Journal of the Egyptian Mathematical Society (2015) 23, 95–101

Egyptian Mathematical Society
Journal of the Egyptian Mathematical Society
www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Fixed point results in ordered metric spaces for rational type expressions with auxiliary functions

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Received 9 August 2013; revised 19 December 2013; accepted 2 February 2014
Available online 15 March 2014

KEYWORDS
Fixed point;
Rational type generalized contraction mappings;
Ordered metric spaces

Abstract In this paper, we establish some fixed point results for mappings involving (ϕ, ψ)-rational type contractions in the framework of metric spaces endowed with a partial order. These results generalize and extend some known results in the literature. Four illustrative examples are given.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 41A50; 47H10; 54H25

1. Introduction

The Banach's contraction mapping principle is one of the most versatile elementary results of mathematical analysis. It is widely applied in different branches of mathematics and is regarded as the source of metric fixed point theory. There is a vast literature dealing with technical extensions and generalizations of Banach's contraction principle, some instances of these works are in [1–16].

In recent times, fixed point theory has developed rapidly in partially ordered metric spaces, that is, metric spaces endowed with a partial ordering. The theory originated at a relatively later point of time. An early result in this direction was established by Turinici in ordered metrizable uniform spaces [17]. Application of fixed point results in partially ordered metric spaces was made subsequently, for example, by Ran and Reurings [18] to solving matrix equations and by Nieto and Rodriguez-Lopez [19] to obtain solutions of certain partial differential equations with periodic boundary conditions. Recently, many researchers have obtained fixed point, common fixed point results in partially ordered metric spaces, some of which are in [20–33].

The purpose of this paper is to establish some fixed point results satisfying a generalized contraction mapping of rational type in metric spaces endowed with partial order using some auxiliary functions. Four illustrative examples are given.

2. Mathematical preliminaries

In [34], Dass and Gupta proved the following fixed point theorem.

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Peer review under responsibility of Egyptian Mathematical Society.

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http://dx.doi.org/10.1016/j.joems.2014.02.002

Theorem 2.1 [34]. *Let $(X,\ d)$ be a complete metric space and $T: X \to X$ a mapping such that there exist $\alpha,\ \beta \geqslant 0$ with $\alpha + \beta < 1$ satisfying*

$$d(Tx,\ Ty) \leqslant \alpha\, \frac{d(y,\ Ty)\ [1 + d(x,\ Tx)]}{1 + d(x,\ y)} + \beta\, d(x,\ y), \quad \text{for all } x,\ y \in X. \tag{2.1}$$

Then T has a unique fixed point.

In [35], Cabrera, Harjani and Sadarangani proved the above theorem in the context of partially ordered metric spaces.

Definition 2.1. Suppose $(X,\ \leqslant)$ is a partially ordered set and $T: X \to X$. T is said to be *monotone nondecreasing* if for all $x,\ y \in X$,

$$x \leqslant y \ \Rightarrow\ Tx \leqslant Ty. \tag{2.2}$$

Theorem 2.2 [35]. *Let $(X,\ \leqslant)$ be a partially ordered set and suppose that there exists a metric d in X such that $(X,\ d)$ is a complete metric space. Let $T: X \to X$ be a continuous and nondecreasing mapping such that (2.1) is satisfied for all $x,\ y \in X$ with $x \leqslant y$. If there exists $x_0 \in X$ such that $x_0 \leqslant Tx_0$ then T has a fixed point.*

Theorem 2.3 [35]. *Let $(X,\ \leqslant)$ be a partially ordered set and suppose that there exists a metric d in X such that $(X,\ d)$ is a complete metric space. Assume that if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \leqslant x$, for all $n \in \mathbf{N}$. Let $T: X \to X$ be a nondecreasing mapping such that (2.1) is satisfied for all $x,\ y \in X$ with $x \leqslant y$. If there exists $x_0 \in X$ such that $x_0 \leqslant Tx_0$ then T has a fixed point.*

Theorem 2.4 [35]. *In addition to the hypotheses of Theorem 2.2 (or Theorem 2.3), suppose that for every $x,\ y \in X$, there exists $u \in X$ such that $u \leqslant x$ and $u \leqslant y$. Then T has a unique fixed point.*

Khan et al. [36] initiated the use of a control function that alters distance between two points in a metric space, which they called an altering distance function.

Definition 2.2 [36]. A function $\phi : [0, \infty) \to [0, \infty)$ is called an *altering distance function* if the following properties are satisfied:

(i) ϕ is monotone increasing and continuous,
(ii) $\phi(t) = 0$ if and only if $t = 0$.

In our results in the following section we will use the following class of functions.

