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Law of large numbers for non-elliptic random walks in dynamic random environments

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Abstract

We prove a law of large numbers for a class of $\mathbb{Z}^d$-valued random walks in dynamic random environments, including non-elliptic examples. We assume for the random environment a mixing property called *conditional cone-mixing* and that the random walk tends to stay inside wide enough space–time cones. The proof is based on a generalization of a regeneration scheme developed by Comets and Zeitouni (2004) [5] for static random environments and adapted by Avena et al. (2011) [2] to dynamic random environments. A number of one-dimensional examples are given. In some cases, the sign of the speed can be determined.

MSC: Primary 60K37; Secondary 60F15; 82C22

Keywords: Random walk; Dynamic random environment; Non-elliptic; Conditional cone-mixing; Regeneration; Law of large numbers

1. Introduction

1.1. Background

Random walk in random environment (RWRE) has been an active area of research for more than three decades. Informally, RWREs are random walks in discrete or continuous space–time

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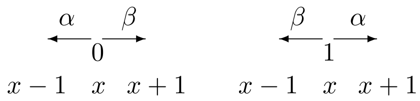

Fig. 1. Jump rates of the (α, β)-walk on top of a hole (=0), respectively, a particle (=1).

whose transition kernels or transition rates are not fixed but are random themselves, constituting a random environment. Typically, the law of the random environment is taken to be translation invariant. Once a realization of the random environment is fixed, we say that the law of the random walk is *quenched*. Under the quenched law, the random walk is Markovian but not translation invariant. It is also interesting to consider the quenched law averaged over the law of the random environment, which is called the *annealed law*. Under the annealed law, the random walk is not Markovian but translation invariant. For an overview on RWRE, we refer the reader to Zeitouni [12,13], Sznitman [10,11], and references therein.

In the past decade, several models have been considered in which the random environment itself evolves in time. These are referred to as *random walk in dynamic random environment* (RWDRE). By viewing time as an additional spatial dimension, RWDRE can be seen as a special case of RWRE, and as such it inherits the difficulties present in RWRE in dimensions two or higher. However, RWDRE can be harder than RWRE because it is an interpolation between RWRE and homogeneous random walk, which arise as limits when the dynamics is slow, respectively, fast. For a list of mathematical papers dealing with RWDRE, we refer the reader to [3]. Most of the literature on RWDRE is restricted to situations in which the space–time correlations of the random environment are either absent or rapidly decaying.

One paper in which a milder space–time mixing property is considered is [2], where a law of large numbers (LLN) is derived for a class of one-dimensional RWDREs in which the role of the random environment is taken by an *interacting particle system* (IPS) with configuration space

$$\Omega := \{0, 1\}^{\mathbb{Z}}. \tag{1.1}$$

In their paper, the random walk starts at 0 and has transition rates as in Fig. 1: on a *hole* (i.e., on a 0) the random walk has rate α to jump one unit to the left and rate β to jump one unit to the right, while on a *particle* (i.e., on a 1) the rates are reversed (w.l.o.g. it may be assumed that $0 < \beta < \alpha < \infty$, so that the random walk has a drift to the left on holes and a drift to the right on particles). Hereafter, we will refer to this model as the (α, β)-model. The LLN is proved under the assumption that the IPS satisfies a space–time mixing property called *cone-mixing* (see Fig. 2), which means that the states inside a space–time cone are almost independent of the states in a space plane far below this cone. The proof uses a regeneration scheme originally developed by Comets and Zeitouni [5] for RWRE and adapted to deal with RWDRE. This proof can be easily extended to $\mathbb{Z}^d$, $d \geq 2$, with the appropriate corresponding notion of cone-mixing.

1.2. Elliptic vs. non-elliptic

The original motivation for the present paper was to study the (α, β)-model in the limit as $\alpha \to \infty$ and $\beta \downarrow 0$. In this limit, which we will refer to as the $(\infty, 0)$-model, the walk is almost a deterministic functional of the IPS; in particular, it is non-elliptic. The challenge was to find a way to deal with the *lack of ellipticity*. As we will see in Section 3, our set-up will be rather general and will include the (α, β)-model, the $(\infty, 0)$-model, as well as various other models. Examples of papers that deal with non-elliptic (actually, deterministic) RW(D)REs are Madras [7] and

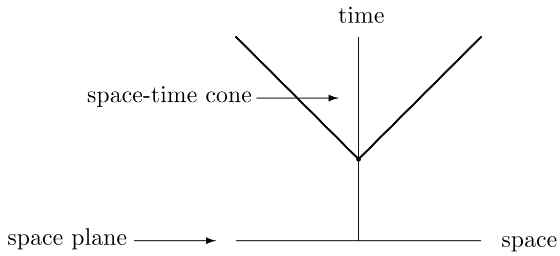

Fig. 2. Cone-mixing property: asymptotic independence of states inside a space–time cone from states inside a space plane.

Matic [9], where a recurrence vs. transience criterion, respectively, a large deviation principle are derived.

In the RW(D)RE literature, ellipticity assumptions play an important role. In the static case, RWRE in $\mathbb{Z}^d$, $d \geq 1$, is called *elliptic* when, almost surely w.r.t. the random environment, all the rates are *finite* and there is a basis $\{e_i\}_{1 \leq i \leq d}$ of $\mathbb{Z}^d$ such that the rate to go from x to $x + e_i$ is *positive* for $1 \leq i \leq d$. It is called *uniformly elliptic* when these rates are *bounded away from infinity*, respectively, *bounded away from zero*. In [5], in order to take advantage of the mixing property assumed on the random environment, it is important to have uniform ellipticity not necessarily in all directions, but in at least one direction in which the random walk is transient. One way to state this "uniform directional ellipticity" in a way that encompasses also the dynamic setting is to require the existence of a deterministic time $T > 0$ and a vector $e \in \mathbb{Z}^d$ such that the quenched probability for the random walk to displace itself along e during time T is uniformly positive for almost every realization of the random environment. This is satisfied by the (α, β)-model for $e = 0$ and any $T > 0$. This model is also transient (indeed, non-nestling) in the time direction, which enables the use of the cone-mixing property of [2]. In the case of the $(\infty, 0)$-model, however, there are in general no such T and e. For example, when the random environment is a spin-flip system with bounded flip rates, any fixed space–time position has positive probability of being unreachable by the random walk. For all such models, the approach in [2] fails.

In the present paper, in order to deal with the possible lack of ellipticity we require a different space–time mixing property for the dynamic random environment, which we call *conditional cone-mixing*. Moreover, as in [5,2], we must require the random walk to have a tendency to stay inside space–time cones. Under these assumptions, we are able to set up a regeneration scheme and prove a LLN. Our result includes the LLN for the (α, β)-model in [2], the $(\infty, 0)$-model for at least two subclasses of IPSs that we will exhibit, as well as models that are intermediate, in the sense that they are neither uniformly elliptic in any direction, nor deterministic as the $(\infty, 0)$-model.

1.3. Outline

The rest of the paper is organized as follows. In Section 2 we discuss, still informally, the $(\infty, 0)$-model and the regeneration strategy. This section serves as a motivation for the formal definition in Section 3 of the class of models we are after, which is based on three *structural assumptions*. Section 4 contains the statement of our LLN under four *hypotheses*, and a description of two classes of one-dimensional IPSs that satisfy these hypotheses for the $(\infty, 0)$-model, namely, spin-flip systems with bounded flip rates that either are in Liggett's $M < \epsilon$ regime, or have finite range and a small enough ratio of maximal/minimal flip rates. Section 5 contains preparation material, given in a general context, that is used in the proof of the LLN given in Section 6. In Section 7 we verify our hypotheses for the two classes of IPSs described in Section 4. We also obtain a criterion to determine the sign of the speed in the LLN, via a comparison with independent spin-flip systems. Finally, in Section 8, we discuss how to adapt the proofs in Section 7 to other models, namely, generalizations of the (α, β)-model and the $(\infty, 0)$-model, and mixtures thereof. We also give an example where our hypotheses fail. The examples in our paper are all one-dimensional, even though our LLN is valid in $\mathbb{Z}^d$, $d \geq 1$.

2. Motivation

2.1. *The* $(\infty, 0)$*-model*

Let

$$\xi := (\xi_t)_{t\geq 0} \quad \text{with } \xi_t := \big(\xi_t(x)\big)_{x\in\mathbb{Z}} \tag{2.1}$$

be a càdlàg Markov process on Ω. We will interpret ξ by saying that at time t site x contains either a *hole* ($\xi_t(x) = 0$) or a *particle* ($\xi_t(x) = 1$). Typical examples are interacting particle systems on Ω, such as independent spin-flips and simple exclusion.

Suppose that we run the (α, β)-model on ξ with $0 < \beta \ll 1 \ll \alpha < \infty$. Then the behavior of the random walk is as follows. Suppose that $\xi_0(0) = 1$ and that the walk starts at 0. The walk rapidly moves to the first hole on its right, typically before any of the particles it encounters manages to flip to a hole. When it arrives at the hole, the walk starts to rapidly jump back and forth between the hole and the particle to the left of the hole: we say that it sits in a *trap*. If $\xi_0(0) = 0$ instead, then the walk rapidly moves to the first particle on its left, where it starts to rapidly jump back and forth in a trap. In both cases, before moving away from the trap, the walk typically waits until one or both of the sites in the trap flip. If only one site flips, then the walk typically moves in the direction of the flip until it hits a next trap, etc. If both sites flip simultaneously, then the probability for the walk to sit at either of these sites is close to $\frac{1}{2}$, and hence it leaves the trap in a direction that is close to being determined by an independent fair coin.

The limiting dynamics when $\alpha \to \infty$ and $\beta \downarrow 0$ can be obtained from the above description by removing the words "rapidly, "typically" and "close to". Except for the extra Bernoulli ($\frac{1}{2}$) random variables needed to decide in which direction to go to when both sites in a trap flip simultaneously, the walk up to time t is a deterministic functional of $(\xi_s)_{0\leq s\leq t}$. In particular, if ξ changes only by single-site flips, then apart from the first jump the walk is completely deterministic. Since the walk spends all of its time in traps where it jumps back and forth between a hole and a particle, we may imagine that it lives on the edges of $\mathbb{Z}$. We implement this observation by associating with each edge its left-most site, i.e., we say that the walk is at x when we actually mean that it is jumping back and forth between x and $x + 1$. See Fig. 3.

Let

$$W := (W_t)_{t\geq 0} \tag{2.2}$$

denote the random walk path. By the description above, W is càdlàg and

$$W_t \text{ is a function of } \big((\xi_s)_{0\leq s\leq t}, Y\big), \tag{2.3}$$

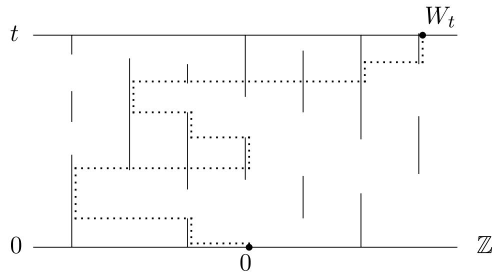

Fig. 3. The vertical lines represent the presence of particles. The dotted line is the path of the $(\infty, 0)$-walk.

where Y is a sequence of i.i.d. Bernoulli($\frac{1}{2}$) random variables independent of ξ. Note that W also has the following three properties:

(1) For any fixed time s, the increment $W_{s+t} - W_s$ is found by applying the same function in (2.3) to the environment shifted in space and time by (W_s, s) and an independent copy of Y; in particular, the pair (W_t, ξ_t) is Markovian.
(2) Given that W stays inside a space–time cone until time t, $(W_s)_{0\leq s\leq t}$ is a functional only of Y and of the states in ξ up to time t inside a slightly larger cone, obtained by adding all neighboring sites to the right.
(3) Each jump of the path follows the same mechanism as the first jump, i.e., $W_t - W_{t-}$ is computed using the same rules as those for W_0 but applied to the environment shifted in space and time by (W_{t-}, t).

The reason for emphasizing these properties will become clearer in Section 2.2.

2.2. Regeneration

The cone-mixing property that is assumed in [2] to prove the LLN for the (α, β)-model can be loosely described as the requirement that all the states of the IPS inside a space–time cone opening upwards depend weakly on the states inside a space plane far below the tip (recall Fig. 2). Let us give a rough idea of how this property can lead to *regeneration*. Consider the event that the walk stands still for a long time. Since the jump times of the walk are independent of the IPS, so is this event. During this pause, the environment around the walk is allowed to mix, which by the cone-mixing property means that by the end of the pause all the states inside a cone with a tip at the space–time position of the walk are almost independent of the past of the walk. If thereafter the walk stays confined to the cone, then its future increments will be almost independent of its past, and so we get an approximate regeneration. Since in the (α, β)-model there is a uniformly positive probability for the walk to stay inside a space–time cone with a large enough inclination, we see that this regeneration strategy can indeed be made to work. See Fig. 4.

For the actual proof of the LLN in [2], cone-mixing must be more carefully defined. For technical reasons, there must be some uniformity in the decay of correlations between events in the space–time cone and in the space plane. This uniformity holds, for instance, for any spin-flip system in the $M < \epsilon$ regime (Liggett [6], Section I.3), but not for the exclusion process or the supercritical contact process. Therefore the approach outlined above works for the first IPS, but not for the other two.

There are three properties of the (α, β)-model that make the above heuristics plausible. First, to be able to apply the cone-mixing property relative to the space–time position of the walk, it

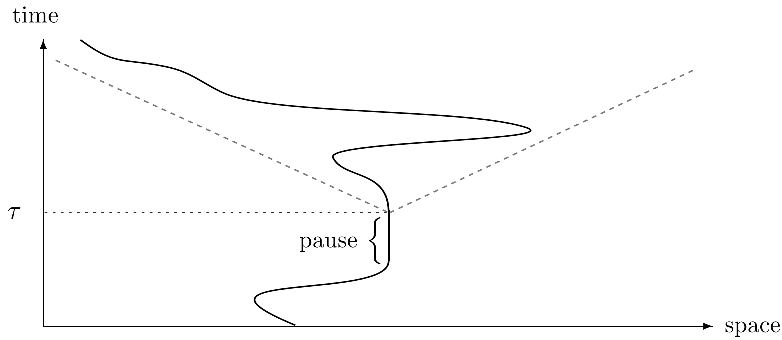

Fig. 4. Regeneration at time τ.

is important that the pair (IPS, walk) is Markovian and that the law of the environment as seen from the walk at any time is comparable to the initial law. Second, there is a uniformly positive probability for the walk to stand still for a long time and afterwards stay inside a space–time cone. Third, once the walk stays inside a space–time cone, its increments depend on the IPS only through the states inside that cone. Let us compare these observations with what happens in the $(\infty, 0)$-model. Property (1) from Section 2.1 gives us the Markov property, while property (2) gives us the measurability inside cones. As we will see, when the environment is translation-invariant, property (3) implies absolute continuity of the law of the environment as seen from the walk at any positive time with respect to its counterpart at time zero. Therefore, as long as we can make sure that the walk has a tendency to stay inside space–time cones (which is reasonable when we are looking for a LLN), the main difference is that the event of standing still for a long time is not independent of the environment, but rather is a *deterministic* functional of the environment. Consequently, it is not at all clear whether cone-mixing is enough to allow for regeneration. On the other hand, the event of standing still is local, since it only depends on the states of the two neighboring sites of the trap where the walk is pausing. For many IPSs, the observation of a local event will not affect the weak dependence between states that are far away in space–time. Hence, if such IPSs are cone-mixing, then states inside a space–time cone remain almost independent of the initial configuration even when we condition on seeing a trap for a long time.

