotient mapping $X \times G \to \bar{X}$ is a relative homeomorphism $(X, A) \times (G, e) \to (\bar{X}, X)$, the second statement follows from [8; 8.4].

Remark. Our definition of a compactly generated space includes the Hausdorff condition. As a quotient space, the enlargement $\bar{X}$ may fail to be Hausdorff. However, when (X, A) is an NDR, $\bar{X}$ is Hausdorff. This follows from Lemma 8.5 of [8] where it is stated that any adjunction space $Y \cup_h X$ is compactly generated if X and Y are compactly generated, and (X, A) is an NDR. Since the proof of the Hausdorff condition was omitted, we give it here. Since $X - A$ maps topologically onto $Y \cup_h X - Y$, this open subset is Hausdorff. Suppose $x_1 \in Y$ and $x_2 \in Y \cup_h X - Y$. Set

$$U = \{x \in X \mid ux < (ux_2)/2\}, \qquad V = \{x \in X \mid ux > (ux_2)/2\}.$$

Then $Y \cup U$, V map onto open sets separating x_1 and x_2. Suppose next that $x_1, x_2 \in Y$. Let U_1, U_2 be open disjoint sets separating x_1, x_2 in Y. Let r: $W \to A$ retract a neighborhood of A in X into A. Then $U_1 \cup r^{-1}h^{-1}U_1$ and $U_2 \cup r^{-1}h^{-1}U_2$ map onto disjoint open sets of $Y \cup_h X$ separating x_1 and x_2.

2.3. LEMMA. *Let G, H be topological monoids, and let h: $G \to H$ be a continuous morphism. Let f: $(X, A) \to (Y, B)$ be a mapping of NDR pairs such that A is a G-space, B is an H-space, and $f \mid A$ is h-equivariant, i.e., $f(ag) = (fa)(hg)$ for all $a \in A$, $g \in G$. Then $f \times h$: $X \times G \to Y \times H$ is h-equivariant relative to right action, and it induces an h-equivariant mapping of quotient spaces $\bar{f}$: $(\bar{X}, A) \to (\bar{Y}, B)$ called the enlargement of f. In this way, enlargement becomes a functor.*

Proof. Two distinct points (x_1, g_1), (x_2, g_2) of $X \times G$ are equivalent in $\bar{X}$ if $x_1, x_2 \in A$ and $x_1 g_1 = x_2 g_2$. Since $f \mid A$ is h-equivariant, we have $f(x_i g_i) = (fx_i)(hg_i)$ for $i = 1, 2$, hence (fx_1, hg_1) and (fx_2, hg_2) are equivalent in $\bar{Y}$. Therefore the composition $X \times G \to Y \times H \to \bar{Y}$ factors into $X \times G \to \bar{X} \to \bar{Y}$. Since $\bar{X}$ has the quotient topology, $\bar{f}$ is continuous. Since the quotient mappings $X \times G \to \bar{X}$ and $Y \times H \to \bar{Y}$ are G- and H-mappings, it follows that $\bar{f}$ is h-equivariant. It is routine to check the functorial properties of enlargement.

2.4. *Remark.* The enlargement $\bar{X} \supset X$ is characterized up to a G-equivalence by the property: if Y is any G-space, and f any map $X \to Y$ such that $f \mid A$ is a G-mapping, then there exists a unique G-mapping f': $\bar{X} \to Y$ extending f.

By a complex we shall mean a CW-complex. A mapping f: $K \to L$ of two complexes is called *skeletal* if f maps the q-skeleton of K into that of L for each $q \geqq 0$. A product of two complexes is regarded as a complex whose cells are the products of cells of the factors.

2.5. LEMMA. *Let G be a complex such that the multiplication $G \times G \to G$ is skeletal. Let (X, A) be a complex and subcomplex, and suppose the action $A \times G \to A$ is skeletal. Then the enlargement $\bar{X}$ inherits a unique structure as a complex from that of $X \times G$, X is a subcomplex of $\bar{X}$, and the mapping $\bar{X} \times G \to \bar{X}$ is skeletal.*

Proof. It is a general proposition that, if L is a subcomplex of K, M is a complex, and f: $L \to M$ is skeletal, then $M \cup_f K$ inherits a unique structure as a complex from the disjoint union $M \cup K$ such that M is a subcomplex and the quotient map $M \cup K \to M \cup_f K$ is skeletal. This becomes obvious if we picture K as being built out of L by successive adjunctions of cells ordered by dimension; for we may build $M \cup_f K$ out of M by adjoining the same cells to M using adjunction maps modified by f. If we apply this proposition to the case $K = X \times G$, $L = A \times G$ and $M = A$, it follows that $\bar{X}$ inherits a structure as a complex such that $X \times G \to \bar{X}$ is skeletal and X is a subcomplex. Since each cell of $\bar{X}$ is the image of a cell of $X \times G$ of the same dimension and $X \times G \times G \to X \times G \to \bar{X}$ are skeletal, it follows that $\bar{X} \times G \to \bar{X}$ is skeletal.

§3. CONTRACTIONS OF SPACES

The unit interval $I = [0, 1]$ is a topological monoid under ordinary multiplication. An I-action $X \times I \to X$ is of course a homotopy which, for $t = 1$, is the identity map of X. It is a special kind of homotopy, the associative law requires that each point on the path of a point follows a path contained in the first path; the homotopy is a 1-parameter semigroup of motions with a reversed parameter. The *base point* of I is defined to be 0.

3.1. *Definition.* A *contraction* of a space X to a base point x_0 is an I-action $h: X \times I \to X$ that factors through the smash product $X \times I \to X \wedge I \to X$. In other words, $h(x, 0) = x_0 = h(x_0, t)$ for all $x \in X$, $t \in I$.

For example, the multiplication mapping $m: I \times I \to I$ is a contraction of I to 0.

We shall restrict ourselves to spaces with base points (X, x_0) that are NDR's. Then, by [8; 6.3], the product pair $(X, x_0) \times (I, 0)$ is an NDR. The reduced cone $X \wedge I$ is just the adjunction space determined by the map of $X \times 0 \cup x_0 \times I$ to a point x_1. Then, by [8; 8.5], $(X \wedge I, x_1)$ is an NDR; in particular $X \wedge I$ is Hausdorff.

The proof of the following lemma is trivial.

3.2. LEMMA. *The right action of I on $X \times I$ induces a contraction on $X \wedge I$ called the canonical contraction. The cone with this contraction is a functor from the category of pointed spaces to the category of pointed spaces with contractions.*

3.3. LEMMA. *Let $x_0 \in A \subset X$ be such that (X, A), (X, x_0) and (A, x_0) are NDR's, and let $h: A \wedge I \to A$ be a contraction of A to x_0. Set $\tilde{X} = A \cup_h (X \wedge I)$ so that $\tilde{X}$ is the quotient space of $X \wedge I$ obtained by collapsing $A \wedge I$ into A by h. Then all of the pairs $(\tilde{X}, x_0)$, $(\tilde{X}, A)$ and $(\tilde{X}, X)$ are NDR's, and the canonical contraction of $X \wedge I$ induces a contraction of $\tilde{X}$ to x_0 which extends h. We call $(\tilde{X}, x_0)$ with this contraction the enlargement to X of the contraction on A. It is functorial for maps $f: (X, A, x_0) \to (Y, B, y_0)$ such that A and B have contractions and $f \mid A$ is an I-mapping.*

Proof. By the product theorem [8; 6.3], $(X, A) \times (I, 0)$ is an NDR. It maps by a relative homeomorphism onto $(X \wedge I, A \wedge I)$, hence, by [8; 8.4], $(X \wedge I, A \wedge I)$ is an NDR. It follows from the lemma [8; 8.5] on adjunction spaces, that $(\tilde{X}, A)$ is an NDR. Since (A, x_0) and $(\tilde{X}, A)$ are NDR's, the lemma [8; 7.2] yields that $(\tilde{X}, x_0)$ is an NDR. If $\dot{I}$ denotes the set of endpoints of I, then $(I, \dot{I})$ is an NDR. Hence, by the product theorem $(X, A) \times (I, \dot{I})$ is an NDR, and, since it maps onto $(\tilde{X}, X)$ by a relative homeomorphism, the latter is also an NDR.

In the diagram

$$\begin{array}{ccc} X \times I \times I & \xrightarrow{1 \times m} & X \times I \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ \tilde{X} \wedge I & \xrightarrow{k} & \tilde{X} \end{array}$$

p and p' are the natural quotient mappings, and m is the multiplication of I. To show that there is a unique function k such that $kp' = p(1 \times m)$, let (x, t, τ) and (x', t', τ') be distinct points of $X \times I \times I$ having the same image under p'. If both map to the base point, we have $x = x_0$ or $t = 0$ or $\tau = 0$; and this implies $x = x_0$ or $t\tau = 0$, hence $(x, t\tau)$ maps to the base point. Similarly $(x', t'\tau')$ maps to the base point. If neither maps to the base point, then we must have that x, x' are in A, $xt = x't'$ and $\tau = \tau' \neq 0$. These imply $xt\tau = x't'\tau'$, hence $(x, t\tau)$ and $(x', t'\tau')$ have the same image in $\tilde{X}$. Thus k is uniquely defined.

