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A lower bound for the Lindelöf function associated to generalized integers

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Abstract

In this paper we study generalized prime systems for which the integer counting function $N_{\mathcal{P}}(x)$ is asymptotically well-behaved, in the sense that $N_{\mathcal{P}}(x) = \rho x + O(x^{\beta})$, where ρ is a positive constant and $\beta < 1/2$. For such systems, the associated zeta function $\zeta_{\mathcal{P}}(s)$ has finite order for $\sigma = \Re s > \beta$, and the Lindelöf function $\mu_{\mathcal{P}}(\sigma)$ may be defined. We prove that for all such systems, $\mu_{\mathcal{P}}(\sigma) \geqslant \mu_0(\sigma)$ for $\sigma > \beta$, where

$$\mu_0(\sigma) = \begin{cases} \frac{1}{2} - \sigma & \text{if } \sigma < \frac{1}{2}, \\ 0 & \text{if } \sigma \geqslant \frac{1}{2}. \end{cases}$$

Introduction

A *generalized prime system* (or *g-prime system*) $\mathcal{P}$ is a sequence of positive reals $p_1, p_2, p_3, \ldots$ satisfying

$$1 < p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n \leqslant \cdots$$

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and for which $p_n \to \infty$ as $n \to \infty$. From these can be formed the system $\mathcal{N}$ of *generalized integers* or *Beurling integers*; that is, the numbers of the form

$$p_1^{a_1} p_2^{a_2} \dots p_k^{a_k},$$

where $k \in \mathbb{N}$ and $a_1, \dots, a_k \in \mathbb{N}_0$.[1]

Such systems were first introduced by Beurling [3] and have been studied by many authors since then (see, in particular, [2]).

Much of the theory concerns connecting the asymptotic behaviour of the g-prime and g-integer counting functions, $\pi_{\mathcal{P}}(x)$ and $N_{\mathcal{P}}(x)$, defined respectively by[2]

$$\pi_{\mathcal{P}}(x) = \sum_{p\in\mathcal{P},\, p\leqslant x} 1 \quad \text{and} \quad N_{\mathcal{P}}(x) = \sum_{n\in\mathcal{N},\, n\leqslant x} 1.$$

The methods invariably involve the associated *Beurling zeta function*, defined formally by

$$\zeta_{\mathcal{P}}(s) = \prod_{p\in\mathcal{P}} \frac{1}{1-p^{-s}} = \sum_{n\in\mathcal{N}} \frac{1}{n^s}. \tag{1}$$

In this paper, we shall be concerned with g-prime systems $\mathcal{P}$ for which

$$N_{\mathcal{P}}(x) = \rho x + O\big(x^{\beta}\big), \tag{2}$$

for some $\beta < \frac{1}{2}$ and $\rho > 0$. (For example, for the rational primes when $\mathcal{N} = \mathbb{N}$, this is true with $\beta = 0$ and $\rho = 1$.)

For such systems, the product and series (1) converge for $\Re s > 1$ and $\zeta_{\mathcal{P}}(s)$ has an analytic continuation to the half-plane $\Re s > \beta$ except for a simple pole at $s = 1$ with residue ρ. Indeed, writing $N_{\mathcal{P}}(x) = \rho x + E(x)$ with $E(x) = O(x^{\beta})$, we have for $\Re s > 1$,

$$\zeta_{\mathcal{P}}(s) = \int_{1-}^{\infty} x^{-s}\, dN_{\mathcal{P}}(x) = s\int_1^{\infty} \frac{N_{\mathcal{P}}(x)}{x^{s+1}}\, dx = s\int_1^{\infty} \frac{\rho x + E(x)}{x^{s+1}}\, dx = \frac{\rho s}{s-1} + s\int_1^{\infty} \frac{E(x)}{x^{s+1}}\, dx.$$

The integral on the right converges for $\Re s > \beta$ and is an analytic function for such s.