Thus, under suitable assumptions, the event "standing still for a long time" is a candidate to induce regeneration. In the (α, β)-model this event does not depend on the environment whereas in the $(\infty, 0)$-model it is a deterministic functional of the environment. If we put the (α, β)-model in the form (2.3) by taking for Y two independent Poisson processes with rates α and β, then we can restate the previous sentence by saying that in the (α, β)-model the regeneration-inducing event depends only on Y, while in the $(\infty, 0)$-model it depends only on ξ. We may therefore imagine that, also for other models of the type (2.3) and that share properties (1)–(3), it will be possible to find more general regeneration-inducing events that depend on both ξ and Y in a non-trivial manner. This motivates our setup in Section 3.

3. Model setting

So far we have mostly been discussing RWDRE driven by an IPS. However, there are convenient constructions of IPSs on richer state spaces (such as graphical representations) that can facilitate the construction of the regeneration-inducing events mentioned in Section 2.2. We will therefore allow for more general Markov processes to represent the dynamic random environment ξ. Notation is set up in Section 3.1. Section 3.2 contains the three structural assumptions that define the class of models we will consider.

3.1. Notation and setup

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of natural numbers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let E be a Polish space and $\xi := (\xi_t)_{t\geq 0}$ a Markov process with state space $E^{\mathbb{Z}^d}$ where $d \in \mathbb{N}$. Let $Y := (Y_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence of random elements independent of ξ. For $I \subset [0, \infty)$, abbreviate $\xi_I := (\xi_u)_{u\in I}$, and analogously for Y. The joint law of ξ and Y when $\xi_0 = \eta \in E^{\mathbb{Z}^d}$ will be denoted by $\mathbb{P}_\eta$. For $n \in \mathbb{N}$, put $\mathscr{Y}_n := \sigma(Y_{[1,n]})$. Let $\mathscr{F}_0 := \sigma(\xi_0)$ and, for $t > 0$, $\mathscr{F}_t := \sigma(\xi_{[0,t]}) \vee \mathscr{Y}_{\lceil t\rceil}$.

For $t \geq 0$ and $x \in \mathbb{Z}^d$, let θ_t and θ_x be the time-shift and space-shift operators given by

$$\theta_t(\xi, Y) := \big((\xi_{t+s})_{s\geq 0}, (Y_{\lfloor t\rfloor+n})_{n\in\mathbb{N}}\big), \qquad \theta_x(\xi, Y) := \big((\theta_x\xi_t)_{t\geq 0}, (Y_n)_{n\in\mathbb{N}}\big), \tag{3.1}$$

where $\theta_x\xi_t(y) = \xi_t(x + y)$. In the sequel, whether θ is a time-shift or a space-shift operator will always be clear from the index.

We assume that ξ is translation-invariant, i.e., $\theta_x\xi$ has under $\mathbb{P}_\eta$ the same distribution as ξ under $\mathbb{P}_{\theta_x\eta}$. We also assume the existence of a (not necessarily unique) translation-invariant equilibrium distribution μ for ξ, and write $\mathbb{P}_\mu(\cdot) := \int \mu(d\eta)\,\mathbb{P}_\eta(\cdot)$ to denote the joint law of ξ and Y when ξ_0 is drawn from μ.

The random walk will be denoted by $W = (W_t)_{t\geq 0}$, and we will write $\bar{\xi} := (\bar{\xi}_t)_{t\geq 0}$ to denote the *environment process as seen from* W, i.e., $\bar{\xi}_t := \theta_{W_t}\xi_t$. Let $\bar{\mu}_t$ denote the law of $\bar{\xi}_t$ under $\mathbb{P}_\mu$. We abbreviate $\bar{\mu} := \bar{\mu}_0$. Note that $\bar{\mu} = \mu$ when $\mathbb{P}_\mu(W_0 = 0) = 1$.

For $m > 0$ and $R \in \mathbb{N}_0$, define the R-enlarged m-cone by

$$C_R(m) := \big\{(x, t) \in \mathbb{Z}^d \times [0, \infty) \colon \|x\| \leq mt + R\big\}, \tag{3.2}$$

where $\|\cdot\|$ is the L^1 norm. Let $\mathscr{C}_{R,t}(m)$ be the σ-algebras generated by the states of ξ up to time t inside $C_R(m)$.

3.2. Structural assumptions

We will assume that W is random translation of a random walk starting at 0. More precisely, we assume that $Z = (Z_t)_{t\geq 0}$ is a càdlàg $\mathscr{F}$-adapted $\mathbb{Z}^d$-valued process with $Z_0 = 0\mathbb{P}_{\bar{\mu}}$-a.s. such that

$$W_t = W_0 + \theta_{W_0} Z_t \quad \forall t \geq 0. \tag{3.3}$$

We also assume that $W_0 \in \mathbb{Z}^d$ and depends on ξ and Y only through ξ_0, i.e.,

$$\mathbb{P}_\mu(W_0 = x \mid \mathscr{F}_\infty) = \mathbb{P}_\mu(W_0 = x \mid \xi_0) \quad \text{a.s. } \forall x \in \mathbb{Z}^d. \tag{3.4}$$

Under these assumptions, $(W_t - W_0)_{t\geq 0}$ has under $\mathbb{P}_\mu$ the same distribution as Z under $\mathbb{P}_{\bar{\mu}}$. In what follows we make *three structural assumptions* on Z:

(A1) **(Additivity)**
For all $n \in \mathbb{N}$,

$$(Z_{t+n} - Z_n)_{t\geq 0} = \theta_{Z_n}\theta_n Z \quad \mathbb{P}_{\bar{\mu}}\text{-a.s.} \tag{3.5}$$

(A2) **(Locality)**
For $m > 0$, let $\mathcal{D}_m := \{\|Z_t\| \leq mt\, \forall t \geq 0\}$. Then there exists $R \in \mathbb{N}_0$ such that, $\forall\, m > 0$, both $\mathcal{D}_m$ and $(1_{\mathcal{D}_m} Z_t)_{t\geq 0}$ are measurable w.r.t. $\mathscr{C}_{R,\infty}(m) \vee \mathscr{Y}_\infty$.

(A3) **(Homogeneity of jumps)**
For all $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$,

$$\mathbb{P}_{\bar{\mu}}\left(Z_n - Z_{n-} = x \mid \xi_{[0,n]}, Z_{[0,n)}\right) = \mathbb{P}_{\theta_{Z_{n-}}\xi_n}\big(W_0 = x\big) \quad \mathbb{P}_{\bar{\mu}}\text{-a.s.} \tag{3.6}$$

These properties are analogs of properties (1)–(3) of the $(\infty, 0)$-model mentioned in Section 2.1, with the difference that we only require them to hold at integer times; this will be enough as our proof relies on integer-valued regeneration times. We also assume the 'extra randomness' Y to be split independently among time intervals of length 1; for example, in the case of the $(\infty, 0)$-model, each Y_n would *not* be a Bernoulli($\frac{1}{2}$) random variable but a whole *sequence* of such variables instead. This is discussed in detail in Section 7.1.
Another remark: assumption (A3) might seem strange since many random walk models have no deterministic jumps, which is indeed the case for the examples described in Section 4. Note however that, in this case, (A3) severely restricts W_0, implying $W_0 = 0$ a.s. when ξ is started from $\theta_{Z_{n-}}\xi_n$. Furthermore, our main theorem (Theorem 4.1 below) is not restricted to this situation and includes also cases with deterministic jumps. For example, one could modify the $(\infty, 0)$-walk to jump exactly at integer times. Additional examples with deterministic jumps are described in item 4 of Section 8. The relevance of assumption (A3) is in showing that the law of the environment as seen by the RW after any jump is absolutely continuous w.r.t. the law after the first jump; this is done in Lemma 6.1 below.

4. Main results

Theorems 4.1 and 4.2 below are the main results of our paper. Theorem 4.1 in Section 4.1 is our LLN. Theorem 4.2 in Section 4.2 verifies the hypotheses in this LLN for the $(\infty, 0)$-model in two classes of one-dimensional IPSs. For these classes some more information is available, namely, convergence in L^p, $p \geq 1$, and a criterion to determine the sign of the speed.

4.1. Law of large numbers

In order to develop a regeneration scheme for a random walk subject to assumptions (A1)–(A3) based on the heuristics discussed in Section 2.2, we need suitable regeneration-inducing events. In the *four hypotheses* stated below, these events appear as a sequence $(\Gamma_L)_{L\in\mathbb{N}}$ such that, for a certain fixed $m \in (0, \infty)$ and R as in (A2), $\Gamma_L \in \mathscr{C}_{R,L}(m) \vee \mathscr{Y}_L$ for all $L \in \mathbb{N}$.

(H1) **(Determinacy)**
On Γ_L, $Z_t = 0$ for all $t \in [0, L]$ $\mathbb{P}_{\bar{\mu}}$-a.s.

(H2) **(Non-degeneracy)**
For L large enough, there exists a $\gamma_L > 0$ such that $\mathbb{P}_\eta(\Gamma_L) \geq \gamma_L$ for $\bar{\mu}$-a.e. η.

(H3) **(Cone constraints)**
Let $\mathcal{S} := \inf\{t > 0\colon \|Z_t\| > mt\}$. Then there exist $a \in (1, \infty)$, $\kappa_L \in (0, 1]$ and $\psi_L \in [0, \infty)$ such that, for L large enough and $\bar{\mu}$-a.e. η,

$$\begin{aligned} &(1) \quad \mathbb{P}_\eta(\theta_L\mathcal{S} = \infty \mid \Gamma_L) \geq \kappa_L, \\ &(2) \quad \mathbb{E}_\eta\left[1_{\{\theta_L\mathcal{S}<\infty\}}(\theta_L\mathcal{S})^a \mid \Gamma_L\right] \leq \psi_L^a. \end{aligned} \tag{4.1}$$

(H4) **(Conditional cone-mixing)**
There exists a sequence of non-negative numbers $(\Phi_L)_{L\in\mathbb{N}}$ satisfying $\lim_{L\to\infty}\kappa_L^{-1}\Phi_L = 0$ such that, for L large enough and for $\bar{\mu}$-a.e. η,

$$\left|\mathbb{E}_\eta\left(\theta_L f \mid \Gamma_L\right) - \mathbb{E}_{\bar{\mu}}(\theta_L f \mid \Gamma_L)\right| \leq \Phi_L\|f\|_\infty \quad \forall f \in \mathscr{C}_{R,\infty}(m),\ f \geq 0. \tag{4.2}$$

We are now ready to state our LLN.

Theorem 4.1. *Under assumptions* (A1)–(A3)*and hypotheses* (H1)–(H4)*, there exists a* $w \in \mathbb{R}^d$ *such that*

$$\lim_{t\to\infty} t^{-1} W_t = w \quad \mathbb{P}_\mu - a.s. \tag{4.3}$$

Remark 1. Hypothesis (H4) above without the conditioning on Γ_L in (4.2) and with constant κ_L is the same as the cone-mixing condition used by Avena et al. [2]. There, $W_0 = 0\,\mathbb{P}_\mu$-a.s., so that $\bar{\mu} = \mu$.

Remark 2. Theorem 4.1 provides no information about the value of w, not even its sign when $d = 1$. Understanding the dependence of w on model parameters is in general a highly non-trivial problem.

4.2. Examples

We next describe two classes of one-dimensional IPSs for which the $(\infty, 0)$-model satisfies hypotheses (H1)–(H4). Further details will be given in Section 7. In both classes, ξ is a spin-flip system in $\Omega = \{0, 1\}^{\mathbb{Z}}$ with bounded and translation-invariant single-site flip rates. We may assume that the flip rates at the origin are of the form

$$c(\eta) = \begin{cases} c_0 + \lambda_0 p_0(\eta) & \text{if } \eta(0) = 1, \\ c_1 + \lambda_1 p_1(\eta) & \text{if } \eta(0) = 0, \end{cases} \qquad \eta \in \Omega, \tag{4.4}$$

for some $c_i, \lambda_i \geq 0$ and $p_i\colon\ \Omega \to [0, 1]$, $i = 0, 1$.

Example 1. $c(\cdot)$ is in the $M < \epsilon$ regime (see Liggett [6], Section I.3).

Example 2. $p(\cdot)$ has finite range and $(\lambda_0+\lambda_1)/(c_0+c_1) < \lambda_c$, where λ_c is the critical infection rate of the one-dimensional contact process with the same range.

Theorem 4.2. *Consider the* $(\infty, 0)$*-model. Suppose that* ξ *is a spin-flip system with flip rates given by* (4.4)*. Then for Examples* 1 *and* 2 *there exist a version of* ξ *and events* $\Gamma_L \in \mathscr{C}_{R,L}(m) \vee \mathscr{Y}_L$*,* $L \in \mathbb{N}$*, satisfying hypotheses* (H1)–(H4)*. Furthermore, the convergence in Theorem* 4.1 *holds also in* L^p *for all* $p \geq 1$*, and*

$$\begin{aligned} w &\geq \frac{c_0 + \lambda_0}{c_1 + c_0 + \lambda_0}(c_1 - c_0 - \lambda_0) \quad \textit{if } c_1 \geq c_0 + \lambda_0, \\ w &\leq -\frac{c_1 + \lambda_1}{c_0 + c_1 + \lambda_1}(c_0 - c_1 - \lambda_1) \quad \textit{if } c_0 \geq c_1 + \lambda_1. \end{aligned} \tag{4.5}$$

For independent spin-flip systems (i.e., when $\lambda_0 = \lambda_1 = 0$), (4.5) shows that w is positive, zero or negative when the density $c_1/(c_0 + c_1)$ is, respectively, larger than, equal to or smaller than $\frac{1}{2}$. The criterion for other ξ is obtained by comparison with independent spin-flip systems.

We expect hypotheses (H1)–(H4) to hold for a very large class of IPSs and walks. For each choice of IPS and walk, the *verification* of hypotheses (H1)–(H4) constitutes a *separate problem*. Typically, (H1)–(H2) are immediate, (H3) requires some work, while (H4) is hard.

Additional models will be discussed in Section 8. We will consider generalizations of the (α, β)-model and the $(\infty, 0)$-model, namely, *internal noise* models and *pattern* models, as well as mixtures of them. The verification of (H1)–(H4) will be analogous to the two examples discussed above and will not be carried out in detail.

This concludes the motivation and the statement of our main results. The remainder of the paper will be devoted to the proofs of Theorems 4.1 and 4.2, with the exception of Section 8, which contains additional examples and remarks.

5. Preparation

The aim of this section is to prove two propositions (Propositions 5.2 and 5.4 below) that will be needed in Section 6 to prove the LLN. In Section 5.1 we deal with approximate laws of large numbers for general discrete- or continuous-time random walks in $\mathbb{R}^d$. In Section 5.2 we specialize to additive functionals of a Markov chain whose transition kernel satisfies a certain absolute-continuity property.