To prove that k is continuous it suffices to show that p' is proclusive (a quotient mapping). Since a composition of proclusions $X \times I \to X \wedge I \to \tilde{X}$ is a proclusion $X \times I \to \tilde{X}$, we may apply [8; 4.4] to conclude that $X \times I \times I \to \tilde{X} \times I$ is proclusive. Composing this with the proclusion $\tilde{X} \times I \to \tilde{X} \wedge I$ gives p', hence p' is also proclusive.

The construction of $\tilde{f}$: $(\tilde{X}, A) \to (\tilde{Y}, B)$ and the verification of functorial properties is routine and will be omitted. This concludes the proof.

3.4. LEMMA. *Let (X, A) be a complex and subcomplex, and let the contraction $A \wedge I \to A$ be a skeletal mapping where I is the complex with two vertices and one edge. Then the enlargement $\tilde{X}$ inherits a unique structure as a complex from that of $X \wedge I$, and the mappings $X \wedge I \to \tilde{X}$ and $\tilde{X} \wedge I \to \tilde{X}$ are skeletal.*

Proof. Apply the argument proving 2.5 with K, L, M replaced by $X \wedge I$, $A \wedge I$, and A, respectively.

Remark. Just as in 2.4, the enlargement $\tilde{X}$ is characterized by the property: if f: $X \to Y$ is a map of X into a space Y having a contraction, and $f \mid A$ is an I-mapping, then f extends to a unique I-mapping $\tilde{X} \to Y$.

§4. CONSTRUCTION OF THE RESOLUTION

For any topological monoid G with unit e such that (G, e) is an NDR, we have the following construction obtained by alternating the constructions of §2 and §3. By an induction on n, we define spaces D_n, E_n such that

$$D_0 \subset E_0 \subset D_1 \subset \cdots \subset D_n \subset E_n \subset D_{n+1} \subset \cdots$$

Moreover each D_n has a contraction $D_n \wedge I \to D_n$ and each E_n is a G-space. Let D_0 consist of the single point e with the obvious contraction. Let E_0 denote the enlargement to D_0 of the G-action on the empty subset of D_0. A check of definition 2.1 shows that $E_0 = D_0 \times G$ is just a copy of G and the action is right translation. Now define D_1 to be the enlargement to E_0 of the contraction of D_0. A check of the definition (see 3.3) shows that D_1 is just the reduced cone on E_0. Define E_1 to be the enlargement to D_1 of the G-action on E_0. In general D_n is the enlargement to E_{n-1} of the contraction of D_{n-1}, and E_n is the enlargement to D_n of the G-action on E_{n-1}. The G- and I-actions are denoted by

$$\phi_n\colon E_n \times G \to E_n \qquad \text{and} \qquad \psi_n\colon D_n \times I \to D_n.$$

We now pass to the limit by setting

$$E_G = \bigcup_{n=0}^{\infty} E_n = \bigcup_{n=0}^{\infty} D_n,$$

and giving E the topology of the union (weak topology). Since the G-action ϕ_n on E_n extends

ϕ_{n-1} for each n, the union of the ϕ_n's defines a G-action $\phi: E \times G \to E$. Since the contraction ψ_n of D_n extends ψ_{n-1} for each n, the union of the ψ_n's is a contraction $\psi: E \wedge I \to E$.

4.1. THEOREM. *If* (G, e) *is an NDR, then the* E_G *constructed above is a G-resolution in the sense of* [7, 1.1]. *Moreover, if* $f: G \to H$ *is a continuous morphism of topological monoids, there is an associated functorial f-mapping of resolutions* $\tilde{f}: E_G \to E_H$.

Proof. Clearly $\{E_n\}$ represents E as a filtered G-space. It is an *acyclic* filtration because the contraction of E contracts E_n to the point e in E_{n+1} for each n. It is a free filtered G-space because, for each n, $E_n = E_{n-1} \cup_\phi (D_n \times G)$, hence the quotient mapping $(D_n, E_{n-1}) \times G \to (E_n, E_{n-1})$ is a relative homeomorphism. Finally we must show that (D_n, E_{n-1}) is an NDR for each n. The proof of this proceeds by induction on n; the case $n = 0$ is trivial. Assume inductively that (D_n, E_{n-1}) is an NDR. Since (G, e) is an NDR, it follows from 2.2 that (E_n, D_n) is an NDR. Then it follows from 3.3 that (D_{n+1}, E_n) is an NDR. This completes the inductive step, and the proof that E_G is a resolution.

The functorial nature of the construction is shown by proving the same for each D_n, E_n using 2.3, 3.3, and passing to the limit.

4.2. LEMMA. *If G is as in* 4.1, *then, for each n,* (E_n, E_{n-1}) *and* (E_G, E_n) *are G-NDR's, i.e. the functions u and h in the definition of an NDR satisfy* $u(xg) = ux$ *and* $h(xg, t) = h(x, t)g$ *for all x, g and t.*

Proof. Let u, h represent (D_n, E_{n-1}) as an NDR. Define $u': D_n \times G \to I$ and $h': D_n \times G \times I \to D_n \times G$ by

$$u'(x, g) = ux, \qquad h'(x, g, t) = (h(x, t), g) \quad \text{for all} \quad x, g, t.$$

Then, with respect to right action in $D_n \gg G$, u' and h' represent $(D_n \times G, E_{n-1} \times G)$ as a G-NDR. Since the quotient mapping

$$\phi: (D_n, E_{n-1}) \times G \to (E_n, E_{n-1})$$

is a relative homeomorphism, it follows from [8; 8.4] that u', h' induce a representation v, k of (E_n, E_{n-1}) as an NDR such that $v\phi = u'$ and $k(\phi \times 1) = \phi h'$. Since ϕ is a G-mapping, it follows that v, k represent (E_n, E_{n-1}) as a G-NDR.

According to [8; 7.1] the NDR property of (E_m, E_{m-1}) is equivalent to the existence of a retraction r_m of $I \times E_m$ into $0 \times E_m \cup I \times E_{m-1}$. It is easily checked that the G-NDR property is equivalent to r being a G-map. Apply now the argument of [8; 9.4] to construct a retraction s of $I \times E$ into $0 \times E \cup I \times E_n$. Since s is essentially a composition of various r_m's, it follows that s is a G-map; hence (E, E_n) is a G-NDR, and the lemma is proved.

Since G is not required to be a group, the orbit of a point of E_G under G need not be a copy of G. However, each point lies in a maximal orbit which is a copy of G because E_G is the union of the sets $E_n - E_{n-1}$ homeomorphic to $(D_n - E_{n-1}) \times G$. These maximal orbits are closed sets.

4.3. *Definition.* The base space B_G of the G-resolution E_G is the quotient space of E_G by its maximal G-orbits. Let $p: E_G \to B_G$ be the natural map. Set $B_n = pE_n$. The base space with this filtration we call *Milgram's classifying space for G.*

It is readily seen that B_n is obtained from B_{n-1} by adjoining D_n by the projection $p: E_{n-1} \to B_{n-1}$. Since (D_n, E_{n-1}) is an NDR, it follows from [8; 8.5] that (B_n, B_{n-1}) is an NDR. Applying [8; 9.4] we obtain that B_G is a Hausdorff space, and each (B, B_n) is an NDR. It follows now from [8; 2.6] that B_G is compactly generated, and by [8; 9.5] that B_G has the topology of the union of $\{B_n\}$.

4.4. *Remark.* The construction of Dold and Lashof differs from ours only in that each D_n is the cone $E_{n-1} \wedge I$ rather than the space obtained from the cone by collapsing $D_{n-1} \wedge I$ into D_{n-1}. It follows that there is a functorial mapping of the Dold–Lashof resolution onto the Milgram resolution. This is a quotient mapping when G is compact, but may not be in general due to the intricate topology Dold and Lashof gave their resolution.

§5. SIMPLICIAL PARAMETERS FOR E_G

The proofs of our main results are based on a parametric representation of E_G, essentially that of Milgram's definition.

Let Δ_n denote the n-simplex of R^n defined by the inequalities $0 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_n \leqq 1$; and let δ_n denote its interior: $0 < t_1 < \cdots < t_n < 1$. The standard imbedding of Δ_n in Δ_{n+1} adjoins the $(n+1)$st coordinate $t_{n+1} = 1$.

A point of $G^n \times \Delta_n$ will be represented by its coordinates in shuffled form $[g_1, t_1, g_2, t_2, \ldots, g_n, t_n]$. Imbed $G^n \times \Delta_n$ in $G^{n+1} \times \Delta_{n+1}$ by adjoining the coordinates $g_{n+1} = e$ and $t_{n+1} = 1$. Let $G^\omega \times \Delta_\omega$ denote the union $\bigcup_{n=0}^{\infty} G^n \times \Delta_n$.