We denote

$\Phi = \{\phi : [0, \infty) \to [0, \infty) : \phi$ an altering distance function$\}$ and

$$\Psi = \{\psi : [0, \infty) \to [0, \infty) : \text{for any sequence } \{x_n\} \text{ in } [0,\ \infty) \text{ with } x_n \to t > 0,\ \underline{\lim}\ \psi(x_n) > 0\}.$$

We note that Ψ is nonempty. For, we define ψ on $[0, \infty)$ by $\psi(t) = e^t,\ t \in [0, \infty)$. Then $\psi \in \Psi$. Here we observe that $\psi(0) = 1 > 0$. On the other hand, if $\psi(t) = t^2,\ t \in [0, \infty)$, then $\psi \in \Psi$ and $\psi(0) = 0$.

Note: For any $\psi \in \Psi$, it is clear that $\psi(t) > 0$ for $t > 0$; and $\psi(0)$ need not be equal to 0.

3. Main results

Theorem 3.1. *Let $(X,\ \leqslant)$ be a partially ordered set and suppose that there exists a metric d on X such that $(X,\ d)$ is a complete metric space. Let $T: X \to X$ be a continuous and nondecreasing mapping such that for all $x,\ y \in X$ with $x \leqslant y$,*

$$\phi(d(Tx,\ Ty)) \leqslant \phi(M(x,\ y)) - \psi(N(x,\ y)), \tag{3.1}$$

where $\phi \in \Phi,\ \psi \in \Psi$,

$$M(x,\ y) = \max\left\{\frac{d(y,\ Ty)\ [1 + d(x,\ Tx)]}{1 + d(x,\ y)},\ \frac{d(y,\ Tx)\ [1 + d(x,\ Ty)]}{1 + d(x,\ y)},\ d(x,\ y)\right\}$$

and

$$N(x,\ y) = \max\left\{\frac{d(y,\ Ty)\ [1 + d(x,\ Tx)]}{1 + d(x,\ y)},\ d(x,\ y)\right\}.$$

If there exists $x_0 \in X$ with $x_0 \leqslant Tx_0$, then T has a fixed point.

Proof. If $Tx_0 = x_0$, then we have the result. Suppose that $x_0 < Tx_0$. Then we construct a sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n, \text{ for every } n \geqslant 0. \tag{3.2}$$

Since T is a nondecreasing mapping, we obtain by induction that

$$x_0 < Tx_0 = x_1 \leqslant Tx_1 = x_2 \leqslant \cdots \leqslant Tx_{n-1} = x_n \leqslant Tx_n = x_{n+1} \leqslant \cdots \tag{3.3}$$

If there exists $n \geqslant 1$ such that $x_{n+1} = x_n$, then from (3.2), $x_{n+1} = Tx_n = x_n$, that is, x_n is a fixed point of T and the proof is finished. Suppose that $x_{n+1} \neq x_n$, that is, $d(x_{n+1},\ x_n) \neq 0$, for all $n \geqslant 1$. Let $R_n = d(x_{n+1},\ x_n)$, for all $n \geqslant 0$.

Since $x_{n-1} < x_n$, for all $n \geqslant 1$, from (3.1), we have

$$\begin{aligned}
\phi(d(x_n, x_{n+1})) &= \phi(d(Tx_{n-1}, Tx_n)) \\
&\leqslant \phi\left(\max\left\{\frac{d(x_n, Tx_n)[1 + d(x_{n-1}, Tx_{n-1})]}{1 + d(x_{n-1}, x_n)},\ \frac{d(x_n, Tx_{n-1})[1 + d(x_{n-1}, Tx_n)]}{1 + d(x_{n-1}, x_n)},\ d(x_{n-1}, x_n)\right\}\right) \\
&\quad - \psi\left(\max\left\{\frac{d(x_n, Tx_n)[1 + d(x_{n-1}, Tx_{n-1})]}{1 + d(x_{n-1}, x_n)},\ d(x_{n-1}, x_n)\right\}\right) \\
&= \phi\left(\max\left\{\frac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)},\ \frac{d(x_n, x_n)[1 + d(x_{n-1}, x_{n+1})]}{1 + d(x_{n-1}, x_n)},\ d(x_{n-1}, x_n)\right\}\right) \\
&\quad - \psi\left(\max\left\{\frac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)},\ d(x_{n-1}, x_n)\right\}\right) \\
&= \phi(\max\{d(x_n, x_{n+1}),\ d(x_{n-1}, x_n)\}) - \psi(\max\{d(x_n, x_{n+1}),\ d(x_{n-1}, x_n)\}),
\end{aligned}$$

that is,

$$\phi(R_n) = \phi(\max\{R_n,\ R_{n-1}\}) - \psi(\max\{R_n,\ R_{n-1}\}). \tag{3.4}$$