5.1. *Approximate law of large numbers*

This section contains two fundamental facts that are the basis of our proof of the LLN. They deal with the notion of an approximate law of large numbers.

Definition 5.1. Let $W = (W_t)_{t\geq 0}$ be a random process in $\mathbb{R}^d$ with $t \in \mathbb{N}_0$ or $t \in [0,\infty)$. For $\varepsilon \geq 0$ and $v \in \mathbb{R}^d$, we say that W has an ε-approximate asymptotic velocity v, written as $W \in AV(\varepsilon, v)$, if

$$\limsup_{t\to\infty} \left\| \frac{W_t}{t} - v \right\| \leq \varepsilon \quad \text{a.s.} \tag{5.1}$$

We take $\|\cdot\|$ to be the L_1-norm. A simple observation is that W a.s. has an asymptotic velocity if and only if for every $\varepsilon > 0$ there exists a $v_\varepsilon \in \mathbb{R}^d$ such that $W \in AV(\varepsilon, v_\varepsilon)$. In this case $\lim_{\varepsilon\downarrow 0} v_\varepsilon$ exists and is equal to the asymptotic velocity.

5.1.1. *First key proposition: skeleton approximate velocity*

The following proposition gives conditions under which an approximate velocity for the process observed along a random sequence of times implies an approximate velocity for the full process.

Proposition 5.2. *Let W be as in Definition 5.1. Set $\tau_0 := 0$, let $(\tau_k)_{k\in\mathbb{N}}$ be an increasing sequence of random times in $(0,\infty)$ (or $\mathbb{N}$) with $\lim_{k\to\infty} \tau_k = \infty$ a.s. and put $X_k := (W_{\tau_k}, \tau_k) \in \mathbb{R}^{d+1}$, $k \in \mathbb{N}_0$. Suppose that the following hold:*

(i) *There exists an $m > 0$ such that*

$$\limsup_{k\to\infty} \sup_{s\in(\tau_k,\tau_{k+1}]} \left\| \frac{W_s - W_{\tau_k}}{s - \tau_k} \right\| \leq m \quad a.s. \tag{5.2}$$

(ii) *There exist $v \in \mathbb{R}^d$, $u > 0$ and $\varepsilon \geq 0$ such that $X \in AV(\varepsilon, (v, u))$.*

Then $W \in AV((3m + 1)\varepsilon/u, v/u)$.

Proof. First, let us check that (i) implies

$$\limsup_{t\to\infty} \frac{\|W_t\|}{t} \le m \quad \text{a.s.} \tag{5.3}$$

Suppose that

$$\limsup_{k\to\infty} \sup_{s>\tau_k} \left\| \frac{W_s - W_{\tau_k}}{s-\tau_k} \right\| \le m \quad \text{a.s.} \tag{5.4}$$

Since, for every k and $t > \tau_k$,

$$\left\|\frac{W_t}{t}\right\| \le \frac{\|W_{\tau_k}\|}{t} + \left\|\frac{W_t - W_{\tau_k}}{t-\tau_k}\right\| \left|1-\frac{\tau_k}{t}\right| \le \frac{\|W_{\tau_k}\|}{t} + \sup_{s>\tau_k} \left\|\frac{W_s - W_{\tau_k}}{s-\tau_k}\right\| \left|1-\frac{\tau_k}{t}\right|, \tag{5.5}$$

(5.3) follows from (5.4) by letting $t\to\infty$ followed by $k\to\infty$.

To check (5.4), define, for $k\in\mathbb{N}_0$ and $l\in\mathbb{N}$,

$$m(k,l) := \sup_{s\in(\tau_k,\tau_{k+l}]} \left\|\frac{W_s - W_{\tau_k}}{s-\tau_k}\right\| \quad \text{and}$$

$$m(k,\infty) := \sup_{s>\tau_k} \left\|\frac{W_s - W_{\tau_k}}{s-\tau_k}\right\| = \lim_{l\to\infty} m(k,l). \tag{5.6}$$

Using the fact that $(x_1+x_2)/(y_1+y_2) \le (x_1/y_1)\vee(x_2/y_2)$ for all $x_1,x_2\in\mathbb{R}$ and $y_1,y_2>0$, we can prove by induction that

$$m(k,l) \le \max\{m(k,1),\dots,m(k+l-1,1)\}, \qquad l\in\mathbb{N}. \tag{5.7}$$

Fix $\varepsilon>0$. By (i), a.s. there exists a k_ε such that $m(k,1)\le m+\varepsilon$ for $k>k_\varepsilon$. By (5.7), the same is true for $m(k,l)$ for all $l\in\mathbb{N}$, and therefore also for $m(k,\infty)$. Since ε is arbitrary, (5.4) follows.

Let us now proceed with the proof of the proposition. Assumption (ii) implies that, a.s.,

$$\limsup_{k\to\infty} \left\|\frac{W_{\tau_k}}{k} - v\right\| \le \varepsilon \quad \text{and} \quad \limsup_{k\to\infty} \left|\frac{\tau_k}{k} - u\right| \le \varepsilon. \tag{5.8}$$

For $t\ge 0$, let k_t be the (random) non-negative integer such that

$$\tau_{k_t} \le t < \tau_{k_t+1}. \tag{5.9}$$

Since $\tau_1<\infty$ a.s., $k_t>0$ for large enough t. From (5.8) and (5.9) we deduce that

$$\limsup_{t\to\infty} \left|\frac{t}{k_t} - u\right| \le \varepsilon \quad \text{and so} \quad \limsup_{t\to\infty} \left|\frac{t}{k_t} - \frac{\tau_{k_t}}{k_t}\right| \le 2\varepsilon. \tag{5.10}$$

For t large enough we may write

$$\begin{aligned}\left\|\frac{uW_t}{t} - v\right\| &\le \frac{\|W_t\|}{t}\left|u-\frac{t}{k_t}\right| + \left\|\frac{W_t - W_{\tau_{k_t}}}{k_t}\right\| + \left\|\frac{W_{\tau_{k_t}}}{k_t} - v\right\| \\ &\le \frac{\|W_t\|}{t}\left|u-\frac{t}{k_t}\right| + \sup_{s\in(\tau_{k_t},\tau_{k_t+1}]} \left\|\frac{W_s - W_{\tau_{k_t}}}{s-\tau_{k_t}}\right\| \left|\frac{t-\tau_{k_t}}{k_t}\right| + \left\|\frac{W_{\tau_{k_t}}}{k_t} - v\right\|,\end{aligned} \tag{5.11}$$

from which we obtain the conclusion by taking the limsup as $t\to\infty$ in (5.11), using (i), (5.3), (5.8) and (5.10), and then dividing by u. □

5.1.2. Conditions for the skeleton to have an approximate velocity

The following lemma states sufficient conditions for a discrete-time process to have an approximate velocity. It will be used in the proof of Proposition 5.4 below.

Lemma 5.3. *Let $X = (X_k)_{k\in\mathbb{N}_0}$ be a sequence of random vectors in $\mathbb{R}^d$ with joint law P such that $P(X_0 = 0) = 1$. Suppose that there exist a probability measure Q on $\mathbb{R}^d$ and numbers $\phi \in [0, 1)$, $a > 1$, $K > 0$ with $\int_{\mathbb{R}^d} \|x\|^a\, Q(dx) \le K^a$, such that, P-a.s. for all $k \in \mathbb{N}_0$,*

(i) $|P(X_{k+1} - X_k \in A \mid X_0, \dots, X_k) - Q(A)| \le \phi$ *for all A measurable;*
(ii) $E[\|X_{k+1} - X_k\|^a | X_0, \dots, X_k] \le K^a$. *Then*

$$\limsup_{n\to\infty} \left\| \frac{X_n}{n} - v \right\| \le 2K\phi^{(a-1)/a} \quad P\text{-a.s.,} \tag{5.12}$$

where $v = \int_{\mathbb{R}^d} x\, Q(dx)$. In other words, $X \in AV(2K\phi^{(a-1)/a}, v)$.

Proof. The proof is an adaptation of the proof of Lemma 3.13 in [5]; we include it here for completeness. With regular conditional probabilities, we can, using (i), couple P and $Q^{\otimes\mathbb{N}_0}$ according to a standard splitting representation (see e.g. Berbee [4]). More precisely, on an enlarged probability space we can construct random variables

$$(\Delta_k, V_k, R_k)_{k\in\mathbb{N}} \tag{5.13}$$

such that

(1) $(\Delta_k)_{k\in\mathbb{N}}$ is an i.i.d. sequence of Bernoulli(ϕ) random variables.
(2) $(V_k)_{k\in\mathbb{N}}$ is an i.i.d. sequence of random vectors with law Q.
(3) $(\Delta_l)_{l\ge k}$ is independent of $(\Delta_l, V_l, R_l)_{0\le l<k}, R_k$.
(4) Setting $\hat{X}_0 := 0$ and, for $k \in \mathbb{N}_0$, $\hat{X}_{k+1} - \hat{X}_k := (1 - \Delta_k)V_k + \Delta_k R_k$, then $\hat{X}$ is equal in distribution to X.
(5) Setting $\mathcal{G}_k := \sigma(\Delta_l, V_l, R_l\colon\ 0 \le l \le k)$, then $E[f(V_k) \mid \mathcal{G}_{k-1}]$ is measurable w.r.t. $\sigma(\hat{X}_l\colon 0 \le l \le k-1)$ for any Borel nonnegative function f.

Using (4), we may write

$$\frac{X_n}{n} \overset{d}{=} \frac{\hat{X}_n}{n} = \frac{1}{n}\sum_{k=1}^{n} V_k - \frac{1}{n}\sum_{k=1}^{n} \Delta_k V_k + \frac{1}{n}\sum_{k=1}^{n} \Delta_k R_k. \tag{5.14}$$

As $n \to \infty$, the first term on the r.h.s. converges a.s. to v by the LLN for i.i.d. random variables. By Hölder's inequality, the norm of the second term is at most

$$\left(\frac{1}{n}\sum_{k=1}^{n} \Delta_k\right)^{(a-1)/a} \left(\frac{1}{n}\sum_{k=1}^{n} \|V_k\|^a\right)^{1/a}, \tag{5.15}$$

which, by (1) and (2), converges a.s. as $n \to \infty$ to

$$\phi^{(a-1)/a} \left(\int_{\mathbb{R}^d} \|x\|^a\, Q(dx)\right)^{1/a} \le K\phi^{(a-1)/a}. \tag{5.16}$$

To control the third term, put $R_k^* := E[R_k \mid \mathcal{G}_{k-1}]$. Since $\|\Delta_k R_k\| \le \|\hat{X}_{k+1} - \hat{X}_k\|$, using (1), (3), (4), (5) and (ii), we get

$$\phi E[\|R_k\|^a \mid \mathcal{G}_{k-1}] = E[\Delta_k \|R_k\|^a \mid \mathcal{G}_{k-1}] \le E[\|\hat{X}_{k+1} - \hat{X}_k\|^a \mid \mathcal{G}_{k-1}] \le K^a. \tag{5.17}$$

Combining (5.17) with Jensen's inequality, we obtain

$$\|R_k^*\| \le E\big[\|R_k\|^a \mid \mathcal{G}_{k-1}\big]^{1/a} \le \frac{K}{\phi^{1/a}}, \tag{5.18}$$

so that

$$\left\|\frac{1}{n}\sum_{k=1}^n \Delta_k R_k^*\right\| \le \frac{K}{\phi^{1/a}}\left(\frac{1}{n}\sum_{k=1}^n \Delta_k\right) \xrightarrow[n\to\infty]{} K\phi^{(a-1)/a}. \tag{5.19}$$

Now fix $y \in \mathbb{R}^d$ and put

$$M_n^y := \sum_{k=1}^n \frac{\Delta_k}{k}\langle R_k - R_k^*, y\rangle \tag{5.20}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product. Then $(M_n^y)_{n\in\mathbb{N}_0}$ is a $(\mathcal{G}_n)_{n\in\mathbb{N}_0}$-martingale whose quadratic variation is

$$\langle M^y\rangle_n = \sum_{k=1}^n \frac{\Delta_k}{k^2}\langle R_k - R_k^*, y\rangle^2. \tag{5.21}$$

By the Burkholder–Gundy inequality and (5.17)–(5.18), we have

$$\begin{aligned} E\left[\sup_{n\in\mathbb{N}} |M_n^y|^{a\wedge 2}\right] &\le C\, E\left[\langle M^y\rangle_\infty^{(a\wedge 2)/2}\right] \\ &\le C\, E\left[\sum_{k=1}^\infty \frac{\Delta_k}{k^{a\wedge 2}}\left|\langle R_k - R_k^*, y\rangle\right|^{a\wedge 2}\right] \le C\,\|y\|^{a\wedge 2} K^{a\wedge 2}, \end{aligned} \tag{5.22}$$

where C is a positive constant that may change after each inequality. This implies that M_n^y is uniformly integrable and therefore converges a.s. as $n\to\infty$. Kronecker's lemma then gives

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n \Delta_k\langle R_k - R_k^*, y\rangle = 0 \quad \text{a.s.} \tag{5.23}$$

Since y is arbitrary, this in turn implies that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n \Delta_k(R_k - R_k^*) = 0 \quad \text{a.s.} \tag{5.24}$$

Therefore, by (5.19) and (5.24), the limsup of the norm of the last term in the r.h.s. of (5.14) is also bounded by $K\phi^{(a-1)/a}$, which finishes the proof. □

5.2. Additive functionals of a discrete-time Markov chain

5.2.1. Notation

Let $\mathcal{X} = (\mathcal{X}_n)_{n\in\mathbb{N}_0}$ be a time-homogeneous Markov chain in the canonical space equipped with the time-shift operators $(\theta_n)_{n\in\mathbb{N}_0}$. For $n \ge 1$, put $\mathcal{F}_n := \sigma(\mathcal{X}_{[1,n]})$ (note that $\mathcal{X}_0 \notin \mathcal{F}_\infty$) and let P_χ denote the law of $(\mathcal{X}_n)_{n\in\mathbb{N}_0}$ when $\mathcal{X}_0 = \chi$. Fix an initial measure ν and suppose that, for any nonnegative $f \in \mathcal{F}_\infty$,

$$P_\nu(E_{\mathcal{X}_n}[f] \in \cdot) \ll P_\nu(E_{\mathcal{X}_0}[f] \in \cdot), \tag{5.25}$$

where $P_\nu := \int \nu(d\chi)P_\chi$.

Let $Z = (Z_n)_{n \in \mathbb{N}_0}$ be a $\mathbb{Z}^d$-valued $\mathcal{F}$-adapted process that is an *additive functional* of $(\mathcal{X}_n)_{n \in \mathbb{N}}$, i.e., $Z_0 = 0$ and, for any $k \in \mathbb{N}_0$,

$$(Z_{k+n} - Z_k)_{n \in \mathbb{N}_0} = \theta_k Z \quad P_\nu\text{-a.s.} \tag{5.26}$$

We are interested in finding random times $(\tau_k)_{k \in \mathbb{N}_0}$ such that $(Z_{\tau_k}, \tau_k)_{k \in \mathbb{N}_0}$ satisfies the hypotheses of Lemma 5.3. In the Markovian setting it makes sense to look for τ_k of the form

$$\tau_0 = 0, \qquad \tau_{k+1} = \tau_k + \theta_{\tau_k} \tau, \quad k \in \mathbb{N}_0, \tag{5.27}$$

where τ is a random time.