5.1. THEOREM. *For each n there is a natural transformation $k_n: G^n \times \Delta_n \to D_n$ with the following properties.*

(a) *Each k_n is proclusive.*
(b) *The restriction of k_n to $G^{n-1} \times \Delta_{n-1}$ is k_{n-1}; hence the union of the k_n's is defined and is a mapping $k: G^\omega \times \Delta_\omega \to E_G$.*
(c) *Each k_n restricts to homeomorphisms*

$$(G - e)^n \times \delta_n \xrightarrow{\alpha_n} D_n - E_{n-1} \qquad \textit{and} \qquad (G - e)^n \times \delta_{n-1} \xrightarrow{\beta_n} E_{n-1} - D_{n-2}.$$

(d) *The restriction of k_n to $G^n \times \Delta_{n-1} \to E_{n-1}$ is a G-map where G acts only on the* nth *G-factor by right translation.*
(e) *If the action of I on $G^n \times \Delta_n$ is defined by*

$$[g_1, t_1, g_2, t_2, \ldots, g_n, t_n]\tau = [g_1, t_1\tau, g_2, t_2\tau, \ldots, g_n, t_n\tau],$$

then k_n is an I-mapping.
(f) *If $x = [g_1, s_1, \ldots, g_n, s_n]$ and $y = [h_1, t_1, \ldots, h_n, t_n]$ in $G^n \times \Delta_n$ are such that, for some $j < n$,*

$$k_j[g_1, s_1, \ldots, g_j, s_j] = k_j[h_1, t_1, \ldots, h_j, t_j],$$

and $g_i = h_i$ and $s_i = t_i$ for $i = j+1, \ldots, n$, then $k_n x = k_n y$.

Proof. The proof proceeds by induction on n. In case $n = 0$, $G^0 \times \Delta_0$ and D_0 are single points. Interpreting Δ_{-1} and E_{-1} to be empty, the six properties hold in a trivial way.

Assume inductively that k_{n-1} has been constructed to satisfy (a)–(f). We shall define k_n so that the following diagram is commutative

$$(5.2)\qquad \begin{array}{ccccc} G^{n-1} \times \Delta_{n-1} \times G \times I & \xrightarrow{T} & G^n \times \Delta_{n-1} \times I & \xrightarrow{1 \times \psi} & G^n \times \Delta_n \\ \downarrow{\scriptstyle k_{n-1} \times 1 \times 1} & & & & \downarrow{\scriptstyle k_n} \\ D_{n-1} \times G \times I & \xrightarrow{\lambda \times 1} & E_{n-1} \times I & \xrightarrow{\mu} & D_n \end{array}$$

The mapping T interchanges the two middle factors, ψ is defined by

$$\psi((t_1, \ldots, t_{n-1}), \tau) = \psi(t_1\tau, \ldots, t_{n-1}\tau, \tau),$$

and λ, μ are the quotient mappings occurring in the definitions of E_{n-1} and D_n. It is readily verified that if $(1 \times \psi)T$ brings two points together then their G-coordinates are equal, and also their I-coordinates; if the latter are non-zero, then all coordinates are equal; and if the I-coordinates are zero, then $\mu(\lambda \times 1)(k_{n-1} \times 1 \times 1)$ carries both points to the base point of D_n. This shows that there is a unique k_n making the diagram commutative. The continuity of k_n follows from the proclusive property of $(1 \times \psi)T$. The functorial property of k_n follows readily from that of the other mappings of the diagram.

To prove (a), we note first that ψ, T, k_{n-1}, λ and μ are proclusive. Since a product of proclusions is a proclusion [8; 4.4], it follows that all mappings of the diagram, other than k_n, are proclusions. Suppose then that $U \subset D_n$ is such that $k_n^{-1}U$ is open; since $(1 \times \psi)T$ is continuous and the diagram commutes, we have that

$$(k_n(1 \times \psi)T)^{-1}U = (\mu(\lambda \times 1)(k_{n-1} \times 1 \times 1))^{-1}U$$

is open. Since the composition $\mu(\lambda \times 1)(k_{n-1} \times 1 \times 1)$ is proclusive, it follows that U is open. Hence k_n is proclusive.

To prove (b), let x be a point of $G^{n-1} \times \Delta_{n-1}$ considered as a point of $G^n \times \Delta_n$ with last two coordinates e, 1. Then $x = (1 \times \psi)T(x, e, 1)$; and, recalling the definitions of λ, μ, we have

$$k_n x = \mu(\lambda \times 1)(k_{n-1} \times 1 \times 1)(x, e, 1) = \mu(\lambda \times 1)(k_{n-1}x, e, 1) = k_{n-1}x.$$

The proof that β_n is a homeomorphism is based on the diagram

$$\begin{array}{ccc} (G-e)^{n-1} \times \delta_{n-1} \times (G-e) & \xrightarrow{1 \times T} & (G-e)^n \times \delta_{n-1} \\ \downarrow{\scriptstyle \alpha_{n-1} \times 1} & & \downarrow{\scriptstyle \beta_n} \\ (D_{n-1} - E_{n-2}) \times (G-e) & \xrightarrow{\lambda'} & E_{n-1} - D_{n-1} \end{array}$$

where T interchanges the last two factors and λ' is the restriction of λ. Since the quotient mapping λ in the construction of E_{n-1} out of E_{n-2} defines a homeomorphism of $(D_{n-1} - E_{n-2}) \times G$ onto $E_{n-1} - E_{n-2}$, it follows that λ' is a homeomorphism. Since T is a homeomorphism, and α_{n-1} is assumed to be so, it follows from the commutativity of the diagram that β_n is a homeomorphism.

To prove the same for α_n, we use the diagram

$$\begin{array}{ccc} (G-e)^n \times \delta_{n-1} \times \delta_1 & \xrightarrow{1\times\psi'} & (G-e)^n \times \delta_n \\ \downarrow{\scriptstyle \beta_n \times 1} & & \downarrow{\scriptstyle \alpha_n} \\ (E_{n-1} - D_{n-1}) \times \delta_1 & \xrightarrow{\mu'} & D_n - E_{n-1} \end{array}$$

where ψ' is the restriction of ψ, and μ' is the restriction of μ. It is readily checked that ψ' and μ' are homeomorphisms. Since β_n is a homeomorphism and the diagram is commutative, it follows that α_n is a homeomorphism. This proves (c).

To prove (d), consider the diagram of subspaces of 5.2 obtained by replacing Δ_n by Δ_{n-1}, each I-factor by the point $1 \in I$, and each mapping by its restriction. With respect to right action of G on the right-hand G-factors, $1 \times \psi$, T and $k_{n-1} \times 1 \times 1$ are clearly G-mappings. Since the G-action in E_{n-1} is induced by that in $D_{n-1} \times G$ through the quotient map λ, it follows that λ and $\lambda \times 1$ are G-maps. Since $\mu \mid E_{n-1} \times 1$ is just the identification of E_{n-1} as a subspace of D_n, it too is a G-map. Since the diagram is commutative and all mappings, other than k_n, are G-maps, it follows that k_n is also a G-map.

To prove (e), let I act on each of the four spaces on the left of 5.2 by standard right action on its factor I. After verifying that all mappings of 5.2, other than k_n, are I-mappings, it follows from the commutativity of the diagram that k_n is also an I-mapping.

To prove (f), note that the hypothesis implies $s_n = t_n$. If both are zero, then k_n maps both to $e \in D_n$. Suppose $s_n = t_n = \tau$ is not zero. Let $s_i' = s_i/\tau$ and $t_i' = t_i/\tau$ for $1 \leqq i \leqq n-1$, and set $x' = [g_1, s_1', \ldots, g_{n-1}, s_{n-1}']$, $y' = [h_1, t_1', \ldots, h_{n-1}, t_{n-1}']$. Since k_j for $j < n$ satisfies (e), we have

$$\begin{aligned} k_j[g_1, s_1', \ldots, g_j, s_j'] &= (k_j[g_1, s_1, \ldots, g_j, s_j])\tau^{-1} \\ &= (k_j[h_1, t_1, \ldots, h_j, t_j])\tau^{-1} = k_j[h_1, t_1', \ldots, h_j, t_j']. \end{aligned}$$

We conclude from this that $k_{n-1}x' = k_{n-1}y'$ because either $j = n-1$, or $j < n-1$ and x', y' satisfy the hypotheses of (f) with n replaced by $n-1$. It follows that $k_{n-1} \times 1 \times 1$ maps (x', g_n, s_n) and $(y', h_n, t_n) = (y', g_n, s_n)$ to the same point. It follows now that $k_n x = k_n y$. This completes the proof of the theorem.