If $R_n > R_{n-1}$, then from (3.4), we have

$$\phi(R_n) \leqslant \phi(R_n) - \psi(R_n), \text{ that is, } \psi(R_n) \leqslant 0,$$

which is a contradiction. So, $R_n \leqslant R_{n-1}$, that is, $\{R_n\}$ is a decreasing sequence. Then the inequality (3.4) yields that

$$\phi(R_n) \leqslant \phi(R_{n-1}) - \psi(R_{n-1}). \tag{3.5}$$

Since $\{R_n\}$ is a decreasing sequence of positive real numbers and is bounded below, there exists $r \geqslant 0$ such that

$$R_n = d(x_{n+1}, x_n) \to r \text{ as } n \to \infty. \tag{3.6}$$

Now, we shall show that $r = 0$. Assume, to the contrary, that $r > 0$. Taking limit supremum in both sides of (3.5), using (3.6), the property of ψ and the continuity of ϕ, we obtain

$$\phi(r) \leqslant \phi(r) + \overline{\lim}\ (-\psi(R_{n-1})).$$

Since $\overline{\lim}\ (-\psi(R_{n-1})) = -\underline{\lim}\ \psi(R_{n-1})$, it follows that

$$\phi(r) \leqslant \phi(r) - \underline{\lim}\ (\psi(R_{n-1})), \text{ that is, } \underline{\lim}\ (\psi(R_{n-1})) \leqslant 0,$$

which, by the property of ψ, is a contradiction unless $r = 0$. Therefore,

$$R_n = d(x_{n+1}, x_n) \to 0 \text{ as } n \to \infty. \tag{3.7}$$

Next we show that $\{x_n\}$ is a Cauchy sequence.

Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$ and $d(x_{m(k)}, x_{n(k)}) \geqslant \epsilon$. Assuming that $n(k)$ is the smallest such positive integer, we get

$$n(k) > m(k) > k,\ d(x_{m(k)}, x_{n(k)}) \geqslant \epsilon \text{ and } d(x_{m(k)}, x_{n(k)-1}) < \epsilon.$$

Now,

$$\epsilon \leqslant d(x_{m(k)}, x_{n(k)}) \leqslant d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}),$$

that is,

$$\epsilon \leqslant d(x_{m(k)}, x_{n(k)}) \leqslant \epsilon + d(x_{n(k)-1}, x_{n(k)}).$$

Letting $k \to \infty$ in the above inequality and using (3.7), we have

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{3.8}$$

Again,

$$d(x_{m(k)-1}, x_{n(k)-1}) \leqslant d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)-1}),$$

and

$$d(x_{m(k)}, x_{n(k)}) \leqslant d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}).$$

Letting $k \to \infty$ in the above inequalities and using (3.7) and (3.8), we have

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon. \tag{3.9}$$

Again,

$$d(x_{n(k)-1}, x_{m(k)}) \leqslant d(x_{m(k)-1}, x_{n(k)-1}) + d(x_{m(k)-1}, x_{m(k)})$$

and

$$d(x_{m(k)-1}, x_{n(k)-1}) \leqslant d(x_{m(k)-1}, x_{m(k)}) + d(x_{n(k)-1}, x_{m(k)}).$$

Letting $k \to \infty$ in the above inequalities and using (3.7) and (3.9), we have

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) = \epsilon. \tag{3.10}$$

Similarly, we have

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon. \tag{3.11}$$

Let

$$M_k = \max\left\{\frac{d(x_{n(k)-1}, x_{n(k)})[1 + d(x_{m(k)-1}, x_{m(k)})]}{1 + d(x_{m(k)-1}, x_{n(k)-1})}, \times \frac{d(x_{n(k)-1}, x_{m(k)})[1 + d(x_{m(k)-1}, x_{n(k)})]}{1 + d(x_{m(k)-1}, x_{n(k)-1})}, d(x_{m(k)-1}, x_{n(k)-1})\right\} \tag{3.12}$$

and

$$N_k = \max\left\{\frac{d(x_{n(k)-1}, x_{n(k)})[1 + d(x_{m(k)-1}, x_{m(k)})]}{1 + d(x_{m(k)-1}, x_{n(k)-1})}, d(x_{m(k)-1}, x_{n(k)-1})\right\}. \tag{3.13}$$

Letting $k \to \infty$ in (3.12) and (3.13), using 3.7, 3.9, 3.10 and 3.11, we have

$$\lim_{k\to\infty} M_k = \max\{0, \epsilon, \epsilon\} = \epsilon \tag{3.14}$$

and

$$\lim_{k\to\infty} N_k = \max\{0, \epsilon\} = \epsilon. \tag{3.15}$$

Since $x_{m(k)-1} \leqslant x_{n(k)-1}$, applying (3.1) and using (3.12) and (3.13), we have

$$\phi(d(x_{m(k)}, x_{n(k)})) = \phi(d(Tx_{m(k)-1}, Tx_{n(k)-1})) \leqslant \phi(M_k) - \psi(N_k).$$