Condition (i) of Lemma 5.3 is a "decoupling condition". It states that the law of an increment of the process depends weakly on the previous increments. Such a condition can be enforced by the occurrence of a "decoupling event" under which the increments of $(Z_{\tau_k}, \tau_k)_{k \in \mathbb{N}_0}$ lose dependence. In this setting, τ is a time at which the decoupling event is observed.

5.2.2. *Second key proposition: approximate regeneration times*

Proposition 5.4 below is a consequence of Lemma 5.3 and is the main result of this section. It will be used together with Proposition 5.2 to prove the LLN in Section 6. It gives a way to construct τ when the decoupling event can be detected by "probing the future" with a stopping time.

For a random variable $\mathcal{T}$ taking values in $\mathbb{N}_0 \cup \{\infty\}$, we define the *image* of $\mathcal{T}$ by $\mathcal{I}_\mathcal{T} := \{n \in \mathbb{N} \colon P_\nu(\mathcal{T} = n) > 0\}$, and its closure under addition by $\bar{\mathcal{I}}_\mathcal{T} := \{n \in \mathbb{N} \colon \exists\, l \in \mathbb{N}, i_1, \ldots, i_l \in \mathcal{I}_\mathcal{T} \colon n = i_1 + \cdots + i_l\}$. Note that $\mathcal{I}_\mathcal{T} = \emptyset$ if and only if $\mathcal{T} \in \{0, \infty\}$ a.s.

Proposition 5.4. *Let $\mathcal{T}$ be a stopping time for the filtration $\mathcal{F}$ taking values in $\mathbb{N} \cup \{\infty\}$. Put $D := \{\mathcal{T} = \infty\}$ and suppose that the following properties hold:*

(i) *For every $n \in \bar{\mathcal{I}}_\mathcal{T}$ there exists a $D_n \in \mathcal{F}_n$ such that*

$$D \cap \theta_n D = D_n \cap \theta_n D \quad P_\nu\text{-a.s.}$$

(ii) *There exist numbers $\rho \in (0, 1]$, $a > 1$, $C > 0$, $m > 0$ and $\phi \in [0, 1)$ such that, P_ν-a.s.,*
   (a) $P_{\mathcal{X}_0}(D) \geq \rho$,
   (b) $E_{\mathcal{X}_0}\left[1_{\{\mathcal{T} < \infty\}} \mathcal{T}^a\right] \leq C^a$,
   (c) *On D, $\|Z_t\| \leq mt$ for all $t \in \mathbb{N}_0$,*
   (d) $\left| E_{\mathcal{X}_0}\left[f(Z, (\theta_n \mathcal{T})_{n \in \bar{\mathcal{I}}_\mathcal{T}}) \mid D\right] - E_\nu\left[f(Z, (\theta_n \mathcal{T})_{n \in \bar{\mathcal{I}}_\mathcal{T}}) \mid D\right]\right| \leq \phi \|f\|_\infty \ \forall f \geq 0$ *measurable.*

Then there exists a random time $\tau \in \mathcal{F}_\infty$ taking values in $\mathbb{N}$ such that, setting τ_k as in (5.27) and $X_k := (Z_{\tau_k}, \tau_k)$, then $X \in AV(\varepsilon, (v, u))$ where $(v, u) = E_\nu[(Z_\tau, \tau) \mid D]$, $u > 0$ and $\varepsilon = 12(m+1)u\phi^{(a-1)/a}$.

5.2.3. *Two further propositions*

In order to prove Proposition 5.4, we will need two further propositions (Propositions 5.5 and 5.6 below).

Proposition 5.5. *Let τ be a random time measurable w.r.t. $\mathcal{F}_\infty$ taking values in $\mathbb{N}$. Put τ_k as in (5.27) and $X_k := (Z_{\tau_k}, \tau_k)$. Suppose that there exists an event $D \in \mathcal{F}_\infty$ such that the following hold P_ν-a.s.:*

(i) *For $n \in \mathcal{I}_\tau$, there exist events H_n and $D_n \in \mathcal{F}_n$ such that*

$$\begin{aligned} &\text{(a)} \quad \{\tau = n\} = H_n \cap \theta_n D, \\ &\text{(b)} \quad D \cap \theta_n D = D_n \cap \theta_n D. \end{aligned} \tag{5.28}$$

(ii) *There exist $\phi \in [0,1)$, $K > 0$ and $a > 1$ such that, on $\{P_{\mathcal{X}_0}(D) > 0\}$,*

$$\begin{aligned} &\text{(a)} \quad E_{\mathcal{X}_0}[\|X_1\|^a | D] \le K^a, \\ &\text{(b)} \quad \left| P_{\mathcal{X}_0}(X_1 \in A|D) - P_\nu(X_1 \in A|D) \right| \le \phi \quad \forall A \text{ measurable.} \end{aligned} \tag{5.29}$$

Then $X \in AV\big(\varepsilon, (v,u)\big)$, where $\varepsilon = 2K\phi^{(a-1)/a}$ and $(v,u) := E_\nu[X_1|D]$.

Proof. Since $\tau < \infty$, by (i)(a) and (5.25) we must have $P_\nu(D) > 0$. Let $\mathcal{F}_{\tau_k}$ be the σ-algebra of the events $B \in \mathcal{F}_\infty$ such that, for all $n \in \mathbb{N}$, there exists $B_n \in \mathcal{F}_n$ with $B \cap \{\tau_k = n\} = B_n \cap \{\tau_k = n\}$. We will show that, P_ν-a.s., for all $k \in \mathbb{N}$,

$$E_\nu\left[\|\theta_{\tau_k} X_1\|^a | \mathcal{F}_{\tau_k}\right] \le K^a \tag{5.30}$$

and

$$\left| P_\nu\left(\theta_{\tau_k} X_1 \in A | \mathcal{F}_{\tau_k}\right) - P_\nu(X_1 \in A|D) \right| \le \phi \quad \forall A \text{ measurable.} \tag{5.31}$$

Then, setting $Q(\cdot) := P_\nu(X_1 \in \cdot | D)$ and noting that $\theta_{\tau_k} X_1 = X_{k+1} - X_k$ and $X_j \in \mathcal{F}_{\tau_k}$ for all $0 \le j \le k$, we will be able to conclude since (5.30)–(5.31) and (ii)(a) imply that the conditions of Lemma 5.3 are all satisfied.

To prove (5.30)–(5.31), first note that, using (i), one can verify by induction that (i)(a) holds also for τ_k, i.e., for every $n \in \mathcal{I}_{\tau_k}$ there exists $H_{k,n} \in \mathcal{F}_n$ such that

$$\{\tau_k = n\} = H_{k,n} \cap \theta_n D \quad P_\nu\text{-a.s.} \tag{5.32}$$

Take $B \in \mathcal{F}_{\tau_k}$ and a measurable nonnegative function f, and write

$$\begin{aligned} E_\nu\left[1_B \theta_{\tau_k} f(X_1)\right] &= \sum_{n \in \mathcal{I}_{\tau_k}} E_\nu\left[1_{B \cap \{\tau_k = n\}} \theta_n f(X_1)\right] = \sum_{n \in \mathcal{I}_{\tau_k}} E_\nu\left[1_{B_n \cap H_{k,n}} \theta_n\big(1_D f(X_1)\big)\right] \\ &= \sum_{n \in \mathcal{I}_{\tau_k}} E_\nu\left[1_{B_n \cap H_{k,n}} P_{\mathcal{X}_n}(D) E_{\mathcal{X}_n}\left[f(X_1)|D\right]\right]. \end{aligned} \tag{5.33}$$

Noting that $P_\nu(B) = \sum_{n \in \mathcal{I}_{\tau_k}} E_\nu\left[1_{B_n \cap H_{k,n}} P_{\mathcal{X}_n}(D)\right]$, obtain (5.30) by taking $f(x) = \|x\|^a$ and using (ii)(a) together with (5.25). For (5.31), choose $f = 1_A$, subtract $P_\nu(B) E_\nu\left[f(X_1)|D\right]$ from (5.33) and use (ii)(b). □

Proposition 5.6. *Let $\mathcal{T}$ be a stopping time as in Proposition 5.4 and suppose that conditions* (ii)(a) *and* (ii)(b) *of that proposition are satisfied. Define a sequence of stopping times $(T_k)_{k \in \mathbb{N}_0}$ as follows. Put $T_0 = 0$ and, for $k \in \mathbb{N}_0$,*

$$T_{k+1} := \begin{cases} \infty & \text{if } T_k = \infty \\ T_k + \theta_{T_k} \mathcal{T} & \text{otherwise.} \end{cases} \tag{5.34}$$

Put

$$N := \inf\{k \in \mathbb{N}_0 \colon T_k < \infty \text{ and } T_{k+1} = \infty\}. \tag{5.35}$$

Then $N < \infty$ a.s. and there exists a constant $\varkappa = \varkappa(a, \rho) \in (0, \infty)$ such that, P_ν-a.s.,

$$E_{\mathcal{X}_0}\left[T_N^a\right] \le (\varkappa C)^a. \tag{5.36}$$

Furthermore, $\mathcal{I}_{T_N} \subset \bar{\mathcal{I}}_{\mathcal{T}}$.

Proof. First, let us check that

$$P_{\mathcal{X}_0}(N \geq n) \leq (1-\rho)^n. \tag{5.37}$$

Indeed, $N \geq n$ if and only if $T_n < \infty$, so that, for $k \in \mathbb{N}_0$,

$$P_{\mathcal{X}_0}(T_{k+1} < \infty) = E_{\mathcal{X}_0}\left[1_{\{T_k<\infty\}} P_{\mathcal{X}_{T_k}}(\mathcal{T} < \infty)\right] \leq (1-\rho) P_{\mathcal{X}_0}(T_k < \infty), \tag{5.38}$$

where we use (ii)(a) and the fact that (5.25) holds also with a stopping time in place of n. Clearly, (5.37) follows from (5.38) by induction. In particular, $N < \infty$ a.s.

From (5.34) we see that, for $0 \leq k \leq n$,

$$T_n = T_k + \theta_{T_k} T_{n-k} \text{ on } \{T_k < \infty\}. \tag{5.39}$$

Using (ii)(a) and (b), with the help of (5.25) again, we can a.s. estimate, for $0 \leq k < n$,

$$\begin{aligned} E_{\mathcal{X}_0}\left[1_{\{T_n<\infty\}} |T_{k+1} - T_k|^a\right] &= E_{\mathcal{X}_0}\left[1_{\{T_{k+1}<\infty\}} |T_{k+1} - T_k|^a \, P_{\mathcal{X}_{T_{k+1}}}(T_{n-k-1} < \infty)\right] \\ &\leq (1-\rho)^{n-k-1} E_{\mathcal{X}_0}\left[1_{\{T_k<\infty, \theta_{T_k}\mathcal{T}<\infty\}} \theta_{T_k} \mathcal{T}^a\right] \\ &= (1-\rho)^{n-k-1} E_{\mathcal{X}_0}\left[1_{\{T_k<\infty\}} E_{\mathcal{X}_{T_k}}\left[1_{\{\mathcal{T}<\infty\}} \mathcal{T}^a\right]\right] \\ &\leq (1-\rho)^{n-k-1} C^a P_{\mathcal{X}_0}(T_k < \infty) \\ &\leq (1-\rho)^{n-1} C^a. \end{aligned} \tag{5.40}$$

Now write

$$T_N = \sum_{k=0}^{N-1} T_{k+1} - T_k. \tag{5.41}$$

By Jensen's inequality,

$$T_N^a \leq N^{a-1} \sum_{k=0}^{N-1} |T_{k+1} - T_k|^a \tag{5.42}$$

so that, by (5.40),

$$E_{\mathcal{X}_0}\left[T_N^a\right] \leq \sum_{n=1}^{\infty} n^{a-1} \sum_{k=0}^{n-1} E_{\mathcal{X}_0}\left[1_{\{N=n\}} |T_{k+1} - T_k|^a\right] \leq C^a \sum_{n=1}^{\infty} n^a (1-\rho)^{n-1} \quad \text{a.s.} \tag{5.43}$$

and (5.36) follows by taking $\varkappa = \left(\sum_{n=1}^{\infty} n^a (1-\rho)^{n-1}\right)^{1/a}$.

As for the claim that $\mathcal{I}_{T_N} \subset \bar{\mathcal{I}}_{\mathcal{T}}$, write, for $n \in \mathbb{N}$,

$$\{T_N = n\} = \sum_{k=1}^{\infty} \{T_k = n, N = k\} \tag{5.44}$$

to see that $\mathcal{I}_{T_N} \subset \bigcup_{k=1}^{\infty} \mathcal{I}_{T_k}$. Using (5.34), we can verify by induction that, for each $k \in \mathbb{N}$, $\mathcal{I}_{T_k} \subset \{n \in \mathbb{N} \colon \exists i_1, \ldots, i_k \in \mathcal{I}_{\mathcal{T}} \colon n = i_1 + \cdots + i_k\} \subset \bar{\mathcal{I}}_{\mathcal{T}}$, and the claim follows. $\square$

5.2.4. Proof of Proposition 5.4

We can now combine Propositions 5.5 and 5.6 to prove Proposition 5.4.

Proof. In the following we will refer to the hypotheses of Proposition 5.5 with the prefix P. For example, P(i)(a) denotes hypothesis (i)(a) in that proposition. The hypotheses in Proposition 5.4 will be referred to without a prefix. Since the hypotheses of Proposition 5.6 are a subset of those of Proposition 5.4, the conclusions of the former are valid.

We will show that, if $\tau := t_0 + \theta_{t_0} T_N$ for a suitable $t_0 \in \mathbb{N}$, then τ satisfies the hypotheses of Proposition 5.5 for a suitable K. There are two cases. If $\mathcal{I}_{\mathcal{T}} = \emptyset$, then $T_N \equiv 0$. Choosing $t_0 = 1$, we basically fall in the context of Lemma 5.3. P(i)(a) and P(i)(b) are trivial, (ii)(c) implies that P(ii)(a) holds with $K = (m+1)$, while P(ii)(b) follows immediately from (ii)(d). Therefore, we may suppose that $\mathcal{I}_{\mathcal{T}} \neq \emptyset$ and put $\iota := \min \mathcal{I}_{\mathcal{T}} \in \mathbb{N}$. Let $\hat{C} := 1 \vee (\varkappa C)$ and $t_0 := \iota \lceil \hat{C} \rho^{-1/a} \rceil$. We will show that τ satisfies the hypotheses of Proposition 5.5 with $K = 6\iota(m+1)\hat{C}\rho^{-1/a}$.