5.3. *Definition.* Let $N_n = \bigcup_{j=0}^n (G-e)^j \times (\delta_j \bigcup \delta_{j-1})$. If $x \in D_n$, $u \in G^n \times \Delta_n$, and $k_n u = x$, then u is said to represent x; if also $u \in N_n$, u is called the representation in *normal form.* Two elements $u, v \in G^n \times \Delta_n$ are called equivalent if $k_n u = k_n v$.

5.4. COROLLARY. *The restriction of k_n to $N_n \to D_n$ is bijective. Thus the representation of an element of D_n in normal form is unique.*

The corollary follows from 5.1b, c and the observation that D_n is the disjoint union $\bigcup_{j=0}^n (D_j - D_{j-1})$.

The condition for $u = [g_1, t_1, \ldots, g_n, t_n]$ to be in N_n is that there is a j such that $g_1, \ldots, g_j \in G - e$, $0 < t_1 < \cdots < t_j \leqq 1$, and $g_i = e$ and $t_i = 1$ for $i = j+1, \ldots, n$.

Starting with a u that is not in normal form we reduce it to its equivalent normal form by a series of elementary reductions of the following two types:

(5.5) If some $g_i = e$ or $t_i = 0$, delete the pair g_i, t_i and adjoint e, 1 on the right.

(5.6) If some $t_{i-1} = 1$, replace g_{i-1} by $g_{i-1}g_i$ and g_i by e.

One verifies these equivalences in the case $i = n$ by checking the definition 5.2 of k_n. The cases $i < n$ follow from the case (i, i) by applying 5.1f.

§6. THE NATURAL EQUIVALENCE $E_{G\times H} \approx E_G \times E_H$

6.1. *Definition.* Let G, H be topological monoids, and let p, q denote the projections of $G \times H$ into G and H respectively. Define $\xi_{G,H}\colon E_{G\times H} \to E_G \times E_H$ to be the mapping whose components are $\tilde{p}$, $\tilde{q}$ (see 4.1).

It is obvious that ξ is continuous, it is a natural transformation of functors, it is a mapping of $(G \times H)$-spaces, and hence it induces a mapping $B_{G\times H} \to B_G \times B_H$.

If K is a third topological monoid, and p, q, r are the projections $G \times H \times K$ into G, H, K respectively, then we have the associative law

$$(1 \times \xi_{H,K})\xi_{G,H\times K} = (\xi_{G,H} \times 1)\xi_{G\times H,K}\colon E_{G\times H\times K} \to E_G \times E_H \times E_K$$

because both sides have the components $\tilde{p}$, $\tilde{q}$, $\tilde{r}$.

If $T\colon G \times H \to H \times G$ interchanges the factors, and also $T'\colon E_G \times E_H \to E_H \times E_G$, then we have the commutative law $T'\xi_{G,H} = \xi_{H,G}\tilde{T}$ because both sides have the components $\tilde{q}$, $\tilde{p}$.

If $d\colon G \to G \times G$ and $d'\colon E_G \to E_G \times E_G$ are diagonal maps, we have $\xi_{G,G}\tilde{d} = d'$. This holds because $\tilde{p}\tilde{d} = (pd)^{\sim} = \tilde{1}$ and similarly $\tilde{q}\tilde{d} = \tilde{1}$.

Let us assign to $E_G \times E_H$ the standard filtration for a product:

$$(E_G \times E_H)_n = \bigcup_{i=0}^{n} E_{G,i} \times E_{H,n-i}.$$

Since $\tilde{p}$ and $\tilde{q}$ preserve filtrations, it follows that ξ maps filtration n into filtration $2n$ for each n.

6.2. THEOREM. *The mapping $\xi_{G,H}$ of 6.1 is a homeomorphism, hence ξ is a natural equivalence. Moreover, $\xi_{G,H}^{-1}$ preserves filtrations.*

Proof. In the diagram

$$\begin{array}{ccc}
(G \times H)^n \times \Delta_n & \xrightarrow{\;k_n\;} & D_n(G \times H) \\
\Big\downarrow{\scriptstyle \xi_n} & & \Big\downarrow{\scriptstyle \xi} \\
(G^n \times \Delta_n) \times (H^n \times \Delta_n) & \xrightarrow{\;k_n \times k_n\;} & D_n(G) \times D_n(H)
\end{array}$$

let k_n be defined as in 5.2, and define ξ_n by

(6.3) $$\xi_n[(g_1, h_1), t_1, \ldots, (g_n, h_n), t_n] = ([g_1, t_1, \ldots, g_n, t_n],\ [h_1, t_1, \ldots, h_n, t_n]).$$

It is readily checked that the diagram is commutative for each n. Define $N_n(G)$ and $N_n(H)$ as in 5.3. If ξ_n in 6.3 is applied to an element in normal form of length $\leqq n$, the components on the right need not be in normal form, but may be reduced to normal form by deleting factors of the form (e, t) (see 5.5). Let ξ_n' be the resulting map of normal forms, giving a commutative diagram:

(6.4)
$$\begin{array}{ccc}
N_n(G \times H) & \xrightarrow{k_n} & D_n(G \times H) \\
\downarrow{\scriptstyle \xi_n'} & & \downarrow{\scriptstyle \xi} \\
N_n(G) \times N_n(H) & \xrightarrow{k_n \times k_n} & D_n(G) \times D_n(H)
\end{array}$$

Define a map $\zeta_{p,q}: N_p(G) \times N_q(H) \to N_{p+q}(G \times H)$ as follows. Let $x = [g_1, s_1, \ldots, g_a, s_a]$ and $y = [h_1, t_1, \ldots, h_b, t_b]$ be in normal form (5.3) where $a \leqq p$ and $b \leqq q$. Let $u_1, \ldots, u_r$ denote the union of the distinct s and t-values of x, y arranged in ascending order $0 < u_1 < \cdots < u_r \leqq 1$. For each $j = 1, \ldots, r$, define g_j' to be g_i if $u_j = s_i$ for some i, otherwise $g_j' = e$. Similarly, $h_j' = h_i$ if $u_j = t_i$ for some i, otherwise $h_j' = e$. Define

$$\zeta(x, y) = [(g_1', h_1'), u_1, \ldots, (g_r', h_r'), u_r]$$

It is readily checked that $\zeta(x, y)$ is in normal form. It is also readily checked that ζ is an inverse of ξ' in the sense that $\xi'_{p+q}\zeta_{p,q}$ is the inclusion of $N_p(G) \times N_q(H)$ in $N_{p+q}(G) \times N_{p+q}(H)$, and $\zeta_{2n}\xi_n'$ is the inclusion of $N_n(G \times H)$ in $N_{2n}(G \times H)$. Since k_n restricted to normal forms is bijective (5.4), it follows now from 6.4 that ξ is bijective. Since ζ maps filtration p, q into filtration $p + q$, it follows that ξ^{-1} preserves filtrations.

We shall now show that ξ^{-1} is continuous. The proof is based on the following diagram:

(6.5)
$$\begin{array}{ccccc}
G^m \times H^n \times \Delta_m \times \Delta_n & \xrightarrow{T} & G^m \times \Delta_m \times H^n \times \Delta_n & \xrightarrow{k_m \times k_n} & D_m(G) \times D_n(H) \\
\uparrow{\scriptstyle i_\alpha} & & & & \downarrow{\scriptstyle \xi^{-1}} \\
G^m \times H^n \times K_\alpha & \xrightarrow{\alpha' \times \alpha} & (G \times H)^{m+n} \times \Delta_{m+n} & \xrightarrow{k_{m+n}} & D_{m+n}(G \times H)
\end{array}$$

The mapping T interchanges the two middle factors. Let α be any (m, n)-shuffle, let K_α be the subset of those elements of $\Delta_m \times \Delta_n$ whose coordinates are brought into (weakly) increasing order by the shuffle α, and let i_α be the indicated inclusion map. The map $\alpha': G^m \times H_n \to (G \times H)^{m+n}$ replaces each $g \in G$ by $(g, e) \in G \times H$, each $h \in H$ by $(e, h) \in G \times H$, and then performs the shuffle α on the resulting factors.

Since $(k_m \times k_n)T$ is proclusive, the continuity of ξ^{-1} will follow from that of $\xi^{-1}(k_m \times k_n)T$. Since each K_α is a closed set and their union is $\Delta_m \times \Delta_n$, it suffices to show that $\xi^{-1}(k_m \times k_n)Ti_\alpha$ is continuous for each α where i_α is the inclusion. The mappings on the bottom row are obviously continuous. Thus we have only to prove that the diagram is commutative. Let $r = m + n$, and form the following diagram

$$\begin{array}{ccccc}
G^m \times H^n \times \Delta_m \times \Delta_n & \xrightarrow{bT} & G^r \times \Delta_r \times H^r \times \Delta_r & \xrightarrow{k_r \times k_r} & D_r(G) \times D_r(H) \\
\uparrow{\scriptstyle i_\alpha} & & \uparrow{\scriptstyle \xi_r} & & \uparrow{\scriptstyle \xi} \\
G^m \times H^n \times K_\alpha & \xrightarrow{\alpha' \times \alpha} & (G \times H)^r \times \Delta_r & \xrightarrow{k_r} & D_r(G \times H)
\end{array}$$

where ξ_r is defined in 6.3 and b is the obvious inclusion mapping. We observed earlier that the right rectangle is commutative. The left rectangle is not commutative; however, the lower route gives an element differing from that of the upper route only in the presence of a number of extra factors (e, t), and these have the same image under $k_r \times k_r$. Thus the long rectangle is commutative. Since it contains the preceding rectangle, it too is commutative. This completes the proof.