Taking limit supremum in both sides of the above inequality, using 3.8, 3.14, 3.15, the property of ψ and the continuity of ϕ, we obtain

$$\phi(\epsilon) \leqslant \phi(\epsilon) + \overline{\lim}\ (-\psi(N_k)).$$

Since $\overline{\lim}\ (-\psi(N_k)) = -\underline{\lim}\ \psi(N_k)$, it follows that

$$\phi(\epsilon) \leqslant \phi(\epsilon) - \underline{\lim}\ (\psi(N_k)), \text{ that is, } \underline{\lim}\ (\psi(N_k)) \leqslant 0,$$

which, by (3.15) and the property of ψ, is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence in a complete metric space X. Therefore, there exits $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Then the continuity of T implies that $Tu = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = u$, that is, u is a fixed point of T. □

In our next theorem we relax the continuity assumption of the mapping T in Theorem 3.1 by imposing the following order condition of the metric space X:

If $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x_n \leqslant x$, for all $n \in \mathbf{N}$.

Theorem 3.2. *Let $(X, \leqslant)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \leqslant x$, for all $n \in \mathbf{N}$. Let $T : X \to X$ be a nondecreasing mapping. Suppose that (3.1) holds, where $M(x, y)$, $N(x, y)$ and the conditions upon (ϕ, ψ) are the same as in Theorem 3.1. If there exists $x_0 \in X$ with $x_0 \leqslant Tx_0$, then T has a fixed point.*

Proof. We take the same sequence $\{x_n\}$ as in the proof of Theorem 3.1. Then we have $x_0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant x_{n+1} \leqslant \ldots$, that is, $\{x_n\}$ is a nondecreasing sequence. Also, this sequence converges to u. Then $x_n \leqslant u$, for all $n \in \mathbf{N}$.

Suppose that $u \neq Tu$, that is, $d(u, Tu) > 0$.

Let

$$M_n = \max\left\{\frac{d(u,Tu)[1+d(x_n, Tx_n)]}{1+d(x_n,u)}, \frac{d(u,Tx_n)[1+d(x_n,Tu)]}{1+d(x_n,u)}, d(x_n, u)\right\}$$
$$= \max\left\{\frac{d(u,Tu)[1+d(x_n, x_{n+1})]}{1+d(x_n,u)}, \frac{d(u,x_{n+1})[1+d(x_n,Tu)]}{1+d(x_n,u)}, d(x_n, u)\right\} \quad (3.16)$$

and

$$N_n = \max\left\{\frac{d(u,Tu)[1+d(x_n, Tx_n)]}{1+d(x_n,u)}, d(x_n, u)\right\}$$
$$= \max\left\{\frac{d(u,Tu)[1+d(x_n, x_{n+1})]}{1+d(x_n,u)}, d(x_n, u)\right\}. \quad (3.17)$$

Letting $n \to \infty$ in (3.16) and (3.17), using the fact that $x_n \to u$ as $n \to \infty$, we have

$$\lim_{n\to\infty} M_n = \max\{d(u, Tu), 0, 0\} = d(u, Tu) > 0 \quad (3.18)$$

and

$$\lim_{n\to\infty} N_n = \max\{d(u, Tu), 0\} = d(u, Tu) > 0. \quad (3.19)$$

Since $x_n \leqslant u$ for all n, applying (3.1) and using (3.16), (3.17), we have

$$\phi(d(x_{n+1}, Tu)) = \phi(d(Tx_n, Tu)) \leqslant \phi(M_n) - \psi(N_n).$$

Taking limit supremum in both sides of the above inequality, using (3.18), (3.19), the property of ψ and the continuity of ϕ, we obtain

$$\phi(d(u, Tu)) \leqslant \phi(d(u, Tu)) + \overline{\lim}\,(-\psi(N_n)).$$

Since $\overline{\lim}\,(-\psi(N_n)) = -\underline{\lim}\,\psi(N_n)$, it follows that

$$\phi(d(u,Tu)) \leqslant \phi(d(u,Tu)) - \underline{\lim}\,(\psi(N_n)), \text{ that is, } \underline{\lim}\,(\psi(N_n)) \leqslant 0,$$

which, by (3.19) and the property of ψ, is a contradiction. Hence, $u = Tu$, that is, u is a fixed point of T. □

Now, we shall prove the uniqueness of the fixed point.

Theorem 3.3. *In addition to the hypotheses of Theorem* 3.1 *(or Theorem* 3.2*), suppose that for every x, $y \in X$, there exists $u \in X$ such that $u \leqslant x$ and $u \leqslant y$. Then T has a unique fixed point.*

Proof. It follows from the Theorem 3.1 (or Theorem 3.2) that the set of fixed points of T is non-empty. We shall show that if x^* and y^* are two fixed points of T, that is, if $x^* = Tx^*$ and $y^* = Ty^*$, then $x^* = y^*$.