Furthermore, $\zeta_{\mathcal{P}}(s)$ has *finite order* for $\Re s > \beta$; i.e. $\zeta_{\mathcal{P}}(\sigma + it) = O(|t|^A)$ as $|t| \to \infty$ for some constant A for $\sigma > \beta$ (indeed, in our case this is true with $A = 1$). We can therefore define, as is usual, the *Lindelöf function* $\mu_{\mathcal{P}}(\sigma)$ to be the infimum of all real numbers λ such that $\zeta_{\mathcal{P}}(\sigma + it) = O(|t|^{\lambda})$. It is well known that, as a function of σ, $\mu_{\mathcal{P}}(\sigma)$ is non-negative, decreasing, and convex (and hence continuous) (see, for example, [5]). Since $\mu_{\mathcal{P}}(\sigma) = 0$ for $\sigma > 1$, and (from above) $\mu_{\mathcal{P}}(\sigma) \leqslant 1$ for $\sigma > \beta$, it follows by convexity that

$$\mu_{\mathcal{P}}(\sigma) \leqslant \frac{1-\sigma}{1-\beta} \quad \text{for } \beta < \sigma \leqslant 1.$$

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[1] Here and henceforth, $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{P} = \{2, 3, 5, \dots\}$—the set of primes.

[2] We write $\sum_{p\in\mathcal{P}}$ to mean a sum over all the g-primes, counting multiplicities. Similarly for $\sum_{n\in\mathcal{N}}$.

For $\mathcal{P}=\mathbb{P}$ (so that $\mathcal{N}=\mathbb{N}$), the Lindelöf hypothesis is the conjecture that $\mu_{\mathbb{P}}(\sigma)=\mu_0(\sigma)$ for all σ, where

$$\mu_0(\sigma)=\begin{cases}\frac{1}{2}-\sigma & \text{if } \sigma<\frac{1}{2},\\ 0 & \text{if } \sigma\geqslant\frac{1}{2}.\end{cases}$$

In this paper we prove that for all g-prime systems satisfying (2), $\mu_{\mathcal{P}}(\sigma)$ must be *at least* as large as $\mu_0(\sigma)$, i.e.,

$$\mu_{\mathcal{P}}(\sigma)\geqslant\mu_0(\sigma)\quad\text{for } \sigma>\beta.$$

This is, of course, trivial for $\sigma\geqslant\frac{1}{2}$, so we shall only concern ourselves with $\beta<\sigma<\frac{1}{2}$.

For the proof we employ the same methods (but strengthened) as those used in [4], where (essentially) it was shown that $\mu_{\mathcal{P}}(\sigma)>0$ for any $\sigma<\frac{1}{2}$, in order to prove that for such systems we have $\psi_{\mathcal{P}}(x)-x=\Omega(x^{\frac{1}{2}-\delta})$ for every $\delta>0$.[3]

Main result

Theorem 1. *Let $\mathcal{P}$ be a g-prime system for which*

$$N_{\mathcal{P}}(x)=\rho x+O\big(x^{\beta}\big),$$

for some $\beta<\frac{1}{2}$ and $\rho>0$. Let $\mu_{\mathcal{P}}(\sigma)$ and $\mu_0(\sigma)$ be as defined above. Then for $\sigma>\beta$, we have

$$\mu_{\mathcal{P}}(\sigma)\geqslant\mu_0(\sigma).$$

Proof. As mentioned above, we need only consider $\beta<\sigma<\frac{1}{2}$.

Suppose, for a contradiction, that we have $\mu_{\mathcal{P}}(\sigma)<\frac{1}{2}-\sigma$ for some $\sigma\in(\beta,\frac{1}{2})$. Then we can write

$$\mu_{\mathcal{P}}(\sigma)=\frac{1}{2}-\sigma-\delta,$$

for some $\delta>0$.