§7. TOPOLOGICAL GROUPS

In this section we assume that G is a topological monoid with a morphism

$$\text{(7.1)} \qquad \text{Ad}: G \to \text{Auto } G$$

such that $gg' = ((\text{Ad } g)g')g$ for all $g, g' \in G$, and $(\text{Ad } g)g'$ is continuous from $G \times G$ to G. If G is a topological group, we have $(\text{Ad } g)g' = gg'g^{-1}$. If G is an abelian H-space, we have $(\text{Ad } g)g' = g'$. For convenience we shall write $gg'g^{-1}$ instead of $(\text{Ad } g)g'$ even when g has no inverse.

Let $\tilde{E}$ be the free associative monoid generated by all pairs $(g, t) \in G \times I$. As a set it is $(G \times I)^\omega = \bigcup_{n=0}^{\infty} (G \times I)^n$, each element being a monomial $(g_1, t_1) \cdots (g_n, t_n)$. Multiplication is defined by the usual identifications $(G \times I)^m \times (G \times I)^n = (G \times I)^{m+n}$ (the juxtaposition of monomials). The unit is the empty monomial corresponding to $n = 0$.

7.2. *Definition.* Let E_G' be the quotient monoid obtained by reducing $\tilde{E}$ by the following three sets of relations:

(1) $(g, 0) = (e, t) =$ the unit e of E_G' for all $g \in G$, $t \in I$,
(2) $(g, t)(g', t) = (gg', t)$ for all $g, g' \in G$, $t \in I$,
(3) if $0 < t' < t \leqq 1$ and $g, g' \in G$, then

$$(g, t)(g', t') = (gg'g^{-1}, t')(g, t).$$

To be precise, two monomials m, m' of $\tilde{E}$ are equivalent if there is a sequence of monomials $m = m_1, m_2, \ldots, m_k = m'$ such that one may pass from any m_i to m_{i+1} by an operation of type 1, 2 or 3 or its inverse applied to some factor or pair of successive factors of m_i. The equivalence classes in $\tilde{E}$ are the elements of E_G'. It is readily seen that the multiplication in $\tilde{E}$ induces one in E_G' so that the natural mapping $\tilde{E} \to E_G'$ preserves products. We do not assign any topology to E_G' until Theorem 7.6 below.

For a fixed $t > 0$, the set of (g, t) for all $g \in G$ forms a submonoid isomorphic to G. We identify G with the submonoid corresponding to $t = 1$.

If G is abelian, it follows from the relations of type 3 that E_G' is abelian. In case G has inverses so also E_G' because, by (2), $(g, t)^{-1} = (g^{-1}, t)$. Thus if G is a group so also is E_G'.

7.3. *Definition.* A monomial $(g_1, t_1) \cdots (g_k, t_k)$ is said to be in *semi-normal form* if $0 \leqq t_1 \leqq \cdots \leqq t_k \leqq 1$. It is said to be in *normal form* if $0 < t_1 < \cdots < t_k \leqq 1$ and each $g_i \in G - e$. The empty monomial representing e is also said to be in normal form.

7.4. LEMMA. *Each monomial is equivalent to one and only one monomial in normal form.*

Proof. Starting with an arbitrary monomial, we may reduce it to semi-normal form using only type 3 operations. Then, if there are any factors with equal t's we combine them by type 2 relations, obtaining thus a monomial such that $0 \leqq t_1 < t_2 < \cdots < t_k \leqq 1$. Finally, using type 1 relations, we may delete all factors of the forms $(g, 0)$ and (e, t). The resulting monomial is in normal form.

To prove uniqueness, we define for each monomial $m = (g_1, t_1) \cdots (g_k, t_k)$ a function $m: (0, 1] \to G$ as follows. For each $j = 1, 2, \ldots, k$, let b_j denote the product in order of those g_i such that $i < j$ and $t_i > t_j$, and set $\bar{g}_j = b_j g_j b^{-1}$. Now set

$$m(t) = \text{the product in order of all } \bar{g}_j \text{ such that } t_j = t.$$

In case $t \neq t_j$ for all j, we set $m(t) = e$. Notice that if m is in normal form, then $m(t_j) = g_j$ for each factor (g_j, t_j), and otherwise $m(t) = e$. We must show that an equivalence $m \equiv m'$ of two monomials implies $m(t) = m'(t)$ for all t. It is enough to show this when m, m' are related by a single application of a relation of one of the three types.

There are four cases to distinguish. In all cases $m = (g_1, t_1) \cdots (g_k, t_k)$ and m' is obtained by an operation involving the factor (g_s, t_s) of m. In case 1, $t_s = 0$ and m' is obtained by deleting (g_s, t_s). In case 2, $g_s = e$ and m' is obtained by the same deletion. In case 3, $t_s = t_{s+1}$ and m' is obtained by replacing the two factors $(g_s, t_s)(g_{s+1}, t_{s+1})$ by one $(g_s g_{s+1}, t_s)$. In case 4, $t_s > t_{s+1}$ and m' is obtained by replacing $(g_s, t_s)(g_{s+1}, t_{s+1})$ by $(g_s g_{s+1} g_s^{-1}, t_{s+1})(g_s, t_s)$. Since the complete proof that $m(t) = m'(t)$ is lengthy and mostly routine, we will outline the main steps and give details for case 4 only.

We compare first the computations of $\bar{g}_j$ and $\bar{g}_j'$ in m and m'. Since m and m' coincide in all factors preceding the sth, and $\bar{g}_j$ depends only on the factors up to and including the jth, it follows that $\bar{g}_j = \bar{g}_j'$ for $j < s$. For the same reason, the factor b_s used to conjugate g_s is also used for g_s'. In case 4, we obtain

$$\bar{g}_s = b_s g_s b_s^{-1}, \qquad \bar{g}_{s+1} = (b_s g_s) g_{s+1} (b_s g_s)^{-1}$$

$$\bar{g}_s' = b_s(g_s g_{s+1} g_s^{-1}) b_s^{-1}, \qquad \bar{g}_{s+1}' = b_s g_s b_s^{-1}$$

Consider now a $j > s + 1$, and case 4. If $t_s \leqq t_j$, then also $t_{s+1} \leqq t_j$, hence the s and $(s + 1)$st factors of m contribute only e's to the factor b_j in $\bar{g}_j = b_j g_j b_j^{-1}$. Interchanging t_s, t_{s+1} does not alter this conclusion, hence $b_j' = b_j$. Since $g_j' = g_j$, we have $\bar{g}_j' = \bar{g}_j$. If $t_{s+1} \leqq t_j < t_s$, b_j obtains the factor g_s from the sth factor of m, and e from the $(s + 1)$st, while b_j' obtains e from the sth factor of m', and g_s from the $(s + 1)$st. Since b_j, b_j' have otherwise the same factors it follows that $b_j = b_j'$, whence $\bar{g}_j = \bar{g}_j'$. If $t_j < t_{s+1}$, b_j obtains the factors g_s and g_{s+1} from the factors s and $s + 1$ of m respectively, while b_j' obtains $g_s g_{s+1} g_s^{-1}$, g_s instead. Since b_j, b_j' have otherwise the same factors, it follows that $b_j = b_j'$, hence $\bar{g}_j = \bar{g}_j'$. Thus in case 4, $\bar{g}_j = \bar{g}_j'$ except for $j = s$ and $s + 1$, and these are given above.

Consider now the computations of $m(t)$ and $m'(t)$. If t is not one of the $t_1, \ldots, t_k$ in m, it also does not occur in m', hence, by definition, $m(t) = e = m'(t)$. If $t = t_j$ for some j but $t \neq t_s$ or t_{s+1} (case 4), we have $\bar{g}_j = \bar{g}_j'$ for every j such that $t = t_j$, hence their products $m(t)$ and $m'(t)$ are equal. If $t = t_s$, we have $\bar{g}_j = \bar{g}_j'$ for $j \neq s$ and $t_j = t$, hence $m(t)$ and $m'(t)$ receive the same factors from corresponding factors of m and m' except that $m(t)$ obtains $b_s g_s b_s^{-1}$ from factor s and an e from factor $s + 1$, while m' obtains an e and $b_s g_s b_s^{-1}$ from the corresponding factors. Hence $m(t_s) = m'(t_s)$. Now take $t = t_{s+1}$. Again $m(t)$ and $m'(t)$ receive the same factors from corresponding factors of m and m' except for the factors s and $s + 1$, and, for these, $m(t)$ obtains e, $\bar{g}_{s+1}$ as given above, and $m'(t)$ obtains $\bar{g}_s'$, e. Since $\bar{g}_s' = \bar{g}_{s+1}$ it follows that $m(t_{s+1}) = m'(t_{s+1})$. This completes our proof that $m(t) = m'(t)$ in case 4. The other cases are less difficult.