P(i)(a): First we show that this property is true for T_N. Indeed,

$$\{T_N = n\} = \sum_{k \in \mathbb{N}_0} \{N = k, T_k = n\} = \sum_{k \in \mathbb{N}_0} \{T_k = n, \theta_n \mathcal{T} = \infty\} \tag{5.45}$$

$$= \theta_n D \cap \left(\bigcup_{k \in \mathbb{N}_0} \{T_k = n\} \right), \tag{5.46}$$

and $\hat{H}_n := \bigcup_{k \in \mathbb{N}_0} \{T_k = n\} \in \mathcal{F}_n$ since the T_k's are all stopping times. Now we observe that $\{\tau = n\} = \theta_{t_0}\{T_N = n - t_0\}$, so we can take $H_n := \emptyset$ if $n < t_0$ and $H_n := \theta_{t_0} \hat{H}_{n-t_0}$ otherwise.

P(i)(b): By (i), it suffices to show that $\mathcal{I}_\tau \subset \bar{\mathcal{I}}_{\mathcal{T}}$. Since $t_0 \in \bar{\mathcal{I}}_{\mathcal{T}}$ (as an integer multiple of ι), this follows from the definition of τ and the last conclusion of Proposition 5.6.

P(ii)(a): By (ii)(c), $\|X_1\|^a = (\|Z_\tau\| + \tau)^a \le ((m+1)\tau)^a$ on D. Therefore, we just need to show that

$$E_{\mathcal{X}_0}\left[\tau^a | D\right] \le (6\iota\hat{C})^a / \rho. \tag{5.47}$$

Now, $\tau^a \le 2^{a-1}\left(t_0^a + \theta_{t_0} T_N^a\right)$ and, by Proposition 5.6 and (5.25),

$$E_{\mathcal{X}_0}\left[\theta_{t_0} T_N^a\right] = E_{\mathcal{X}_0}\left[E_{\mathcal{X}_{t_0}}\left[T_N^a\right]\right] \le \hat{C}^a. \tag{5.48}$$

Using (ii)(a), we obtain

$$E_{\mathcal{X}_0}\left[\theta_{t_0} T_N^a | D\right] \le \hat{C}^a / \rho. \tag{5.49}$$

Since $t_0 \le 2\iota\hat{C}\rho^{-1/a}$ and $\iota \ge 1$, (5.47) follows.

P(ii)(b): Let $S = (S_n)_{n \in \bar{\mathcal{I}}_{\mathcal{T}}}$ with $S_n := \theta_n \mathcal{T}$. By (ii)(d), it is enough to show that $X_1 = (Z_\tau, \tau) \in \sigma(Z, S)$ a.s. Since $Z_\tau = \sum_{n=0}^{\infty} 1_{\{\tau = n\}} Z_n \in \sigma(Z, \tau)$, it suffices to show that $\tau \in \sigma(S)$ a.s. Using the definition of the T_k's, we verify by induction that each T_k is a.s. measurable in $\sigma(S)$. Since $N \in \sigma((T_k)_{k \in \mathbb{N}_0})$, both N and T_N are also a.s. in $\sigma(S)$. Therefore, a.s. $\tau \in \sigma(\theta_{t_0} S) \subset \sigma(S)$.

With all hypotheses verified, Proposition 5.5 implies that $X \in AV(\hat{\varepsilon}, (v, u))$, where $(v, u) = E_\nu[X_1 | D]$ and $\hat{\varepsilon} = 2K\phi^{(a-1)/a}$. To conclude, observe that $u = E_\nu[\tau | D] \geq t_0 \geq \iota \hat{C} \rho^{-1/a} > 0$, so that $K = 6(m+1)\iota \hat{C} \rho^{-1/a} \leq 6(m+1)u$. Therefore, $\hat{\varepsilon} \leq \varepsilon$ and the proposition follows. In the case $\mathcal{I}_\mathcal{T} = \emptyset$, we conclude similarly since $u = 1$ and $K = (m+1)$. □

6. Proof of Theorem 4.1

In this section we show how to put the model defined in Section 3 in the context of Section 5, and we prove the LLN using Propositions 5.2 and 5.4.

6.1. Two further lemmas

Before we start, we first derive two lemmas (Lemmas 6.1 and 6.2 below) that will be needed in Section 6.2. The first lemma relates the laws of the environment as seen from W_n and from W_0. The second lemma is an extension of the conditional cone-mixing property for functions that depend also on Y.

Lemma 6.1. *$\bar{\mu}_n \ll \bar{\mu}$ for all $n \in \mathbb{N}$.*

Proof. For $t \geq 0$, let $\bar{\mu}_{t-}$ denote the law of $\theta_{W_{t-}}\xi_t$ under $\mathbb{P}_\mu$. First we will show that $\bar{\mu}_{t-} \ll \mu$. This is a consequence of the fact that μ is translation-invariant equilibrium, and remains true if we replace W_{t-} by any random variable taking values in $\mathbb{Z}^d$. Indeed, if $\mu(A) = 0$ then $\mathbb{P}_\mu(\theta_x\xi_t \in A) = 0$ for every $x \in \mathbb{Z}^d$, so

$$\bar{\mu}_{t-}(A) = \mathbb{P}_\mu(\theta_{W_{t-}}\xi_t \in A) = \sum_{x \in \mathbb{Z}^d} \mathbb{P}_\mu(W_{t-} = x, \theta_x\xi_t \in A) = 0. \tag{6.1}$$

Now take $n \in \mathbb{N}$ and let $g_n := \frac{d\bar{\mu}_{n-}}{d\mu}$. For any measurable $f \geq 0$,

$$\begin{aligned}
\mathbb{E}_\mu\left[f(\theta_{W_n}\xi_n)\right] = \mathbb{E}_{\bar{\mu}}\left[f(\theta_{Z_n}\xi_n)\right] &= \sum_{x \in \mathbb{Z}^d} \mathbb{E}_{\bar{\mu}}\left[1_{\{Z_n - Z_{n-} = x\}} f(\theta_x\theta_{Z_{n-}}\xi_n)\right] \\
&= \sum_{x \in \mathbb{Z}^d} \mathbb{E}_{\bar{\mu}}\left[\mathbb{P}_{\theta_{Z_{n-}}\xi_n}(W_0 = x) f(\theta_x\theta_{Z_{n-}}\xi_n)\right] \\
&= \sum_{x \in \mathbb{Z}^d} \mathbb{E}_\mu\left[\mathbb{P}_{\theta_{W_{n-}}\xi_n}(W_0 = x) f(\theta_x\theta_{W_{n-}}\xi_n)\right] \\
&= \sum_{x \in \mathbb{Z}^d} \mathbb{E}_\mu\left[g_n(\xi_0)\mathbb{P}_{\xi_0}(W_0 = x) f(\theta_x\xi_0)\right] \\
&= \sum_{x \in \mathbb{Z}^d} \mathbb{E}_\mu\left[g_n(\xi_0)1_{\{W_0 = x\}} f(\theta_x\xi_0)\right] \\
&= \mathbb{E}_\mu\left[g_n(\xi_0) f(\theta_{W_0}\xi_0)\right]
\end{aligned} \tag{6.2}$$

where, for the second equality, we use (A3). □

Lemma 6.2. *For L large enough and for all nonnegative $f \in \mathscr{C}_{R,\infty}(m) \vee \mathscr{Y}_\infty$,*

$$\left|\mathbb{E}_\eta\left[\theta_L f \mid \Gamma_L\right] - \mathbb{E}_{\bar{\mu}}[\theta_L f \mid \Gamma_L]\right| \leq \Phi_L \|f\|_\infty \quad \textit{for } \bar{\mu}\textit{-a.e. } \eta. \tag{6.3}$$

Proof. Put $f_y(\eta) = f(\eta, y)$ and abbreviate $Y^{(L)} = (Y_k)_{k>L}$. Then $\theta_L f = \theta_L f_{Y^{(L)}}$. Since Γ_L depends on Y only through $(Y_k)_{k\le L}$, we have

$$\mathbb{E}_\eta[\theta_L f\, 1_{\Gamma_L} \mid Y^{(L)}] = \mathbb{E}_\eta\big[\theta_L f_{(\cdot)}\, 1_{\Gamma_L}\big] \circ (Y^{(L)}), \tag{6.4}$$

and (6.3) follows from (H4) applied to f_y. $\square$

6.2. *Proof of Theorem 4.1*

Proof. Extend ξ and Z for times $t \in [-1, 0]$ by taking them constant in this interval, and let Y_0 be a copy of Y_1 independent of $\mathscr{F}_\infty$. Put

$$\begin{aligned} &\mathcal{X}_0 := \big(\xi_{[-1,0]}, Z_{[-1,0]}, Y_0\big), \\ &\mathcal{X}_{n+1} := \big(\theta_{Z_n}\xi_{[n,n+1]}, (Z_{t+n} - Z_n)_{0\le t\le 1}, Y_{n+1}\big), \quad n \in \mathbb{N}_0. \end{aligned} \tag{6.5}$$

Then $(\mathcal{X}_n)_{n\in\mathbb{N}_0}$ is a time-homogeneous Markov chain; to avoid confusion, we will denote its time-shift operator by $\bar\theta_n$. Note that $\mathcal{F}_n = \mathscr{F}_n \forall n \in \mathbb{N} \cup \{\infty\}$ and that, for functions $f \in \mathcal{F}_\infty$, $\bar\theta_n f = \theta_{Z_n}\theta_n f\ \forall n \in \mathbb{N}_0$.

Fix $L \in \mathbb{N}$ large enough and put

$$\mathcal{T}_L := L + 1_{\Gamma_L} \lceil \theta_L \mathcal{S} \rceil. \tag{6.6}$$

By (3.5) and since $\Gamma_L \in \mathscr{F}_L$ and Z is $\mathscr{F}$-adapted, $\mathcal{T}_L$ is an $\mathcal{F}$-stopping time and $(Z_n)_{n\in\mathbb{N}_0}$ is an additive functional of $(\mathcal{X}_n)_{n\in\mathbb{N}}$ as in Section 5.2.

Next, we will verify (5.25) for $\mathcal{X}$ and the hypotheses of Proposition 5.4 for Z and $\mathcal{T}_L$ under $\mathbb{P}_{\bar\mu}$. These hypotheses will be referred to with the prefix P. The notation here is consistent in the sense that parameters in Section 3 are named according to their role in Section 5; the presence/absence of a subscript L indicates whether the parameter depends on L or not.

(5.25): Noting that, for nonnegative $f \in \mathcal{F}_\infty$ and $n \in \mathbb{N}_0$,

$$E_{\mathcal{X}_n}[f] = \mathbb{E}_{\theta_{Z_n}\xi_n}[f] \quad \mathbb{P}_{\bar\mu}\text{-a.s.}, \tag{6.7}$$

this follows from Lemma 6.1 and (3.3)–(3.4).

P(i): We will find D_n for $n \ge L$. This is enough, since both $\mathcal{I}_{\mathcal{T}_L}$ and $\bar{\mathcal{I}}_{\mathcal{T}_L}$ are subsets of $[L, \infty)\cap\mathbb{N}$. Using (A1) and (H1), we may write

$$\begin{aligned} &D = \Gamma_L \cap \{\|Z_{t+L}\| \le mt\ \forall t \ge 0\}, \\ &\bar\theta_n D = \bar\theta_n \Gamma_L \cap \{\|Z_{t+n+L} - Z_n\| \le mt\ \forall t \ge 0\}. \end{aligned} \tag{6.8}$$

Intersecting the two above events, we get

$$D \cap \bar\theta_n D = \Gamma_L \cap \{\|Z_t\| \le mt\ \forall t \in [0, n]\} \cap \bar\theta_n D, \tag{6.9}$$

i.e., P(i) holds with $D_n := \Gamma_L \cap \{\|Z_t\| \le mt\ \forall t \in [0, n]\} \in \mathcal{F}_n$ for $n \ge L$.

For the remaining items, note that, by (6.7), the distribution of $(Z, \mathcal{T}_L)$ under $P_{\mathcal{X}_0}$ is $\mathbb{P}_{\bar\mu}$-a.s. the same as under $\mathbb{P}_{\xi_0}$.

P(ii)(a): Since $\{\mathcal{T}_L = \infty\} = \{\theta_L\mathcal{S} = \infty\} \cap \Gamma_L$, we get from (H2) and (H3)(1) that, $\mathbb{P}_{\bar\mu}$-a.s.,

$$\mathbb{P}_{\xi_0}(\mathcal{T}_L = \infty) = \mathbb{P}_{\xi_0}(\theta_L\mathcal{S} = \infty \mid \Gamma_L)\,\mathbb{P}_{\xi_0}(\Gamma_L) \ge \kappa_L\gamma_L > 0, \tag{6.10}$$

so that we can take $\rho_L := \kappa_L\gamma_L$.

P(ii)(b): By the definition of $\mathcal{T}_L$, we have

$$\begin{aligned}\mathcal{T}_L^a 1_{\{\mathcal{T}_L<\infty\}} &= L^a 1_{\Gamma_L^c} + (L + \lceil \theta_L \mathcal{S} \rceil)^a \, 1_{\Gamma_L \cap \{\theta_L \mathcal{S} < \infty\}} \\ &\le L^a 1_{\Gamma_L^c} + (L + 1 + \theta_L \mathcal{S})^a \, 1_{\Gamma_L \cap \{\theta_L \mathcal{S} < \infty\}} \\ &\le 2^{a-1}(L+1)^a + 2^{a-1} \left((\theta_L \mathcal{S})^a 1_{\{\theta_L \mathcal{S} < \infty\}} \right) 1_{\Gamma_L}. \end{aligned} \tag{6.11}$$

Therefore, by (H3)(2), we get

$$\mathbb{E}_{\xi_0} \left[\mathcal{T}_L^a 1_{\{\mathcal{T}_L<\infty\}} \right] \le 2^a ((L+1)^a + (1 \vee \psi_L)^a) \le [2(L+1+1 \vee \psi_L)]^a \quad \mathbb{P}_{\bar{\mu}}\text{-a.s.,} \tag{6.12}$$

so that we can take $C_L := 2(L + 1 + 1 \vee \psi_L)$.

P(ii)(c): This follows from (H1) and the definition of $\mathcal{S}$.

P(ii)(d): First note that, for any $n \in \bar{\mathcal{I}}_{\mathcal{T}_L}$, $\bar{\theta}_n \mathcal{T}_L \in \sigma(Z, \bar{\theta}_n \Gamma_L)$. Since $n \ge L$, on $\{\mathcal{T}_L = \infty\} = \Gamma_L \cap \{\theta_L \mathcal{S} = \infty\}$, Z, $\bar{\theta}_n \Gamma_L$ and $\{\theta_L \mathcal{S} = \infty\}$ are all measurable in $\theta_L(\mathscr{C}_{R,\infty}(m) \vee \mathscr{Y}_\infty)$; this follows from (A2), (H1) and the assumptions on Γ_L. Noting that, for any two probability measures ν_1, ν_2 and an event A,

$$\|\nu_1(\cdot \mid A) - \nu_2(\cdot \mid A)\|_{TV} \le 2 \frac{\|\nu_1 - \nu_2\|_{TV}}{\nu_1(A) \vee \nu_2(A)} \tag{6.13}$$

where $\| \cdot \|_{TV}$ stands for total variation distance, we see that P(ii)(d) follows from Lemma 6.2 and (H3)(1) with $\phi_L := 2\Phi_L/\kappa_L \to 0$ as $L \to \infty$ by (H4).