Let $\zeta_N(s)=\sum_{n\leqslant N}n^{-s}$, where the sum ranges over $n\in\mathcal{N}$ (for clarity, we shall drop the subscript $\mathcal{P}$ throughout this proof). By identical arguments as those used in [4], we find that there exists constants $c_1,c_2>0$ such that for $R\geqslant c_1N$,

$$\sum_{r=1}^{R}\int_{0}^{2r-1}\big|\zeta_N(\sigma+it)\big|^2\,dt\geqslant c_2R^2N^{1-2\sigma}. \tag{3}$$

Also, writing $s=\sigma+it$, and following the arguments in [4], we have

[3] Here $\psi_{\mathcal{P}}(x)$ is the generalized Chebychev function $\psi_{\mathcal{P}}(x)=\sum_{p^k\leqslant x,\,p\in\mathcal{P},\,k\in\mathbb{N}}\log p$ (counting multiplicities).

$$\zeta_N(s) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\zeta_{\mathcal{P}}(s+w)N^w}{w}\,dw + O\left(\frac{N^c}{T(c+\sigma-1)}\right) + O\left(\frac{N^{1-\sigma}}{T}\sum_{\substack{\frac{N}{2}<n<2N\\ n\in\mathcal{N}}}\frac{1}{|n-N|}\right),$$

for $|t| < T$, $c > 1-\sigma$ and $N \notin \mathcal{N}$.

Now push the contour in the integral to the left as far as $\Re w = -\eta$, where $\eta > 0$, picking up the residues at $w = 0$ and $w = 1-s$ (since $|t| < T$). Here, η is chosen sufficiently small such that $\sigma - \eta > \beta$ and $\mu_{\mathcal{P}}(\sigma - \eta) < \frac{1}{2} - \sigma$. This is possible since $\mu_{\mathcal{P}}(\cdot)$ is continuous. Thus $\zeta_{\mathcal{P}}(\sigma - \eta + it) = O(|t|^{\frac{1}{2}-\sigma-\delta'})$ for some $\delta' > 0$.

The contribution along the horizontal line $[-\eta + iT, c + iT]$ is, in modulus, less than

$$\frac{1}{2\pi}\int_{-\eta}^{c} \frac{N^y|\zeta_{\mathcal{P}}(\sigma + y + i(t+T))|}{\sqrt{y^2+T^2}}\,dy = O\big(N^c T^{-\frac{1}{2}-\sigma-\delta'}\big).$$

Similarly on $[-\eta - iT, c - iT]$. For the integral along $\Re w = -\eta$, we have

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{-\eta-iT}^{-\eta+iT} \frac{\zeta_{\mathcal{P}}(s+w)N^w}{w}\,dw\right| &\leqslant \frac{N^{-\eta}}{2\pi}\int_{-T}^{T} \frac{|\zeta_{\mathcal{P}}(\sigma-\eta+i(t+y))|}{\sqrt{\eta^2+y^2}}\,dy \\ &= O\left(N^{-\eta}\int_{-T}^{T}\frac{T^{\frac{1}{2}-\sigma-\delta'}}{\sqrt{\eta^2+y^2}}\,dy\right) \\ &= O\big(N^{-\eta}T^{\frac{1}{2}-\sigma-\delta'}\log T\big).\end{aligned}$$

The residues at $w = 0$ and $w = 1-s$ are, respectively, $\zeta_{\mathcal{P}}(s)$ and $\rho N^{1-s}/(1-s) = O(\frac{N^{1-\sigma}}{|t|+1})$. Putting these observations together and letting $c = 1 - \sigma + \frac{1}{\log N}$ (so that $N^c = eN^{1-\sigma}$), we have

$$\begin{aligned}\zeta_N(\sigma+it) = \zeta_{\mathcal{P}}(\sigma+it) &+ O\left(\frac{N^{1-\sigma}}{|t|+1}\right) + O\big(N^{1-\sigma}T^{-\frac{1}{2}-\sigma-\delta'}\big) + O\big(N^{-\eta}T^{\frac{1}{2}-\sigma-\delta'}\log T\big) \\ &+ O\left(\frac{N^{1-\sigma}\log N}{T}\right) + O\left(\frac{N^{1-\sigma}}{T}\sum_{\substack{\frac{N}{2}<n<2N\\ n\in\mathcal{N}}}\frac{1}{|n-N|}\right),\end{aligned} \tag{4}$$

for $|t| < T$ and $N \notin \mathcal{N}$.