As observed earlier, a monomial m in normal form can be reconstructed from its $m(t)$ because $m(t_i) = g_i$ for each of its factors (g_i, t_i) and $m(t) = e$ for other t's. The invariance of $m(t)$ under equivalence implies therefore the uniqueness of the normal form. This completes the proof of lemma 7.4.

Remark. In case G is a group, a simpler proof is obtained by defining $m(t)$ to be the product of those g_i occurring in m such that $t_i > t$. Inverses are needed to reconstruct from this $m(t)$ the normal form of m.

7.5. *Definition.* Define $k: E_G' \to E_G$ by assigning to the element of E_G' whose normal form is $(g_1, t_1) \cdots (g_m, t_m)$ the element $k_m[g_1, t_1, \ldots, g_m, t_m]$ of E_G (see 5.1). In the special case $m = 0$, k maps $e \in E_G'$ into $e = D_0$ in E_G.

7.6. THEOREM. *Assuming that G satisfies* 7.1, *then the following hold.*

(a) *The mapping k defined in* 7.5 *is bijective.*

(b) *For each m,* $D_m - D_{m-1}$ *corresponds bijectively under k to precisely those elements whose normal forms have length m, and* $E_{m-1} - D_{m-1}$ *corresponds to the subset with* $t_m = 1$.

(c) *Under k the submonoid G of* E_G' *corresponds to* E_0, *and the action mapping* $\phi: E_G \times G \to E_G$ *of* §4 *coincides under k with right translation.*

(d) *Let* ϕ *denote the multiplication defined in* E_G *by taking over the multiplication in* E_G' *under k. Then* ϕ *is continuous, hence* E_G *is a topological monoid.*

(e) *If G is a group (i.e. G has a continuous inverse), then* E_G *is also a group.*

(f) *If both G and H satisfy* 7.1 *and* $f: G \to H$ *is a morphism, then the natural mapping* $\bar{f}: E_G \to E_H$ *is a morphism of monoids.*

(g) *If both G and H satisfy* 7.1, *then the natural equivalence* $\xi: E_{G\times H} \to E_G \times E_H$ *is an isomorphism of monoids.*

Proof. (b) is an immediate consequence of 5.1c; and (a) follows from (b). It is easy to verify (c).

To prove (d) it suffices to show that the multiplication mapping $\phi: D_m \times D_n \to D_{m+n}$ is continuous for all m, n because $E_G \times E_G$ has the topology of the union of the sets $D_m \times D_n$ (see [8; 10.3]). For each (m, n)-shuffle α, let K_α denote the set of those points of $\Delta_m \times \Delta_n$ whose coordinates $((s_1, \ldots, s_m), (t_1, \ldots, t_n))$ are brought into weakly increasing order by the shuffle α. We have then the diagram

(7.7)
$$\begin{array}{ccccc}
D_m \times D_n & \xleftarrow{k_m \times k_n} & G^m \times \Delta_m \times G^n \times \Delta_n & \xleftarrow{T} & G^m \times G^n \times \Delta_m \times \Delta_n \\
\Big\downarrow{\scriptstyle \phi} & & & & \Big\uparrow{\scriptstyle i_\alpha} \\
D_{m+n} & \xleftarrow{k_{m+n}} & G^{m+n} \times \Delta_{m+n} & \xleftarrow{\alpha' \times \alpha} & G^m \times G^n \times K_\alpha
\end{array}$$

where i_α is an inclusion, T interchanges the two middle factors G^n and Δ_m, and $\alpha'(g_1, \ldots, g_m; h_1, \ldots, h_n)$ is obtained by first replacing each h_j by $c_j h_j c_j^{-1}$ where c_j is the product of the g_i's that h_j must pass in the shuffle α, and then performing the shuffle α.

Commutativity of the diagram is seen as follows. Starting with $u \in G^m \times G^n \times K_\alpha$, we obtain $(1 \times T \times 1)i_\alpha u = (u_1, u_2)$ where u_1, u_2 represent the elements $k_m u_1$ and $k_n u_2$ in semi-normal form. Similarly $(\alpha' \times \alpha)u$ represents $k_{m+n}(\alpha' \times \alpha)u$ in semi-normal form. Now the product $\phi(k_m u_1, k_n u_2)$ of these elements in the semi-normal forms u_1, u_2 can be reduced to a semi-normal form by applying type 3 operations alone, and the result is seen to be $(\alpha' \times \alpha)u$. Therefore $k_{m+n}(\alpha' \times \alpha)u = \phi(k_m u_1, k_n u_2)$ as required.

By 5.1a, the maps k_m, k_n are proclusive, hence also their product [8; 4.4], and also $(k_m \times k_n)T$. Thus to prove that ϕ is continuous, it suffices to prove that $\phi(k_m \times k_n)T$ is continuous. Since the sets $G^m \times G^n \times K_\alpha$, for all shuffles α, cover $G^m \times G^n \times \Delta_m \times \Delta_n$ and are closed, it suffices to prove the continuity on each of them. But commutativity of the diagram implies that $\phi(k_m \times k_n)T$ restricted to $G^m \times G^n \times K_\alpha$ is $(\alpha' \times \alpha)k_{m+n}$, and this mapping is clearly continuous.

To prove (e), it suffices to prove the continuity of $\lambda x = x^{-1}$ on each D_m because E_G has the topology of their union. This is based on the diagram

$$\begin{array}{ccc} G^m \times \Delta_m & \xrightarrow{k_m} & D_m \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \lambda} \\ G^m \times \Delta_m & \xrightarrow{k_m} & D_m \end{array}$$

The mapping μ is defined by $\mu[g_1, t_1, \ldots, g_m, t_m] = [g_1', t_1, \ldots, g_m', t_m]$ where

$$g_k' = (g_{k+1} \cdots g_m)^{-1} g_k^{-1} (g_{k+1} \cdots g_m) \quad \text{for} \quad k = 1, 2, \ldots, m.$$

It is readily checked that the diagram commutes. Since k_m is a proclusion and μ is obviously continuous, it follows that λ is continuous.

To prove (f), it is enough to show that $\bar{f}$ (see 4.1) preserves products. This is a triviality one has only to check that the construction from G to E_G' is a functor, and that the mapping $k: E_G' \to E_G$ is a natural transformation of functors.

To prove (g), it suffices to show that ξ preserves products. Since the projections of $G \times H$ into G and H preserve products, it follows from (f) that the associated mappings $E_{G \times H}$ into E_G and E_H also preserve products. Since these are the components of ξ and $E_G \times E_H$ is a direct product, the assertion follows.

§8. THE FIBRATION $E_G \to B_G$

Recall the definition of Dold and Thom [2]: a mapping $p: E \to B$ is called a *quasifibration* if $pE = B$, and

$$p_*: \pi_i(E, p^{-1}x, y) \approx \pi_i(B, x) \quad \text{for all} \quad x \in B, y \in p^{-1}x, i \geqq 0.$$

8.1. THEOREM. *Let G be a topological monoid with unit e such that (G, e) is an NDR.*

Assume also that each left translation of G induces isomorphisms of all homotopy groups. Then $p: E_G \to B_G$ *is a quasifibration.*

Proof. Our proof, in outline, is the same as that of Dold and Lashof [1; Prop. 2.3]. Since B_0 is a point, $E_0 \to B_0$ is a quasifibration. Assume inductively that, for some n, $E_n \to B_n$ is a quasifibration. We shall show that $E_{n+1} \to B_{n+1}$ is a quasifibration. By 4.2, there is a representation $\bar{u}$, $\bar{h}$ of (E_{n+1}, E_n) as a G-NDR; let $\tilde{u}$, $\tilde{h}$ denote the induced representation of the quotient (B_{n+1}, B_n) as an NDR (see 4.3). Set $V = B_{n+1} - B_n$ and $U = \tilde{u}^{-1}[0, 1)$. Then $B_{n+1} = U \cup V$. Since $E_{n+1} - E_n = p^{-1}V \to V$ is the projection of a product structure, V is a distinguished set (i.e. $p^{-1}V \to V$ is a quasifibration). For the same reason $U \cap V$ is a distinguished set.