Thus, for large enough L, we can conclude by Proposition 5.4 that there exists a sequence of times $(\tau_k)_{k \in \mathbb{N}_0}$ with $\lim_{k\to\infty} \tau_k = \infty$ a.s. such that $(Z_{\tau_k}, \tau_k)_{k \in \mathbb{N}_0} \in AV(\varepsilon_L, (v_L, u_L))$, where

$$\begin{aligned} v_L &= \mathbb{E}_{\bar{\mu}}[Z_{\tau_1} | D], \\ u_L &= \mathbb{E}_{\bar{\mu}}[\tau_1 | D] > 0, \\ \varepsilon_L &= 12(m+1) u_L \phi_L^{(a-1)/a}. \end{aligned} \tag{6.14}$$

From (6.14) and P(ii)(c), Proposition 5.2 implies that $Z \in AV(\delta_L, w_L)$, where

$$\begin{aligned} w_L &= v_L / u_L, \\ \delta_L &= (3m+1) 12(m+1) \phi_L^{(a-1)/a}. \end{aligned} \tag{6.15}$$

By (H4), $\lim_{L\to\infty} \delta_L = 0$. As was observed after Definition 5.1, this implies that $w := \lim_{L\to\infty} w_L$ exists and that $\lim_{t\to\infty} t^{-1} Z_t = w \mathbb{P}_{\bar{\mu}}$-a.s., which, by (3.3)–(3.4), implies the same for W, $\mathbb{P}_\mu$-a.s. □

We have at this point finished the proof of our LLN. In the following sections, we will look at examples that satisfy (H1)–(H4). Section 7 is devoted to the $(\infty, 0)$-model for two classes of one-dimensional spin-flip systems. In Section 8 we discuss three additional models where the hypotheses are satisfied, and one where they are not.

7. Proof of Theorem 4.2

We begin with a proper definition of the $(\infty, 0)$-model in Section 7.1, where we identify Z and W_0 of Section 3.2. In Section 7.2, we first define suitable versions of spin-flip systems with bounded rates. After checking assumptions (A1)–(A3), we define events Γ_L satisfying (H1) and

(H2) for which we then verify (H3). We also derive uniform integrability properties of $t^{-1}W_t$ which are the key for convergence in L^p once we have the LLN. In Sections 7.3 and 7.4, we specialize to particular constructions in order to prove (H4), which is the hardest of the four hypotheses. Section 7.5 is devoted to proving a criterion for positive or negative speed.

7.1. Definition of the model

Assume that ξ is a càdlàg process with state space $E := \{0,1\}^{\mathbb{Z}}$. We will define the walk W in several steps, and a monotonicity property will follow.

7.1.1. Identification of Z and W_0

First, let $Tr^+ = Tr^+(\eta)$ and $Tr^- = Tr^-(\eta)$ denote the locations of the closest traps to the right and to the left of the origin in the configuration $\eta \in E$, i.e.,

$$\begin{aligned} Tr^+(\eta) &:= \inf\{x \in \mathbb{N}_0 \colon\ \eta(x) = 1, \eta(x+1) = 0\}, \\ Tr^-(\eta) &:= \sup\{x \in -\mathbb{N}_0 \colon\ \eta(x) = 1, \eta(x+1) = 0\}, \end{aligned} \tag{7.1}$$

with the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. For $i, j \in \{0,1\}$, abbreviate $\langle i, j\rangle := \{\eta \in E \colon\ \eta(0) = i, \eta(1) = j\}$. Let $\bar{E} := \langle 1, 0\rangle$, i.e., the set of all the configurations with a trap at the origin.

Next, we define the functional J that gives the jumps in W. For $b \in \{0,1\}$ and $\eta \in E$, let

$$J(\eta, b) := Tr^+\left(1_{\langle 1,1\rangle} + b 1_{\langle 0,1\rangle}\right) + Tr^-\left(1_{\langle 0,0\rangle} + (1-b)1_{\langle 0,1\rangle}\right), \tag{7.2}$$

i.e., J is equal to either the left or the right trap, depending on the configuration around the origin. In the case of an inverted trap ($\langle 0,1\rangle$), the direction of the jump is decided by the value of b. Observe that $J = Tr^+ = Tr^- = 0$ when $\eta \in \bar{E}$, independently of the value of b.

Let b_0 be a Bernoulli($\frac{1}{2}$) random variable independent of ξ and set

$$W_0 = X_0 := J(\xi_0, b_0). \tag{7.3}$$

Now let $(b_{n,k})_{n,k\in\mathbb{N}}$ be a double-indexed i.i.d. sequence of Bernoulli($\frac{1}{2}$) r.v.'s independent of (ξ, b_0). Put $\tau_0 := 0$ and, for $k \geq 0$,

$$\begin{aligned} \tau_{k+1} &:= \begin{cases} \infty & \text{if } |X_k| = \infty, \\ \inf\{t > \tau_k \colon\ (\xi_t(X_k), \xi_t(X_k+1)) \neq (1,0)\} & \text{otherwise}, \end{cases} \\ X_{k+1} &:= \begin{cases} X_k & \text{if } \tau_{k+1} = \infty, \\ X_k + J\left(\theta_{X_k}\xi_{\tau_k}, b_{\lceil \tau_{k+1}\rceil, k+1}\right) & \text{otherwise}. \end{cases} \end{aligned} \tag{7.4}$$

Since ξ is càdlàg, for any $k \in \mathbb{N}_0$ we either have $\tau_k = \infty$ or $\tau_{k+1} > \tau_k$. We define $(W_t)_{t\geq 0}$ as the path that jumps $X_{k+1} - X_k$ at time τ_{k+1} and is constant between jumps, i.e.,

$$W_t := \sum_{k=0}^{\infty} 1_{\{\tau_k \leq t < \tau_{k+1}\}} X_k. \tag{7.5}$$

With this definition, it is clear that W_t is càdlàg and, by (7.3)–(7.4),

$$W_{n+t} - W_n = \theta_{W_n}\theta_n W_t \quad \text{on } \{W_n < \infty\}\ \forall n \in \mathbb{N}_0, t \geq 0. \tag{7.6}$$

Therefore, defining Z by

$$Z_t := 1_{\{\xi_0 \in \bar{E}\}} W_t, \quad t \geq 0, \tag{7.7}$$

we get $W_t = W_0 + \theta_{W_0} Z_t$ on $\{W_0 < \infty\}$ since, in this case, $\theta_{W_0}\xi_0 \in \bar{E}$, and $W_0 = 0$ on $\bar{E}$.

7.1.2. Monotonicity

The following monotonicity property will be helpful in checking (H3). In order to state it, we first endow both E and $D([0,\infty), E)$ with the usual partial ordering, i.e., for $\eta_1, \eta_2 \in E$, $\eta_1 \leq \eta_2$ means that $\eta_1(x) \leq \eta_2(x)$ for all $x \in \mathbb{Z}$, while, for $\xi^{(1)}, \xi^{(2)} \in D([0,\infty), E)$, $\xi^{(1)} \leq \xi^{(2)}$ means that $\xi_t^{(1)} \leq \xi_t^{(2)}$ for all $t \geq 0$.

Lemma 7.1. *Fix a realization of b_0 and $(b_{n,k})_{n,k\in\mathbb{N}}$. If $\xi^{(1)} \leq \xi^{(2)}$, then*

$$W_t\left(\xi^{(1)}, b_0, (b_{n,k})_{n,k\in\mathbb{N}}\right) \leq W_t\left(\xi^{(2)}, b_0, (b_{n,k})_{n,k\in\mathbb{N}}\right) \tag{7.8}$$

for all $t \geq 0$.

Proof. This is a straightforward consequence of the definition. We need only to understand what happens when the two walks separate and, at such moments, the second walk is always to the right of the first. □

7.2. Spin-flip systems with bounded flip rates

7.2.1. Dynamic random environment

From now on we will take ξ to be a single-site spin-flip system with translation-invariant and bounded flip rates. We may assume that the rates at the origin are of the form

$$c(\eta) = \begin{cases} c_0 + \lambda_0 p_0(\eta) & \text{when } \eta(0) = 1, \\ c_1 + \lambda_1 p_1(\eta) & \text{when } \eta(0) = 0, \end{cases} \tag{7.9}$$

where $c_i, \lambda_i > 0$ and $p_i \in [0,1]$. We assume the existence conditions of Liggett [6], Chapter I, which in our setting amounts to the additional requirement that $c(\cdot)$ has finite triple norm. This is automatically satisfied in the $M < \epsilon$ regime or when $c(\cdot)$ has finite range.

From (7.9), we see that the IPS is stochastically dominated by the system ξ^+ with rates

$$c^+(\eta) = \begin{cases} c_0 & \text{when } \eta(0) = 1, \\ c_1 + \lambda_1 & \text{when } \eta(0) = 0, \end{cases} \tag{7.10}$$

while it stochastically dominates the system ξ^- with rates

$$c^-(\eta) = \begin{cases} c_0 + \lambda_0 & \text{when } \eta(0) = 1, \\ c_1 & \text{when } \eta(0) = 0. \end{cases} \tag{7.11}$$

These are the rates of two independent spin-flip systems with respective densities $\rho^+ := (c_1 + \lambda_1)/\lambda^+$ and $\rho^- := c_1/\lambda^-$ where $\lambda^+ := c_0 + c_1 + \lambda_1$ and $\lambda^- := c_0 + \lambda_0 + c_1$. Consequently, any equilibrium for ξ is stochastically dominated by ν_{ρ^+} and dominates ν_{ρ^-}, where ν_ρ is a Bernoulli product measure with density ρ.

We will take as the dynamic random environment the triple $\Xi := (\xi^-, \xi, \xi^+)$ starting from the same initial configuration and coupled together via the basic (or Vasershtein) coupling, which implements the stochastic ordering as an a.s. partial ordering. More precisely, Ξ is the IPS with state space E^3 whose rates are translation invariant and at the origin are given schematically by (the configuration of the middle coordinate is η),

$$
\begin{aligned}
(000) \quad &\to \quad \begin{cases} (111) & c_1, \\ (011) & c(\eta) - c_1, \\ (001) & c_1 + \lambda_1 - c(\eta), \end{cases} \\
(001) \quad &\to \quad \begin{cases} (111) & c_1, \\ (011) & c(\eta) - c_1, \\ (000) & c_0, \end{cases} \\
(011) \quad &\to \quad \begin{cases} (111) & c_1, \\ (000) & c_0, \\ (001) & c(\eta) - c_0, \end{cases} \\
(111) \quad &\to \quad \begin{cases} (000) & c_0, \\ (001) & c(\eta) - c_0, \\ (011) & c_0 + \lambda_0 - c(\eta). \end{cases}
\end{aligned}
\tag{7.12}
$$

7.2.2. Verification of (A1)–(A3)

Under our assumptions, $\lim_{k\to\infty} \tau_k = \infty$ and $X_0 < \infty$ $\mathbb{P}_\mu$-a.s., as ξ has bounded flip rates per site and μ dominates and is dominated by non-trivial product measures. By induction, $X_k < \infty$ a.s. for every $k \in \mathbb{N}$ as well, since the law of $\theta_{X_{k-1}}\xi_{\tau_k}$ is absolutely continuous w.r.t. μ, which can be verified by approximating τ_k from above by times taking values in a countable set. Therefore, W_t is finite for all $t \geq 0$.

Set $Y_n := (b_{n,k})_{k\in\mathbb{N}}$. Then Z is $\mathscr{F}$-adapted as it is independent of b_0. (A1) follows by (7.6) and (7.7), and (A3) follows either from the recursive construction (7.4) or by noting that Z has no deterministic jumps and $\theta_{Z_n}\theta_n W_0 = 0$. To verify (A2), note that $\{J = x\}$ depends on η only through $(\eta(y))_{y\in\{0\wedge x,\ldots,0\vee x+1\}}$ so we may take $R = 1$.

7.2.3. Definition of Γ_L and verification of (H1)–(H3)

Using Ξ, we can define the events Γ_L by

$$\Gamma_L := \left\{\xi_t^{\pm}(x) = \xi_0^{\pm}(x)\, \forall t \in [0, L], x = 0, 1\right\}. \tag{7.13}$$

Then $\Gamma_L \in \mathscr{C}_{1,L}(m)$ for any $m > 0$. When $\xi_0^{\pm} \in \bar{E}$, Γ_L implies that there is a trap at the origin between times 0 and L; since $\bar{\mu}(\bar{E}) = 1$, (H1) holds. The probability of Γ_L is positive and depends on Ξ_0 only through the states at 0 and 1, so (H2) is also satisfied.

In order to verify (H3), we will take advantage of Lemma 7.1 and the stochastic domination in Ξ to define two auxiliary processes $H^{\pm} = (H_t^{\pm})_{t\geq 0}$ which we can control and which will bound Z. This will also allow us to deduce uniform integrability properties.

In the following we will suppose that $\xi_0^{\pm} \in \bar{E}$. Let $G_0 = U_0 := 0$ and, for $k \geq 0$,

$$
\begin{aligned}
U_{k+1} &:= \inf\left\{t > U_k :\ \xi_t^{+}(G_k + 1) = 1\right\}, \\
G_{k+1} &:= G_k + Tr^{+}\left(\theta_{G_k}\xi^{+}_{U_{k+1}}\right)
\end{aligned}
\tag{7.14}
$$

and put

$$H_t^{+} := \sum_{k=0}^{\infty} 1_{\{U_k \leq t < U_{k+1}\}} G_{k+1}. \tag{7.15}$$

Define H^- analogously, using Tr^- and ξ^- instead and switching 1's to 0's in (7.14). Then H^+ (H^-) is the process that, observing ξ^+ (ξ^-), waits to the left of a hole (on a particle) until it flips to a particle (hole), and then jumps to the right (left) to the next trap. Therefore, by Lemma 7.1 and the definition of Z, $H_t^- \le Z_t \le H_t^+ \forall t \ge 0$. Note that H^+ depends only on $(\xi^+(x))_{x\ge1}$, and analogously for H^-.

In the following, we will write $\mathbb{Z}_{\le x} := \mathbb{Z} \cap (-\infty, x]$ and analogously for $\mathbb{Z}_{\ge x}$.

Lemma 7.2. *Fix $\rho_* \in (0, \rho^-]$ and $\rho^* \in [\rho^+, 1)$. There exist $m, a, \psi_* \in (0,\infty)$ and $\kappa_* \in (0, 1)$, depending on ρ_*, ρ^* and $\lambda^\pm$, such that, for any probability measure $\bar\nu$ on $\bar E$ that stochastically dominates ν_{ρ_*} on $\mathbb{Z}_{\le -1}$ and is dominated by ν_{ρ^*} on $\mathbb{Z}_{\ge 2}$,*

$$\text{(a)}\ \sup_{t\ge1} \mathbb{E}_{\bar\nu}\left[e^{a\left(t^{-1}|H_t^\pm|\right)}\right] \le \psi_* \tag{7.16}$$

and, setting

$$\mathcal{S}^\pm := \inf\{t>0 \colon |H_t^\pm| > mt\}, \qquad \widehat{\mathcal{S}}^\pm := \sup\{t>0\colon |H_t^\pm| > mt\}, \tag{7.17}$$

then

$$\begin{aligned} &\text{(b)}\ \mathbb{P}_{\bar\nu}\left(\mathcal{S}^\pm = \infty\right) \ge \kappa_*,\\ &\text{(c)}\ \mathbb{E}_{\bar\nu}\left[e^{a\widehat{\mathcal{S}}^\pm}\right] \le \psi_*. \end{aligned} \tag{7.18}$$

Before proving this lemma, let us see how it leads to (H3). We will show that there exist $m, a, \psi \in (0,\infty)$ and $\kappa \in (0, 1)$ such that, for all $L \ge 1$ and $\eta \in \bar E$,

$$\mathbb{P}_\eta\left(\theta_L \mathcal{S} = \infty \mid \Gamma_L\right) \ge \kappa \tag{7.19}$$

and

$$\mathbb{E}_\eta\left[e^{a(\theta_L\mathcal{S})} 1_{\{\theta_L\mathcal{S}<\infty\}} \mid \Gamma_L\right] \le \psi, \tag{7.20}$$

which clearly imply (H3).