By the assumption, there exists $u_0 \in X$ such that $u_0 \leqslant x^*$ and $u_0 \leqslant y^*$. Then, similarly as in the proof of Theorem 3.1, we define the sequence $\{u_n\}$ such that

$$u_{n+1} = Tu_n = T^{n+1}u_0, \ n = 0, 1, 2, \ldots \quad (3.20)$$

Monotonicity of T implies that

$$T^n u_0 = u_n \leqslant x^* = T^n x^* \text{ and } T^n u_0 = u_n \leqslant y^* = T^n y^*.$$

If there exists a positive integer m such that $x^* = u_m$, then $x^* = Tx^* = Tu_n = u_{n+1}$, for all $n \geqslant m$. Then $u_n \to x^*$ as $n \to \infty$. Now we suppose that $x^* \neq u_n$, for all $n \geqslant 0$. So $u_n < x^*$, for all $n \geqslant 0$. Then $d(u_n, x^*) \neq 0$, for all $n \geqslant 0$.

Let $P_n = d(u_n, x^*)$, for all $n \geqslant 0$. Since $u_n < x^*$, for all $n \geqslant 0$, applying (3.1), we have

$$\phi(d(u_{n+1}, x^*)) = \phi(d(Tu_n, Tx^*))$$
$$\leqslant \phi\left(\max\left\{\frac{d(x^*, Tx^*)[1+d(u_n, Tu_n)]}{1+d(u_n, x^*)}, \frac{d(x^*, Tu_n)[1+d(u_n, Tx^*)]}{1+d(u_n, x^*)}, d(u_n, x^*)\right\}\right)$$
$$- \psi\left(\max\left\{\frac{d(x^*, Tx^*)[1+d(u_n, Tu_n)]}{1+d(u_n, x^*)}, d(u_n, x^*)\right\}\right)$$
$$= \phi\left(\max\left\{0, \frac{d(x^*, u_{n+1})[1+d(u_n, x^*)]}{1+d(u_n, x^*)}, d(u_n, x^*)\right\}\right) - \psi(\max\{0, d(u_n, x^*)\})$$
$$= \phi(\max\{d(x^*, u_{n+1}), d(u_n, x^*)\}) - \psi(d(u_n, x^*)),$$

that is,

$$\phi(P_{n+1}) = \phi(\max\{P_{n+1}, P_n\}) - \psi(P_n). \quad (3.21)$$

If $P_{n+1} > P_n$, then from the above inequality, we have

$$\phi(P_{n+1}) \leqslant \phi(P_{n+1}) - \psi(P_n), \text{ that is, } \psi(P_n) \leqslant 0,$$

which is a contradiction. So, $P_{n+1} \leqslant P_n$, that is, $\{P_n\}$ is a decreasing sequence. Then it follows from the inequality (3.21) that

$$\phi(P_{n+1}) \leqslant \phi(P_n) - \psi(P_n). \quad (3.22)$$

Since $\{P_n\}$ is a decreasing sequence of positive real numbers and is bounded below, there exists $r \geqslant 0$ such that

$$P_n = d(u_n, x^*) \to r \text{ as } n \to \infty. \quad (3.23)$$

Arguing similarly as in the proof of Theorem 3.1 we can show that $r = 0$. Then,

$$P_n = d(u_n, x^*) \to 0 \text{ as } n \to \infty, \text{ that is, } u_n \to x^* \text{ as } n \to \infty. \quad (3.24)$$

Using a similar argument, we can prove that

$$u_n \to y^* \text{ as } n \to \infty. \quad (3.25)$$

Finally, the uniqueness of the limit implies $x^* = y^*$. Hence T has a unique fixed point. □

Example 3.4. Let $X = \{(0, 1), (1, 0), (1, 1)\} \subset \mathbb{R}^2$ with the Euclidean distance d. We consider the partial order R in X as follows:

$$R = \{(x, x) : x \in X\} \cup \{((0, 1), (1, 1))\}.$$

Let $T : X \to X$ be given by

$$T(0, 1) = (0, 1), \ T(1, 0) = (1, 0) \text{ and } T(1, 1) = (0, 1).$$

Let ϕ, $\psi : [0, \infty) \to [0, \infty)$ be given respectively by the formulas

$$\phi(t) = t^2, \quad \psi(t) = \begin{cases} \frac{[t]^2}{2}, & \text{if } 3 < t < 4, \\ \frac{t^2}{2}, & \text{otherwise,} \end{cases}$$

where $[t]$ denotes the greatest integer not exceeding t.

Here all the conditions of Theorems 3.1 and 3.2 are satisfied and T has two fixed points which are $(0, 1)$ and $(1, 0)$.