The homotopy $\bar{h}$ restricted to $p^{-1}U \times I$ is a deformation retraction of $p^{-1}U$ into E_n, and covers the deformation $\tilde{h}|(U \times I)$ of U into B_n. Let $\bar{h}_1 = \bar{h} | E_{n+1} \times 1$ and $\tilde{h}_1 = \tilde{h} | B_{n+1} \times 1$. We claim that

(8.2) $$(\bar{h}_1 | p^{-1}x)_*: \pi_i(p^{-1}x) \approx \pi_i(p^{-1}\tilde{h}_1 x) \quad \text{for all} \quad x \in B_{n+1} \quad \text{and} \quad i \geqq 0.$$

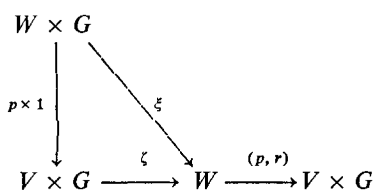
For $x \in B_n$, this is trivial since $\bar{h}_1$ and $\tilde{h}_1$ restrict to identities. For $x \in B_{n+1} - B_n$, $p^{-1}x$ is a copy of G under its action on the point $y = p^{-1}x \cap D_{n+1}$. Since $p^{-1}\tilde{h}_1 x$ is a copy of G, it has the form bG for some $b \in p^{-1}\tilde{h}_1 x$, and then $\bar{h}_1 y = bg_0$ for some $g_0 \in G$. Since $\bar{h}_1$ is a G-mapping, we have $\bar{h}_1(yg) = bg_0 g$. Thus $\bar{h}_1$ on $p^{-1}x$ is just a copy of the left translation of G by g_0; hence 8.2 holds. Since B_n is a distinguished set by the inductive hypothesis, it follows now from [2; 2.10] that U is a distinguished set. Since U, V and $U \cap V$ are distinguished, we may apply [2; 2.2] to conclude that $U \cup V = B_{n+1}$ is distinguished. This concludes the inductive step; hence B_n is distinguished for every n. Since B_G has the topology of the union $\bigcup_0^\infty B_n$, it follows from [2; 2.15] that B_G itself is distinguished. This completes the proof.

8.3. THEOREM. *Let G be a topological group such that* (G, e) *is an NDR. Then* E_G *is a principal G-bundle over* B_G *with the action* $E_G \times G \to E_G$ *as principal map.*

Proof. Since E_G is a topological group and G is a closed subgroup, it suffices to prove that G has a neighborhood W which is a product space over $pW = V$. By 4.2, the pair (E_G, E_0) has a representation as a G-NDR by mappings u, h. Set $W = u^{-1}[0, 1)$. Since W is open and is G-invariant, it follows that $V = pW$ is open in B_G. Define $r: W \to G$ by $ry = h(y, 1)$. Note that r is a G-mapping. Define

$$\xi: W \times G \to W \quad \text{by} \quad \xi(y, g) = y(ry)^{-1}g \quad \text{for all} \quad (y, g) \in W \times G.$$

(It is in this definition of ξ that the existence of inverses in G is needed.) All the maps of the diagram



have been defined excepting ζ. We shall show that ξ induces a map ζ of its quotient space

$V \times G$ such that $\zeta(p \times 1) = \xi$. Clearly a point of $W \times G$ has the same image as (y, g) under $p \times 1$ if and only if it has the form (yg', g) for some $g' \in G$. Then

$$\begin{aligned}\xi(yg', g) &= yg'(r(yg'))^{-1}g = yg'((ry)g')^{-1}g = yg'g'^{-1}(ry)^{-1}g \\ &= y(ry)^{-1}g = \xi(y, g).\end{aligned}$$

Therefore ξ induces a unique function ζ such that $\zeta(p \times 1) = \xi$. Since V is a quotient space of W, it follows from [8; 4.4] that $V \times G$ is a quotient space of $W \times G$, and this implies that ζ is continuous.

Since ξ is a G-mapping, so also is ζ. Now $p\xi(y, g) = p(y(ry)^{-1}g) = py$, and

$$r\xi(y, g) = r(y(ry)^{-1}g) = (ry)((ry)^{-1}g) = g.$$

Therefore ζ composed with (p, r) is the identity of $V \times G$. On the other hand, if $y \in W$, $\xi(y, ry) = y(ry)^{-1}(ry) = y$, and this shows that (p, r) composed with ζ is the identity of W. Therefore ζ is the required representation of W as a product $V \times G$. This concludes the proof.

§9. COMPLEXES ON G, E_G AND B_G, AND THE BAR RESOLUTION

We assume in this section that G is also a complex such that e is a vertex and the multiplication $G \times G \to G$ is skeletal. Let $I = [0, 1]$ have the cellular structure consisting of two vertices $\bar{0}$, $\bar{1}$ and one edge denoted by $\delta_1 = (0, 1)$. We shall construct now the associated complexes (or reticulations) of D_n, E_n and B_n.

The reticulation of E_0 comes from its identification with G, each cell of G is a cell of E_0. The I-structure on $D_0 = e$ is given by a skeletal map $e \times I \to e$, hence, by 3.5, D_1 has a reticulation such that the natural maps $E_0 \times I \to D_1$ and $D_1 \times I \to D_1$ are skeletal. Recall that the first map is a homeomorphism from $(E_0 - D_0) \times \delta_1$ to $D_1 - E_0$. We denote by $[\sigma]$ the image cell of $\sigma \times \delta_1$ in $D_1 - E_0$. The general stage is described as follows.

9.1. THEOREM. *Starting with the reticulation of $E_0 = G$ and alternating the constructions of Lemmas 3.4 and 2.5, we obtain reticulations of D_n, E_n for each n; their union is a functorial reticulation of E_G such that the action $E_G \times G \to E_G$ and the contraction $E_G \times I \to E_G$ are skeletal. The cells of $D_n - E_{n-1}$ are in 1–1 correspondence with sequences of cells of $G - e$ of length n; the cell corresponding to $\sigma_1, \dots, \sigma_n$ is denoted by $[\sigma_1 | \cdots | \sigma_n]$. The cells of $E_n - D_n$ are in 1–1 correspondence with sequences of cells of $G - e$ of length $n + 1$; the cell corresponding to $\sigma_1, \dots, \sigma_{n+1}$ is denoted by $[\sigma_1 | \cdots | \sigma_n]\sigma_{n+1}$. These cells are defined by the inductive conditions*

$$[\sigma_1 | \cdots | \sigma_n] = \mu(([\sigma_1 | \cdots | \sigma_{n-1}]\sigma_n) \times \delta_1) \tag{9.2}$$

$$[\sigma_1 | \cdots | \sigma_n]\sigma_{n+1} = \lambda([\sigma_1 | \cdots | \sigma_n] \times \sigma_{n+1}) \tag{9.3}$$

where μ: $E_{n-1} \times I \to D_n$ and λ: $D_n \times G \to E_n$ are the quotient maps occurring in the constructions of D_n and E_n. In case $n = 0$, the cell $[\ ]$ corresponding to the empty sequence is e, and $[\ ]\sigma$ is the cell σ of $G - e = E_0 - D_0$. Moreover, k_n maps the cell $\sigma_1 \times \cdots \times \sigma_n \times \delta_n$ of $G^n \times \Delta_n$ homeomorphically onto $[\sigma_1| \cdots | \sigma_n]$, and k_{n+1} maps the cell $\sigma_1 \times \cdots \times \sigma_{n+1} \times \delta_n$ of $G^{n+1} \times \Delta_{n+1}$ homeomorphically onto $[\sigma_1| \cdots | \sigma_n]\sigma_{n+1}$.

Proof. The proofs of the statements of the first sentence are straightforward. To prove that the cells are as described, assume inductively that the cells of $E_{n-1} - D_{n-1}$ have the form $[\sigma_1|\cdots|\sigma_{n-1}]\sigma_n$. Now μ maps $(E_{n-1} - D_{n-1}) \times \delta_1$ homeomorphically onto $D_n - E_{n-1}$, hence each cell of $D_n - E_{n-1}$ has the form $\mu(\tau \times \delta_1)$ where τ is a cell of $E_{n-1} - D_{n-1}$. Since τ has the form $[\sigma_1|\cdots|\sigma_{n-1}]\sigma_n$, it follows from the definition 9.2, that each cell of $D_n - E_{n-1}$ has the form $[\sigma_1|\cdots|\sigma_n]$. Now λ maps $(D_n - E_{n-1}) \times (G - e)$ homeomorphically onto $E_n - D_n$, hence each cell of $E_n - D_n$ has the form $\lambda(\rho \times \sigma)$ where ρ is a cell of $D_n - E_{n-1}$ and σ is a cell of $G - e$. Since ρ has the form $[\sigma_1|\cdots|\sigma_n]$, it follows from the definition 9.2 that each cell of $E_n - D_n$ has the form $[\sigma_1|\cdots|\sigma_n]\sigma$.