Fix $\alpha \in (0, \frac{1}{4\rho})$, and let $N \to \infty$ in such a way that $(N-\alpha, N+\alpha) \cap \mathcal{N} = \emptyset$. This is possible for if not, then $n' < n + 4\alpha$ (where n and n' are consecutive g-integers), which leads to $N(x) \gtrsim \frac{1}{4\alpha}x$—a contradiction as $\frac{1}{4\alpha} > \rho$.

For such N, we can bound the final sum in (4) as follows. We have

$$\sum_{\substack{\frac{N}{2}<n<2N\\ n\in\mathcal{N}}}\frac{1}{|n-N|} = \sum_{\substack{\alpha\leqslant|n-N|<\sqrt{N}\\ n\in\mathcal{N}}}\frac{1}{|n-N|} + \sum_{\substack{\sqrt{N}\leqslant|n-N|<\frac{N}{2}\\ n\in\mathcal{N}}}\frac{1}{|n-N|} + O(1)$$

$$= O\big(N\big(N+\sqrt{N}\big) - N\big(N-\sqrt{N}\big)\big) + O\bigg(\frac{N\big(\frac{3}{2}N\big)}{\sqrt{N}}\bigg) + O(1) = O\big(\sqrt{N}\big),$$

using $N(x) = \rho x + O(x^{\beta+\varepsilon})$ with $\beta < \frac{1}{2}$. (In fact, the better estimate $O(N^{\beta+\varepsilon})$ is possible by splitting the sum over smaller ranges, but $O(\sqrt{N})$ suffices for our purposes.) Hence (4) becomes

$$\zeta_N(\sigma+it) = \zeta_{\mathcal{P}}(\sigma+it) + O\bigg(\frac{N^{1-\sigma}}{|t|+1}\bigg) + O\bigg(\frac{N^{1-\sigma}}{T^{\frac{1}{2}+\sigma+\delta'}}\bigg)$$
$$+ O\big(N^{-\eta}T^{\frac{1}{2}-\sigma-\delta'}\log T\big) + O\bigg(\frac{N^{\frac{3}{2}-\sigma}}{T}\bigg). \tag{5}$$

Choosing $T = N^{1+\eta}$ makes the last three O-terms all $O(N^{\frac{1}{2}-\sigma-\eta'})$ for some $\eta' > 0$. Using the hypothetical bound $\zeta_{\mathcal{P}}(\sigma+it) = O(|t|^{\frac{1}{2}-\sigma-\delta'})$, (5) becomes

$$\zeta_N(\sigma+it) = O\big(|t|^{\frac{1}{2}-\sigma-\delta'}\big) + O\bigg(\frac{N^{1-\sigma}}{|t|+1}\bigg) + O\big(N^{\frac{1}{2}-\sigma-\eta'}\big).$$

Using the Cauchy–Schwarz inequality, we have

$$\sum_{r=1}^{R}\int_0^{2r-1}\big|\zeta_N(\sigma+it)\big|^2\,dt = O\Bigg(\sum_{r=1}^{R}\int_0^{2r-1} t^{1-2\sigma-2\delta'}\,dt\Bigg) + O\Bigg(\sum_{r=1}^{R}\int_0^{2r-1}\frac{N^{2-2\sigma}}{(t+1)^2}\,dt\Bigg)$$
$$+ O\Bigg(\sum_{r=1}^{R}\int_0^{2r-1} N^{1-2\sigma-2\eta'}\,dt\Bigg)$$
$$= O\big(R^{3-2\sigma-2\delta'}\big) + O\big(RN^{2-2\sigma}\big) + O\big(R^2N^{1-2\sigma-2\eta'}\big).$$