Let us verify (7.19). First note that $\theta_L\mathcal{S} \ge \theta_L(\mathcal{S}^+ \wedge \mathcal{S}^-)$, and that the latter is nonincreasing in $(\eta(x))_{x\ge2}$ and nondecreasing in $(\eta(x))_{x\le-1}$. Therefore we may assume that $\eta = \eta_{01}$ which is the configuration in $\bar E$ with all 0's on $\mathbb{Z}_{\le-1}$ and all 1's on $\mathbb{Z}_{\ge2}$. In this case, ξ_L^- is distributed as $\nu_{\rho_0^L}$ on $\mathbb{Z}_{\le-1}$ and ξ_L^+ as $\nu_{\rho_1^L}$ on $\mathbb{Z}_{\ge2}$, where $\rho_0^L = \rho^-(1-e^{-\lambda^- L})$ and $\rho_1^L = \rho^+ + e^{-\lambda^+ L}(1-\rho^+)$. Furthermore, on Γ_L, $\xi_L^\pm \in \bar E$.

Let now m, a, ψ_* and κ^* as in Lemma 7.2 for $\rho_* := \rho_0^1$ and $\rho^* := \rho_1^1$, and let $\bar\nu_L$ be the distribution of $\bar\eta_L \in \bar E$ given by ξ_L^- on $\mathbb{Z}_{\le-1}$ and ξ_L^+ on $\mathbb{Z}_{\ge2}$. Noting that $\bar\eta_L$ is independent of Γ_L and that $\mathcal{S}^+$ and $\mathcal{S}^-$ are independent, we use the previous observations, the Markov property and Lemma 7.2(b) to write

$$\begin{aligned} \mathbb{P}_\eta\left(\theta_L\mathcal{S} = \infty \mid \Gamma_L\right) &\ge \mathbb{P}_{\eta_{01}}\left(\theta_L(\mathcal{S}^+\wedge\mathcal{S}^-) = \infty \mid \Gamma_L\right)\\ &= \mathbb{E}_{\eta_{01}}\left[1_{\Gamma_L}\mathbb{P}_{\bar\eta_L}\left(\mathcal{S}^+\wedge\mathcal{S}^- = \infty\right)\right]\mathbb{P}_{\eta_{01}}\left(\Gamma_L\right)^{-1}\\ &= \mathbb{P}_{\bar\nu_L}\left(\mathcal{S}^+ = \infty\right)\mathbb{P}_{\bar\nu_L}\left(\mathcal{S}^- = \infty\right) \ge \kappa_*^2 \in (0,1), \end{aligned} \tag{7.21}$$

and we may take $\kappa := \kappa_*^2$. For (7.20), note now that, when finite, $\theta_L\mathcal{S} < \theta_L(\widehat{\mathcal{S}}^+ \vee \widehat{\mathcal{S}}^-)$ and the latter is nondecreasing in $(\eta(x))_{x\ge2}$ and nonincreasing in $(\eta(x))_{x\le-1}$. Therefore we may again assume $\eta = \eta_{01}$ and write, using Lemma 7.2(c),

$$\mathbb{E}_\eta\Big[\theta_L\big(e^{a\mathcal{S}}1_{\{\mathcal{S}=\infty\}}\big) \mid \Gamma_L\Big] \le \mathbb{E}_{\eta_{01}}\Big[\theta_L e^{a\left(\widehat{\mathcal{S}}^+ + \widehat{\mathcal{S}}^-\right)} \mid \Gamma_L\Big]$$
$$= \mathbb{E}_{\bar{\nu}_L}\left[e^{a\widehat{\mathcal{S}}^+}\right]\mathbb{E}_{\bar{\nu}_L}\left[e^{a\widehat{\mathcal{S}}^-}\right] \le \psi_*^2 \in (0,\infty), \tag{7.22}$$

and we can take $\psi := \psi_*^2$. All that is left to do is to prove Lemma 7.2.

Proof of Lemma 7.2. By symmetry, it is enough to prove (a)–(c) for H^+. Since H^+, $\mathcal{S}^+$ and $\widehat{\mathcal{S}}^+$ are monotone, we may assume that ξ^+ has rates $\lambda^+\rho^*$ to flip from holes to particles and $\lambda^+(1-\rho^*)$ from particles to holes and starts from ν_{ρ^*}, which is the equilibrium measure. In this case, the increments $G_{k+1} - G_k$ are i.i.d. Geom$(1-\rho^*)$, and $U_{k+1} - U_k$ are i.i.d. Exp$(\lambda^+\rho^*)$, independent from $(G_k)_{k\in\mathbb{N}_0}$. Therefore, H^+ is a càdlàg Lévy process and H_1^+ has an exponential moment, so (a) promptly follows. Moreover, H^+ satisfies a large deviation estimate of the type

$$\mathbb{P}_{\nu_{\rho^*}}\left(\exists s > t \text{ such that } H_s^+ > ms\right) \le K_1 e^{-K_2 t} \quad \text{for all } t > 0, \tag{7.23}$$

where m, K_1 and K_2 are functions of (ρ^*, λ^+), which proves (c). In particular, $\widehat{\mathcal{S}}^+ < \infty$ a.s., which implies that $\mathbb{P}_{\nu_{\rho^*}}(H_s^+ \le m(s+n^*)\ \forall s \ge 0) \ge \frac{1}{2}$ for some n^* large enough; then

$$\mathbb{P}_{\nu_{\rho^*}}\left(\mathcal{S}^+ = \infty\right) \ge \mathbb{P}_{\nu_{\rho^*}}\left(H_{n^*}^+ = 0,\, H_{n^*+s}^+ - H_{n^*}^+ \le m(s+n^*)\ \forall s \ge 0\right)$$
$$= \mathbb{P}_{\nu_{\rho^*}}\left(H_{n^*}^+ = 0\right)\mathbb{P}_{\nu_{\rho^*}}\left(H_s^+ \le m(s+n^*)\ \forall s \ge 0\right) =: \kappa_* > 0, \tag{7.24}$$

proving (b). □

7.2.4. Uniform integrability

The following corollary implies that, for systems given by (7.9), $(t^{-1}|W_t|^p)_{t\ge1}$ is uniformly integrable for any $p \ge 1$, so that, whenever we have a LLN, the convergence holds also in L^p.

Corollary 7.3. *Let ξ be a spin-flip system with rates as in (7.9), starting from equilibrium. Then $(t^{-1}W_t)_{t\ge1}$ is bounded in L^p for all $p \ge 1$.*

Proof. The claim for Z under $\mathbb{P}_{\bar{\mu}}$ follows from Lemma 7.2(a) by noting that $\bar{\mu}$ stochastically dominates ν_{ρ^-} on $\mathbb{Z}_{\le -1}$ and is dominated by ν_{ρ^+} on $\mathbb{Z}_{\ge 2}$; this can be verified noting that $W_0 \ge 0$ corresponds to finding particles to the left of W_0, and $W_0 \le 0$ to holes to its right. The same for W follows from (3.3)–(3.4) since W_0 has exponential moments under $\mathbb{P}_\mu$. □

We still need to verify (H4). This will be done in Sections 7.3 and 7.4 below. As κ in (7.19) could be taken independently of L for (H3), we only need $\lim_{L\to\infty} \Phi_L = 0$ in (H4).

7.3. Example 1: $M < \epsilon$

We recall the definition of M and ϵ for a translation-invariant spin-flip system:

$$M := \sum_{x\ne0} \sup_\eta \left|c(\eta^x) - c(\eta)\right|, \tag{7.25}$$

$$\epsilon := \inf_\eta \left\{c(\eta) + c(\eta^0)\right\}, \tag{7.26}$$

where η^x is the configuration obtained from η by flipping the x-coordinate.

7.3.1. Mixing for ξ

If ξ is in the $M < \epsilon$ regime, then there is exponential decay of space–time correlations (see Liggett [6], Section I.3). In fact, if ξ, ξ' are two copies starting from initial configurations η, η' and coupled according to the Vasershtein coupling, then, as was shown in Maes and Shlosman [8], the following estimate holds uniformly in $x \in \mathbb{Z}$ and in the initial configurations:

$$\mathbb{P}_{\eta,\eta'}\left(\xi_t(x) \neq \xi'_t(x)\right) \leq e^{-(\epsilon-M)t}. \tag{7.27}$$

Since the system has uniformly bounded flip rates, it follows that there exist constants $K_1, K_2 \in (0,\infty)$, independent of $x \in \mathbb{Z}$ and of the initial configurations, such that

$$\mathbb{P}_{\eta,\eta'}\left(\exists s > t \text{ s.t. } \xi_s(x) \neq \xi'_s(x)\right) \leq K_1 e^{-K_2 t}. \tag{7.28}$$

For $A \subset \mathbb{Z} \times \mathbb{R}_+$ measurable, let $\mathrm{Discr}(A)$ be the event in which there is a discrepancy between ξ and ξ' in A, i.e., $\mathrm{Discr}(A) := \{\exists\,(x,t) \in A\colon\ \xi_t(x) \neq \xi'_t(x)\}$. Recall the definition of $C_R(m)$ in Section 3.1, and let $C_{R,t}(m) := C_R(m) \cap \mathbb{Z} \times [0,t]$. From (7.28) we deduce that, for any fixed $m > 0$ and $R \in \mathbb{N}_0$, there exist (possibly different) constants $K_1, K_2 \in (0,\infty)$ such that

$$\mathbb{P}_{\eta,\eta'}(\mathrm{Discr}(C_R(m) \setminus C_{R,t}(m))) \leq K_1 e^{-K_2 t}. \tag{7.29}$$

7.3.2. Mixing for Ξ

Bounds of the same type as (7.27)–(7.29) hold for $\xi^{\pm}$, since $M = 0$ and $\epsilon > 0$ for independent spin-flips. Therefore, in order to have such bounds for the triple Ξ, we need only couple a pair Ξ, Ξ' in such a way that each coordinate is coupled with its primed counterpart by the Vasershtein coupling. A set of coupling rates for Ξ, Ξ' that accomplishes this goal is given in (A.1), in Appendix A. Redefining $\mathrm{Discr}(A) := \{\exists\,(x,t) \in A\colon\ \Xi_t(x) \neq \Xi'_t(x)\}$, by the previous results we see that (7.29) still holds for this coupling, with possibly different constants. As a consequence, we get the following lemma.

Lemma 7.4. *Define $d(\eta,\eta') := \sum_{x\in\mathbb{Z}} 1_{\{\eta(x)\neq\eta'(x)\}} 2^{-|x|-1}$. For any $m > 0$ and $R \in \mathbb{N}_0$,*

$$\lim_{d(\Xi_0,\Xi'_0)\to 0} \mathbb{P}_{\Xi_0,\Xi'_0}\left(\mathit{Discr}(C_R(m))\right) = 0. \tag{7.30}$$

Proof. For any $t > 0$, we may split $\mathrm{Discr}(C_R(m)) = \mathrm{Discr}(C_{R,t}(m)) \cup \mathrm{Discr}(C_R(m) \setminus C_{R,t}(m))$, so that

$$\mathbb{P}_{\eta,\eta'}\left(\mathrm{Discr}(C_R(m))\right) \leq \mathbb{P}_{\eta,\eta'}\left(\mathrm{Discr}(C_{R,t}(m))\right) + \mathbb{P}_{\eta,\eta'}\left(\mathrm{Discr}(C_R(m) \setminus C_{R,t}(m))\right). \tag{7.31}$$

Fix $\varepsilon > 0$. By (7.29), for t large enough the second term in (7.31) is smaller than ε uniformly in η,η'. For this fixed t, the first term goes to zero as $d(\eta,\eta') \to 0$, since $C_{R,t}(m)$ is contained in a finite space–time box and the coupling in (A.1) is Feller with uniformly bounded total flip rates per site. (Note that the metric d generates the product topology, under which the configuration space is compact.) Therefore $\limsup_{d(\eta,\eta')\to 0} \mathbb{P}_{\eta,\eta'}(\mathrm{Discr}(C_R(m))) \leq \varepsilon$. Since ε is arbitrary, (7.30) follows. $\square$

7.3.3. Conditional mixing

Next, we define an auxiliary process $\bar{\Xi}$ that, for each L, has the law of Ξ conditioned on Γ_L up to time L. We restrict to initial configurations $\eta \in \bar{E}$. In this case, $\bar{\Xi}$ is a process on $\left(\{0,1\}^{\mathbb{Z}\setminus\{0,1\}}\right)^3$ with rates that are equal to those of Ξ, evaluated with a trap at the origin. More precisely, for $\bar{\eta} \in \{0,1\}^{\mathbb{Z}\setminus\{0,1\}}$, denote by $(\bar{\eta})_{1,0}$ the configuration in $\{0,1\}^{\mathbb{Z}}$ that is equal to $\bar{\eta}$ in $\mathbb{Z}\setminus\{0,1\}$ and has a trap at the origin. Then set $\bar{C}_x(\bar{\eta}) := C_x((\bar{\eta})_{1,0})$, where $\bar{C}_x$ are the rates of $\bar{\Xi}$ and C_x the rates of Ξ at a site $x \in \mathbb{Z}$. Observe that the latter depend only on the middle configuration η, and not on $\eta^{\pm}$. These rates give the correct law for $\bar{\Xi}$ because Ξ conditioned on Γ_L is Markovian up to time L. Indeed, the probability of Γ_L does not depend on η (for $\eta \in \bar{E}$) and, for $s < L$, $\Gamma_L = \Gamma_s \cap \theta_s \Gamma_{L-s}$. Thus, the rates follow by uniqueness. Observe that they are no longer translation-invariant.

Two copies of the process $\bar{\Xi}$ can be coupled analogously to Ξ by restricting the rates in (A.1) to $\bar{E}$. Since each coordinate of $\bar{\Xi}$ has similar properties as the corresponding coordinate in Ξ (i.e., $\bar{\xi}^{\pm}$ are independent spin-flip systems and $\bar{\xi}$ is in the $M < \epsilon$ regime), it satisfies an estimate of the type

$$\bar{\mathbb{P}}_{\eta,\eta'}\left(\mathrm{Discr}([-t,t]\times\{t\})\right) \le K_1 e^{-K_2 t} \quad \forall \eta,\eta' \in \bar{E}, \tag{7.32}$$

for appropriate constants $K_1, K_2 \in (0,\infty)$. From this estimate we see that $d(\bar{\Xi}_t, \bar{\Xi}'_t) \to 0$ in probability as $t \to \infty$, uniformly in the initial configurations. By Lemma 7.4, this is also true for $\mathbb{P}_{(\bar{\Xi}_t)_{1,0},(\bar{\Xi}'_t)_{1,0}}(\mathrm{Discr}(C_R(m)))$. Since the latter is bounded, the convergence holds in L_1 as well, uniformly in η, η'.