To prove the cellular property of k_n, let $\sigma_1, \ldots, \sigma_n$ be a sequence of cells of $G - e$. Consider the cell $\rho = \sigma_1 \times \cdots \times \sigma_{n-1} \times \delta_{n-1} \times \sigma_n \times \delta_1$ of $G^{n-1} \times \Delta_{n-1} \times G \times I$ (see 5.2). Since ψ maps $\delta_{n-1} \times \delta_1$ homeomorphically onto δ_n, we have that $(1 \times \psi)T$ maps ρ homeomorphically onto $\sigma_1 \times \cdots \times \sigma_n \times \delta_n$. Thus the k_n-image of $\sigma_1 \times \cdots \times \sigma_n \times \delta_n$ is the $\mu(\lambda \times 1)(k_{n-1} \times 1 \times 1)$ – image of ρ. The inductive hypothesis on k_{n-1} and 9.2, 9.3 give

$$\begin{aligned}\mu(\lambda \times 1)(k_{n-1} \times 1 \times 1)\rho &= \mu(\lambda \times 1)([\sigma_1|\cdots|\sigma_{n-1}] \times \sigma_n \times \delta_1)\\ &= \mu([\sigma_1|\cdots|\sigma_{n-1}]\sigma_n \times \delta_1) = [\sigma_1|\cdots|\sigma_n],\end{aligned}$$

and this is the required form.

Corresponding to a sequence $\sigma_1, \ldots, \sigma_{n+1}$ of cells of $G - e$, form the cell $\tau = \sigma_1 \times \cdots \times \sigma_n \times \delta_n \times \sigma_{n+1} \times 1$ of $G^n \times \Delta_n \times G \times I$ (see 5.2 with n replaced by $n + 1$). Since $(1 \times \psi)T$ maps τ homeomorphically onto $\sigma_1 \times \cdots \times \sigma_{n+1} \times \delta_n$, the k_{n+1}-image of this latter cell is the $\mu(\lambda \times 1)(k_n \times 1 \times 1)$-image of τ. Using what was proved above for k_n and 9.2, 9.3, we obtain

$$\begin{aligned}\mu(\lambda \times 1)(k_n \times 1 \times 1)\tau &= \mu(\lambda \times 1)([\sigma_1|\cdots|\sigma_n] \times \sigma_{n+1} \times 1)\\ &= \mu([\sigma_1|\cdots|\sigma_n]\sigma_{n+1} \times 1) = [\sigma_1|\cdots|\sigma_n]\sigma_{n+1},\end{aligned}$$

which is the stated form. This completes the proof.

9.4. THEOREM. *There is a unique reticulation of B_G satisfying the conditions*: (1) *each B_n is a subcomplex*, (2) *each projection $E_n \to B_n$ is skeletal, and* (3) *B_n is the complex formed by attaching the complex D_n to B_{n-1} by the* (*skeletal*) *projection $E_{n-1} \to B_{n-1}$*.

Proof. Clearly the conditions provide an inductive definition of the reticulations of the B_n's provided we show at each stage that p: $E_n \to B_n$ is skeletal. Let τ be any cell of E_n; by 9.3, either τ is in D_n or it has the form $\lambda(\rho \times \sigma)$ where $\rho \times \sigma$ is a cell of $D_n \times G$ of the same dimension as τ. If τ is in D_n, we also have $\lambda(\tau \times e) = \tau$. Now $p\lambda$: $D_n \times G \to B_n$ can be factored into the projection q: $D_n \times G \to D_n$ followed by $p' = p \mid D_n$. Since p' and q are skeletal, so is $p\lambda$, hence $p\tau = p\lambda(\rho \times \sigma)$ lies in the r-skeleton of B_n where $r = \dim(\rho \times \sigma) = \dim \tau$. Therefore p is skeletal, and the proof is complete.

9.5. THEOREM. *If G and H have reticulations such that the multiplications $G \times G \to G$ and $H \times H \to H$ are skeletal, then the mapping ξ^{-1}: $E_G \times E_H \to E_{G\times H}$ of* 6.2 *is skeletal.*

To prove the theorem it suffices to show that ξ^{-1} in the diagram 6.5 is skeletal. Using 9.1, any cell $\sigma \times \tau$ of $D_m(G) \times D_n(H)$ has the form $k_m\sigma' \times k_n\tau'$; hence it is the image of a cell of $G^m \times H^n \times \Delta_m \times \Delta_n$ of the same dimension. Thus it suffices to show that $k_{m+n}(\alpha' \times \alpha)i_\alpha^{-1}$ is skeletal for each α. Since k_{m+n} is skeletal, we are reduced to studying $(\alpha' \times \alpha)i_\alpha^{-1}$. Since α' imbeds $G^m \times H^n$ as a subcomplex of $(G \times H)^{m+n}$, we need only study α on K_α.

Let σ be a face of Δ_m of dimension q, and τ a face of Δ_n of dimension q. Let $s = (s_1, \ldots, s_m)$ be a point of σ, and $t = (t_1, \ldots, t_n)$ a point of τ such that $\alpha(s, t)$ is in increasing order, i.e. $(s, t) \in K_\alpha \cap (\sigma \times \tau)$. Of the possible equalities that may hold among the coordinates of s, namely, $0 = s_1, s_1 = s_2, \ldots, s_{m-1} = s_m, s_m = 1$, let $N(s)$ be the number that do hold. Then the smallest face of Δ_m containing s has dimension $m - N(s)$. Since σ is a face containing s, we have $q \geqq m - N(s)$, or $N(s) \geqq m - q$. Similarly, $N(t) \geqq n - r$. Set $u = \alpha(s, t)$. It is easily seen that $N(u) \geqq N(s) + N(t)$ because any equality of elements in s or t still holds after shuffling. Therefore $\alpha(s, t)$ is on a face of Δ_{m+n} of dimension

$$m + n - N(u) \leqq m - N(s) + n - N(t) \leqq q + r.$$

This completes the proof.

9.6. THEOREM. *If G is abelian, then the multiplications in E_G and B_G are skeletal mappings. If also G is a group, and $v: G \to G$, defined by $vg = g^{-1}$, is skeletal, then the induced maps $\tilde{v}$ and $\bar{v}$, defining inverses in E_G and B_G, are likewise skeletal.*

Proof. To prove that the multiplication ϕ for E_G is skeletal it suffices to prove that its restriction to $D_m \times D_n$ is skeletal for each m, n. Let σ be a q-cell of D_m, and τ an r-cell of D_n. By 9.1, $\sigma = k_m(\sigma_1 \times \sigma_2)$ where σ_1, σ_2 are cells of G^m, Δ_m of dimensions q_1 and $q_2 = q - q_1$, respectively. Similarly, $\tau = k_n(\tau_1 \times \tau_2)$. Referring to the diagram 7.7, we have

$$\phi(\sigma \times \tau) = \phi(k_m \times k_n)T(\sigma_1 \times \tau_1 \times \sigma_2 \times \tau_2).$$

Let $s = (s_1, \ldots, s_m)$ be a point of σ_2, $t = (t_1, \ldots, t_n)$ a point of τ_2, and α an (m, n)-shuffle such that $\alpha(s, t)$ is in increasing order, i.e. $(s, t) \in K_\alpha \cap (\sigma_2 \times \tau_2)$. Define $N(s)$ and $N(t)$ as in the proof of 9.5. Arguing exactly as in 9.5, we conclude that $\alpha(s, t)$ is on a face of Δ_{m+n} of dimension at most $q_2 + r_2$. Let $x \in \sigma_1$ and $y \in \tau_1$. Since G is abelian, $\alpha': G^m \times G^n \to G^{m+n}$ is just the shuffle α of the factors; since this is a skeletal mapping, $\alpha'(x, y)$ lies on a cell of dimension $q_1 + r_1$. Therefore $(\alpha' \times \alpha)i_\alpha^{-1}(x, y, s, t)$ lies on a $(q + r)$-cell. Since k_{m+n} is skeletal,

$$k_{m+n}(\alpha' \times \alpha)i_\alpha^{-1}(x, y, s, t) = \phi(k_m \times k_n)T(x, y, s, t)$$

lies in the $(q + r)$-skeleton. It follows that $\phi(\sigma \times \tau)$ is in the $(q + r)$-skeleton; hence ϕ is skeletal.

Let σ, τ be cells of B_G of dimensions q, r, respectively. By 9.2, there are cells σ', τ' of E_G of dimensions q, r, respectively, mapped by $p: E_G \to B_G$ onto σ and τ. Since B_G is a quotient group of E_G, $p\phi(\sigma' \times \tau')$ coincides with the image $\sigma\tau$ of $\sigma \times \tau$ under multiplication. Since ϕ and p are skeletal, $\sigma\tau = p\phi(\sigma' \times \tau')$ lies in the $(q + r)$-skeleton. Hence the multiplication in B_G is skeletal.

When G is an abelian group, the mapping v is a morphism of groups. If v is also skeletal, it follows from the functorial nature of the reticulations that $\tilde{v}$ and $\bar{v}$ are skeletal.

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