Taking R to be of slightly larger order than N, say $R = N\log N$, the RHS becomes $o(R^2N^{1-2\sigma})$, which contradicts (3). □

Remark. The result is best possible—at least if we assume the Lindelöf Hypothesis. If $\mathcal{P} = \mathbb{P}$, then (2) holds with $\beta = 0$ and, on the Lindelöf hypothesis, $\mu_{\mathcal{P}} = \mu_0$. However, it is conceivable that the result might be subject to further improvements if (2) holds with $\beta > 0$. The example below shows this is not the case—again on the assumption of the Lindelöf hypothesis.

Let $\beta \in (0, \frac{1}{2})$ and denote by $\mathcal{P}$ the g-prime system made up of p and $p^{1/\beta}$ where p varies over all the primes, i.e.,

$$\mathcal{P} = \mathbb{P} \cup \big\{p^{\frac{1}{\beta}}:\ p \in \mathbb{P}\big\}.$$

For this system, $N_{\mathcal{P}}(x)$ satisfies (2). Indeed,

$$N_{\mathcal{P}}(x) = \sum_{n\leqslant x^\beta}\bigg[\frac{x}{n^{1/\beta}}\bigg] = \sum_{n\leqslant a^\beta}\bigg[\frac{x}{n^{1/\beta}}\bigg] + \sum_{n\leqslant b}\bigg[\bigg(\frac{x}{n}\bigg)^\beta\bigg] - \big[a^\beta\big][b],$$

for any $ab = x$ (see [1] for such manipulations). Putting $a = x^\lambda$, we obtain

$$\begin{aligned}
N_{\mathcal{P}}(x) &= x \sum_{n \leqslant x^{\lambda\beta}} \frac{1}{n^{1/\beta}} + x^\beta \sum_{n \leqslant x^{1-\lambda}} \frac{1}{n^\beta} - x^{\lambda\beta+1-\lambda} + O\big(x^{\lambda\beta}\big) + O\big(x^{1-\lambda}\big) \\
&= x\left(\zeta\left(\frac{1}{\beta}\right) - \frac{\beta}{1-\beta} x^{-\lambda\beta(\frac{1}{\beta}-1)} + O\big(x^{-\lambda\beta(\frac{1}{\beta})}\big)\right) \\
&\quad + x^\beta\left(\frac{x^{(1-\lambda)(1-\beta)}}{1-\beta} + \zeta(\beta) + O\big(x^{-(1-\lambda)\beta}\big)\right) - x^{\lambda\beta+1-\lambda} + O\big(x^{\lambda\beta}\big) + O\big(x^{1-\lambda}\big) \\
&= \zeta\left(\frac{1}{\beta}\right)x + \zeta(\beta)x^\beta + O\big(x^{\lambda\beta}\big) + O\big(x^{1-\lambda}\big).
\end{aligned}$$

Choosing $\lambda = \frac{1}{1+\beta}$ so that $\lambda\beta = 1 - \lambda$ minimizes the error. This gives

$$N_{\mathcal{P}}(x) = \zeta\left(\frac{1}{\beta}\right)x + \zeta(\beta)x^\beta + O\big(x^{\frac{\beta}{1+\beta}}\big).$$

The associated Beurling zeta function is $\zeta(s)\zeta(s/\beta)$. On the Lindelöf Hypothesis, it follows that $\mu_{\zeta(\cdot/\beta)}(\sigma) = 0$ for $\sigma \geqslant \frac{\beta}{2}$. Thus $\mu_{\mathcal{P}}(\sigma) \leqslant \frac{1}{2} - \sigma$ for $\beta < \sigma < \frac{1}{2}$. By Theorem 1, we must have $\geqslant$ as well, so in fact there is equality, i.e.,

$$\mu_{\mathcal{P}}(\sigma) = \frac{1}{2} - \sigma,$$

for $\beta < \sigma < \frac{1}{2}$.

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