Example 3.5. Let $X = \{(0, 0), (\frac{1}{2}, 0), (0, 1)\}$ be a subset of R^2 with the order "$\leqslant$" defined as: for $(x_1, y_1), (x_2, y_2) \in X$, $(x_1, y_1) \leqslant (x_2, y_2)$ if and only if $x_1 \leqslant x_2, y_1 \leqslant y_2$. Let $d : X \times X \to \mathbb{R}$ be given as

$$d(x, y) = \max\{|x_1 - x_2|, |y_1 - y_2|\},$$

for $x = (x_1, y_1), y = (x_2, y_2) \in X$.

Let $T : X \to X$ be defined as follows:

$$T(0, 0) = (0, 0), \ T(0, 1) = (\frac{1}{2}, 0) \text{ and } T\left(\frac{1}{2}, 0\right) = (0, 0).$$

Let us consider the functions ϕ and ψ as defined in Example 3.4. Here all the conditions of Theorems 3.1, 3.2 and 3.3 are satisfied and $(0, 0)$ is the unique fixed point of T.

Example 3.6. Let $X = [1.5, 2]$ with usual partial order "$\leqslant$" and usual metric "d" be a partially ordered metric space. Let $T : X \to X$ be defined as follows:

$$Tx = \begin{cases} 1.81, & \text{if } 1.5 \leqslant x < 1.75, \\ x + \frac{1}{x} - \frac{1}{2}, & \text{if } 1.75 \leqslant x \leqslant 2. \end{cases}$$

Let $\phi, \psi : [0, \infty) \to [0, \infty)$ be given respectively by the formulas

$$\phi(t) = t, \quad \psi(t) = \begin{cases} \frac{[t]}{1000}, & \text{if } 3 < t < 4, \\ \frac{t}{1000}, & \text{otherwise,} \end{cases}$$

where $[t]$ denotes the greatest integer not exceeding t.

Here all the conditions of Theorems 3.2 and 3.3 are satisfied and $x = 2$ is the unique fixed point of T.

Note. In the above example the mapping T is not continuous. Therefore, the above example is not applicable to Theorem 3.1.

Corollary 3.7. *Let $(X, \leqslant)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping such that for all $x, y \in X$ with $x \leqslant y$,*

$$\phi(d(Tx, Ty)) \leqslant \phi(N(x, y)) - \psi(N(x, y)), \tag{3.26}$$

where $N(x, y)$ and the conditions upon (ϕ, ψ) are the same as in Theorem 3.1. If there exists $x_0 \in X$ with $x_0 \leqslant Tx_0$, then T has a fixed point.

Proof. Since the inequality (3.26) implies the inequality (3.1), by Theorem 3.1, we have the required proof. □

Corollary 3.8. *Let $(X, \leqslant)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \leqslant x$, for all $n \in \mathbf{N}$. Let $T : X \to X$ be a nondecreasing mapping. Suppose that (3.26) holds, where $N(x, y)$ and the conditions upon (ϕ, ψ) are the same as in Theorem 3.1. If there exists $x_0 \in X$ with $x_0 \leqslant Tx_0$, then T has a fixed point.*

Proof. Since the inequality (3.26) implies the inequality (3.1), by Theorem 3.2, we have the required proof. □

Corollary 3.9. *In addition to the hypotheses of Corollary 3.7 (or Corollary 3.8), suppose that for every $x, y \in X$, there exists $u \in X$ such that $u \leqslant x$ and $u \leqslant y$. Then T has a unique fixed point.*

Example 3.10. Let $X = C[0, 1] = \{x : [0, 1] \to \mathbb{R}, \text{continuous}\}$ with the partial order given by $x \leqslant y \Longleftrightarrow x(t) \leqslant y(t)$, for $t \in [0, 1]$. Let the metric d on X be given by

$$d(x, y) = \text{Sup}\{|x(t) - y(t)| : t \in [0, 1]\}, \text{ for } x, y \in X.$$

Let $T : X \to X$ be defined as $Tx = \frac{x}{3}$, $x \in X$.

Let $\phi, \psi : [0, \infty) \to [0, \infty)$ be given respectively by the formulas

$$\phi(t) = t^2, \quad \psi(t) = \begin{cases} \frac{[t]^2}{10} & \text{if } 3 < t < 4, \\ \frac{t^2}{10}, & \text{otherwise,} \end{cases}$$

where $[t]$ denotes the greatest integer not exceeding t.

Here, all the conditions of Corollaries 3.7 and 3.8 are satisfied. Moreover, as for $x, y \in X = C[0, 1]$, the function $\min(x, y)(t) = \min\{x(t), y(t)\}$ is continuous, the conditions of Corollary 3.9 are satisfied. It is seen that $x = 0$ is the unique fixed point of T in X.

In the Corollaries 3.7 and 3.8, taking ϕ to be the identity mapping and $\psi(t) = (1 - k)t$ for all $t \in [0, \infty)$, where $k \in (0, 1)$, we have the following results.

Corollary 3.11. *Let $(X, \leqslant)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping. Suppose there exists $k \in (0, 1)$ such that for all $x, y \in X$ with $x \leqslant y$,*

$$d(Tx, Ty) \leqslant k \max\left\{\frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}, d(x, y)\right\}. \tag{3.27}$$

If there exists $x_0 \in X$ with $x_0 \leqslant Tx_0$, then T has a fixed point.

Corollary 3.12. *Let $(X, \leqslant)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \leqslant x$, for all $n \in \mathbf{N}$. Let $T : X \to X$ be a nondecreasing mapping. Suppose that (3.27) holds. If there exists $x_0 \in X$ with $x_0 \leqslant Tx_0$, then T has a fixed point.*

Remark 3.1. Theorems 3.1, 3.2 and 3.3 are respectively generalizations of Theorems 2, 3 and 4 in [35] which are also noted here as Theorems 2.2, 2.3 and 2.4 respectively.

Other consequences of our results are the following for the mappings involving contractions of integral type.

Denote by Λ the set of functions $\mu : [0, \infty) \to [0, \infty)$ satisfying the following hypotheses:

(h1) μ is a Lebesgue-integrable mapping on each compact subset of $[0, \infty)$;
(h2) for any $\epsilon > 0$, we have $\int_0^\epsilon \mu(t)dt > 0$.

Corollary 3.13. *Let $(X, \leqslant)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping. Suppose that there exist $\tau \in \Lambda$ and for all $x, y \in X$ with $x \leqslant y$,*

$$\int_0^{\phi(d(Tx, Ty))} \tau(t)\, dt \leqslant \int_0^{\phi(M(x, y))} \tau(t)\, dt - \int_0^{\psi(N(x, y))} \tau(t)\, dt, \tag{3.28}$$

where $M(x, y)$, $N(x, y)$ and the conditions upon (ϕ, ψ) are the same as in Theorem 3.1. *If there exists $x_0 \in X$ with $x_0 \leqslant Tx_0$, then T has a fixed point.*

Corollary 3.14. *Let $(X, \leqslant)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \leqslant x$, for all $n \in \mathbf{N}$. Let $T : X \to X$ be a nondecreasing mapping. Suppose that there exist $\tau \in \Lambda$ for which* (3.28) *holds, where $M(x, y)$, $N(x, y)$ and the conditions upon (ϕ, ψ) are the same as in Theorem* 3.1. *If there exists $x_0 \in X$ with $x_0 \leqslant Tx_0$, then T has a fixed point.*

Corollary 3.15. *Let $(X, \leqslant)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping. Suppose that there exist $\tau \in \Lambda$ and for all $x, y \in X$ with $x \leqslant y$,*

$$\int_0^{\phi(d(Tx, Ty))} \tau(t)\, dt \leqslant \int_0^{\phi(N(x, y))} \tau(t)\, dt - \int_0^{\psi(N(x, y))} \tau(t)\, dt, \tag{3.29}$$

where $N(x, y)$ and the conditions upon (ϕ, ψ) are the same as in Theorem 3.1. *If there exists $x_0 \in X$ with $x_0 \leqslant Tx_0$, then T has a fixed point.*

Corollary 3.16. *Let $(X, \leqslant)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \leqslant x$, for all $n \in \mathbf{N}$. Let $T : X \to X$ be a nondecreasing mapping. Suppose that there exist $\tau \in \Lambda$ for which* (3.29) *holds, where $N(x, y)$ and the conditions upon (ϕ, ψ) are the same as in Theorem* 3.1. *If there exists $x_0 \in X$ with $x_0 \leqslant Tx_0$, then T has a fixed point.*

Acknowledgment

The authors gratefully acknowledge the suggestions made by the learned referee.

References

[1] A.D. Arvanitakis, A proof of the generalized Banach contraction conjecture, Proc. Amer. Math. Soc. 131 (2003) 3647–3656.
[2] V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, Nonlinear Anal. Forum 9 (2004) 43–53.
[3] D.W. Boyd, T.S.W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. 20 (1969) 458–464.
[4] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci. 29 (2002) 531–536.
[5] S. Chandok, Some common fixed point theorems for generalized f-weakly contractive mappings, J. Appl. Math. Inf. 29 (2011) 257–265.
[6] S. Chandok, Some common fixed point theorems for generalized nonlinear contractive mappings, Comput. Math. Appl. 62 (2011) 3692–3699.
[7] S. Chandok, Common fixed points for generalized nonlinear contractive mappings in metric spaces, Mat. Vesnik 65 (2013) 29–34.
[8] B.S. Choudhury, P. Konar, B.E. Rhoades, N. Metiya, Fixed point theorems for generalized weakly contractive mappings, Nonlinear Anal. 74 (2011) 2116–2126.
[9] L.B. Ćirić, A generalization of Banachs contraction principle, Proc. Am. Math. Soc. 45 (1974) 267–273.
[10] M. Edelstein, An extension of Banachs contraction principle, Proc. Amer. Math. Soc. 12 (1961) 7–10.
[11] D.S. Jaggi, Some unique fixed point theorems, Indian J. Pure Appl. Math. 8 (1977) 223–230.
[12] D.S. Jaggi, B.K. Das, An extension of Banach's fixed point theorem through rational expression, Bull. Cal. Math. Soc. 72 (1980) 261–264.
[13] A. Meir, E. Keeler, A theorem on contraction mappings, J. Math. Anal. Appl. 28 (1969) 326–329.
[14] J. Merryfield, B. Rothschild, J.D. Stein Jr, An application of Ramseys theorem to the Banach contraction principle, Proc. Amer. Math. Soc. 130 (2002) 927–933.
[15] B.E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal. 47 (2001) 2683–2693.
[16] T. Zamfirescu, Fixed point theorems in metric spaces, Arch. Math. (Basel) 23 (1972) 292–298.
[17] M. Turinici, Abstract comparison principles and multivariable Gronwall–Bellman inequalities, J. Math. Anal. Appl. 117 (1986) 100–127.
[18] A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2004) 1435–1443.
[19] J.J. Nieto, R.R. López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order 22 (2005) 223–239.
[20] R.P. Agarwal, M.A. El-Gebeily, D. O'Regan, Generalized contractions in partially ordered metric spaces, Appl. Anal. 87 (2008) 109–116.
[21] M. Arshad, E. Karapinar, J. Ahmad, Some unique fixed point theorems for rational contractions in partially ordered metric spaces, J. Inequal. Appl. 2013 (2013) 248.
[22] G.V.R. Babu, G.N. Alemayehu, Existence of common fixed points for weakly biased maps satisfying weakly contractive condition involving rational expressions, Proc. Jangjeon Math. Soc. 14 (1) (2011) 115–133.
[23] T.G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (2006) 1379–1393.
[24] S. Chandok, J.K. Kim, Fixed point theorem in ordered metric spaces for generalized contractions mappings satisfying rational

type expressions, J. Nonlinear Functional Anal. Appl. 17 (2012) 301–306.
[25] S. Chandok, Some common fixed point results for generalized weak contractive mappings in partially ordered metric spaces, J. Nonlinear Anal. Opt. 4 (2013) 45–52.
[26] S. Chandok, M.S. Khan, K.P.R. Rao, Some coupled common fixed point theorems for a pair of mappings satisfying a contractive condition of rational type, J. Nonlinear Anal. Appl. 2013 (2013) 1–6.
[27] S. Chandok, T.D. Narang, M.A. Taoudi, Some common fixed point results in partially ordered metric spaces for generalized rational type contraction mappings, Veitnam J. Math. 41 (2013) 323–331.
[28] B.S. Choudhury, N. Metiya, Multivalued and singlevalued fixed point results in partially ordered metric spaces, Arab J. Math. Sci. 17 (2011) 135–151.
[29] B.S. Choudhury, A. Kundu, (ψ, α, β) – weak contractions in partially ordered metric spaces, Appl. Math. Lett. 25 (2012) 6–10.
[30] B.S. Choudhury, N. Metiya, Fixed point theorems for almost contractions in partially ordered metric spaces, Ann. Univ. Ferrara 58 (2012) 21–36.
[31] B.S. Choudhury, N. Metiya, M. Postolache, A generalized weak contraction principle with applications to coupled coincidence point problems, Fixed Point Theory Appl. 2013 (2013) 152.
[32] J. Harjani, K. Sadarangani, Fixed point theorems for weakly contractive mappings in partially ordered sets, Nonlinear Anal. 71 (2009) 3403–3410.
[33] J. Harjani, B. López, K. Sadarangani, A fixed point theorem for mappings satisfying a contractive condition of rational type on a partially ordered metric space, Abstract Appl. Anal. (2010) (Article ID 190701).
[34] B.K. Dass, S. Gupta, An extension of Banach contraction principle through rational expressions, Inidan J. Pure Appl. Math. 6 (1975) 1455–1458.
[35] I. Cabrera, J. Harjani, K. Sadarangani, A fixed point theorem for contractions of rational type in partially ordered metric spaces, Ann. Univ. Ferrara. 59 (2013) 251–258.
[36] M.S. Khan, M. Swaleh, S. Sessa, Fixed points theorems by altering distances between the points, Bull. Austral. Math. Soc. 30 (1984) 1–9.