7.3.4. Proof of (H4)

Let f be a bounded function measurable in $\mathscr{C}_{R,\infty}(m)$ and estimate

$$\begin{aligned}\left|\mathbb{E}_\eta[\theta_L f \mid \Gamma_L] - \mathbb{E}_{\eta'}[\theta_L f \mid \Gamma_L]\right| &\le 2\|f\|_\infty \mathbb{P}_{\eta,\eta'}\big(\theta_L \mathrm{Discr}(C_R(m)) \mid \Gamma_L\big)\\ &\le 2\|f\|_\infty \sup_{\eta,\eta'} \bar{\mathbb{E}}_{\eta,\eta'}\left[\mathbb{P}_{(\bar{\Xi}_L)_{1,0},(\bar{\Xi}'_L)_{1,0}}\left(\mathrm{Discr}(C_R(m))\right)\right],\end{aligned} \tag{7.33}$$

where $\bar{\mathbb{E}}$ denotes expectation under the (coupled) law of $\bar{\Xi}$. Therefore (H4) follows with

$$\Phi_L := 2\sup_{\eta,\eta'} \bar{\mathbb{E}}_{\eta,\eta'}\left[\mathbb{P}_{(\bar{\Xi}_L)_{1,0},(\bar{\Xi}'_L)_{1,0}}\left(\mathrm{Discr}(C_R(m))\right)\right], \tag{7.34}$$

which converges to zero as $L \to \infty$ by the previous discussion. This is enough since κ_L could be taken constant in the verification of (H3)(1), as we saw in (7.19).

7.4. Example 2: subcritical dependence spread

In this section, we suppose that the rates $c(\eta)$ have a finite range of dependence $r \in \mathbb{N}_0$. In this case, the system can be constructed via a graphical representation as follows.

7.4.1. Graphical representation

For each $x \in \mathbb{Z}$, let $I_t^j(x)$ and $\Lambda_t^j(x)$ be independent Poisson processes with rates c_j and λ_j respectively, where $j = 0, 1$. At each event of $I_t^j(x)$, put a j-cross on the corresponding space–time point. At each event of $\Lambda^j(x)$, put two j-arrows pointing at x, one from each side, extending over the whole range of dependence. Start with an arbitrary initial configuration $\xi_0 \in \{0,1\}^{\mathbb{Z}}$. Then obtain the subsequent states $\xi_t(x)$ from ξ_0 and the Poisson processes by, at each j-cross, choosing the next state at site x to be j and, at each j-arrow pair, choosing the next state to be j if an independent Bernoulli($p_j(\theta_x \xi_s)$) trial succeeds, where s is the time of the j-arrow event. This algorithm is well defined since, because of the finite range, up to each fixed positive time it can a.s. be performed locally.

Any collection of processes with the same range and with rates of the form (7.9) with c_i, λ_i fixed ($i = 0, 1$) can be coupled together via this representation by fixing additionally for each site x a sequence $(U_n(x))_{n\in\mathbb{N}}$ of independent Uniform[0, 1] random variables to evaluate the Bernoulli trials at j-arrow events. In particular, $\xi^{\pm}$ can be coupled together with ξ in the graphical representation by noting that, for ξ^-, $p_0 \equiv 1$ and $p_1 \equiv 0$ and the opposite is true for ξ^+. For example, ξ^- is the process obtained by ignoring all 1-arrows and using all 0-arrows. This gives the same coupling as the one given by the rates (7.12). In particular, we see that in this setting the events Γ_L are given by (when $\xi_0 \in \bar{E}$)

$$\Gamma_L := \left\{ I_L^0(0) = \Lambda_L^0(0) = I_L^1(1) = \Lambda_L^1(1) = 0 \right\}. \tag{7.35}$$

7.4.2. Coupling with a contact process

We will couple Ξ with a contact process $\zeta = (\zeta_t)_{t\geq 0}$ in the following way. We keep all Poisson events and start with a configuration $\zeta_0 \in \{i, h\}^{\mathbb{Z}}$, where i stands for "infected" and h for "healthy". We then interpret every cross as a recovery, and every arrow pair as infection transmission from any infected site within a neighborhood of radius r to the site the arrows point to. This gives rise to a 'threshold contact process' (TCP), i.e., a process with transitions at a site x given by

$$\begin{aligned} i &\to h \quad \text{with rate } c_0 + c_1, \\ h &\to i \quad \text{with rate } (\lambda_0 + \lambda_1) 1_{\{\exists \text{ infected site within range } r \text{ of } x\}}. \end{aligned} \tag{7.36}$$

In the graphical representation for ξ, we can interpret crosses as moments of memory loss and arrows as propagation of influence from the neighbors. Therefore, looking at the pair $(\Xi_t(x), \zeta_t(x))$, we can interpret the second coordinate being healthy as the first coordinate being independent of the initial configuration.

Proposition 7.5. *Let $\underline{i}$ represent the configuration with all sites infected, and let Ξ_0, $\Xi_0' \in E^3$. Couple Ξ, Ξ' and ζ by fixing a realization of all crosses, arrows and uniform random variables, where Ξ and Ξ' are obtained from the respective initial configurations and ζ is started from $\underline{i}$. Then a.s. $\Xi_t(x) = \Xi_t'(x)$ for all $t > 0$ and $x \in \mathbb{Z}$ such that $\zeta_t(x) = h$.*

Proof. Fix $t > 0$ and $x \in \mathbb{Z}$. With all Poisson and uniform random variables fixed, an algorithm to find the state at (x, t), simultaneously for any collection of systems of type (7.9) with fixed c_i, λ_i and finite range r from their respective initial configurations runs as follows. Find the first Poisson event before t at site x. If it is a j-cross, then the state is j. If it is a j-arrow, then to decide the state we must evaluate p_j and, therefore, we must first take note of the states at this time at each site within range r of x, including x itself. In order to do so, we restart the algorithm for each of these sites. This process ends when time 0 or a cross is reached along every possible path from (x, t) to $\mathbb{Z} \times \{0\}$ that uses arrows (transversed in the direction opposite to which they point) and vertical lines. In particular, if along each of these paths time 0 is never reached, then the state at (x, t) does not change when we change the initial configuration. On the other hand, time 0 is not reached if and only if every path ends in a cross, which is exactly the description of the event $\{\zeta_t(x) = h\}$. $\square$

7.4.3. Cone-mixing in the subcritical regime

The process $(\zeta_t)_{t\geq 0}$ is stochastically dominated by a standard (linear) contact process (LCP) with the same range and rates. Therefore, if the LCP is subcritical, i.e., if $\lambda := (\lambda_0 + \lambda_1)/(c_0 + c_1) < \lambda_c$ where λ_c is the critical parameter for the corresponding LCP, then the TCP will die out as well. Moreover, we have the following lemma:

Lemma 7.6. *Let A_t be the set of infected sites at time t. If $\lambda < \lambda_c$, then there exist positive constants K_1, K_2, K_3, K_4 such that*

$$\mathbb{P}_{\underline{i}}\big(\exists\, s > t\colon\ A_s \cap [-K_1 e^{K_2 s}, K_1 e^{K_2 s}] \neq \emptyset\big) \leq K_3 e^{-K_4 t}. \tag{7.37}$$

Proof. This is a straightforward consequence of Proposition 1.1 in [1]. □

According to Lemma 7.6, the infection disappears exponentially fast around the origin. For $r = 1$, a proof can be found in Liggett [6], Chapter VI, but it relies strongly on the nearest-neighbor nature of the interaction.

Let us now prove cone-mixing for ξ when the rates are subcritical. Pick a cone C_t with any inclination and tip at time t, and let $\mathcal{H}_t := \{$all sites inside C_t are healthy$\}$. This event is independent of ξ_0 and, because of Lemma 7.6, has large probability if t is large. Furthermore, by Proposition 7.5, on $\mathcal{H}_t$ the states of ξ in C_t are equal to a random variable that is independent of ξ_0, which implies the cone-mixing property.

7.4.4. Proof of (H4)

In order to prove the *conditional* cone-mixing property, we couple the conditioned process to a conditioned contact process as follows. First, let

$$\tilde{\Gamma}_L := \big\{I_L^j(i) = \Lambda_L^j(i) = 0\colon\ j, i \in \{0, 1\}\big\}. \tag{7.38}$$

Proposition 7.7. *Let $\hat{i}$ represent the configuration with all sites infected except for $\{0, 1\}$, which are healthy. Let $\Xi_0, \Xi_0' \in \bar{E}^3$. Couple Ξ, Ξ' conditioned on Γ_L and ζ conditioned on $\tilde{\Gamma}_L$ by fixing a realization of all crosses, arrows and uniform random variables as in Proposition 7.5 and starting, respectively, from Ξ_0, Ξ_0' and $\hat{i}$, but, for Ξ and Ξ', remove the Poisson events that characterize Γ_L and, for ζ, remove all Poisson events up to time L at sites 0 and 1, which characterizes $\tilde{\Gamma}_L$. Then a.s. $\Xi_t(x) = \Xi_t'(x)$ for all $t > 0$ and $x \in \mathbb{Z}$ such that $\zeta_t(x) = h$.*

Proof. On Γ_L, the states at sites 0 and 1 are fixed for time $[0, L]$. Therefore, in order to determine the state at (x, t), we need not extend paths that touch $\{0, 1\} \times [0, L]$: when every path from (x, t) either ends in a cross or touches $\{0, 1\} \times [0, L]$, the state at (x, t) does not change when the initial configuration is changed in $\mathbb{Z} \setminus \{0, 1\}$. But this is precisely the characterization of $\{\eta_t(x) = h\}$ on $\tilde{\Gamma}_L$ when started from $\hat{i}$. □

The proof of (H4) is finished by noting that $(\eta_t)_{t\geq 0}$ starting from $\hat{i}$ and conditioned on $\tilde{\Gamma}_L$ is stochastically dominated by $(\eta_t)_{t\geq 0}$ starting from $\underline{i}$. Therefore, by Lemma 7.6, the "dependence infection" still dies out exponentially fast, and we conclude as for the unconditioned cone-mixing.

7.5. The sign of the speed

For independent spin flips, we are able to characterize with the help of a coupling argument the regimes in which the speed is positive, zero or negative. By the stochastic domination described in Section 7.2, this gives us a criterion for positive (or negative) speed in the two classes addressed in Sections 7.3 and 7.4 above.

7.5.1. Lipschitz property of the speed for independent spin-flip systems

Let ξ be an independent spin-flip system with rates d_0 and d_1 to flip to holes and particles, respectively. Since it fits both classes of IPS considered in Sections 7.3 and 7.4, by Theorem 4.1

there exists a $w(d_0, d_1) \in \mathbb{R}$ that is the a.s. speed of the $(\infty, 0)$-walk in this environment. This speed has the following local Lipschitz property.

Lemma 7.8. *Let $d_0, d_1, \delta > 0$. Then*

$$w(d_0, d_1 + \delta) - w(d_0, d_1) \geq \frac{d_0}{d_0 + d_1}\delta. \tag{7.39}$$

Proof. Our proof strategy is based on the proof of Theorem 2.24, Chapter VI in [6]. Construct ξ from a graphical representation by taking, for each site $x \in \mathbb{Z}$, two independent Poisson processes $N^i(x)$ with rates d_i, $i = 0, 1$, with each event of N^i representing a flip to state i. For a fixed $\delta > 0$, a second system ξ^δ with rates d_0 and $d_1 + \delta$ can be coupled to ξ by starting from a configuration $\xi_0^\delta \geq \xi_0$ and adding to each site x an independent Poisson process $N^\delta(x)$ with rate δ, whose events also represent flips to particles, but only for ξ^δ. Let us denote by W and W^δ the walks in these respective environments. Under this coupling, $\xi \leq \xi^\delta$, so, by monotonicity, $W_t \leq W_t^\delta$ for all $t \geq 0$ as well. We aim to prove that

$$\mathbb{E}_{\mu^\delta}\left[W_t^\delta\right] - \mathbb{E}_\mu\left[W_t\right] \geq \frac{d_0}{d_0 + d_1}\delta t, \tag{7.40}$$

where μ and μ^δ are the equilibria of the respective systems. From this the conclusion will follow after dividing by t and letting $t \to \infty$.

Define a third walk W^* that is allowed to use one and only one event of N^δ. More precisely, let S be the first time when there is an event of N^δ at $W_S + 1$. Take W^* equal to W on $[0, S)$ and, for times $\geq S$, let W^* evolve by the same rules as W but adding a particle at $W_S + 1$ at time S, and using no more N^δ events. By construction, we have $W_t \leq W_t^* \leq W_t^\delta$ $\forall t \geq 0$.

Let $\eta_1 := \theta_{W_S}\xi_S \in \bar{E}$ and $\eta_2 := (\eta_1)^1$ be the configurations around W_S and W_{S-}^*, respectively. Then

$$\begin{aligned}\mathbb{E}_{\mu^\delta}\left[W_t^\delta\right] - \mathbb{E}_\mu\left[W_t\right] &\geq \mathbb{E}_\mu\left[W_t^* - W_t, S \leq t\right] \geq \mathbb{E}_\mu\left[W_t^* - W_t, S \leq t, \eta_1(2) = 0\right] \\ &= \mathbb{E}_\mu\left[\mathbb{E}_{\eta_1,\eta_2}\left[W_{t-S}^2 - W_{t-S}^1\right], \eta_1(2) = 0, S \leq t\right],\end{aligned} \tag{7.41}$$

where W^i, $i = 1, 2$ are copies of W starting from η_i and coupled via the graphical representation. We claim that, if $\eta_1(2) = 0$,

$$\mathbb{E}_{\eta_1,\eta_2}\left[W_s^2 - W_s^1\right] \geq 1 \quad \forall s \geq 0. \tag{7.42}$$

Indeed, we will argue that the difference $W_s^2 - W_s^1$ can only decrease when we flip all states of η_1, η_2 on $\mathbb{Z}_{\leq -1}$ to particles and on $\mathbb{Z}_{\geq 2}$ to holes; but after doing these operations, we find that W^2 has the same distribution as $W^1 + 1$, which gives (7.42). It is enough to consider a single $x > 2$. Let $\tau := \inf\{t > 0\colon\ N_t^0(x) + N_t^1(x) > 0\} \wedge s$, and put $T := \inf\{t > 0\colon\ W_t^1 = x - 1\}$. There are two cases: either $T > \tau$ or not. In the first case, W_s^1 remains constant if we set $\eta_{1,2}(x) = 0$, while W_s^2 does not increase. In the second case, if $\eta_{1,2}(x) = 0$, then $W_T^1 = W_T^2$; but then they must remain equal thereafter since, for them to meet, the state at site 1 must have flipped, and therefore they see the same configuration in the environment at time T. Hence, in this case, $W_s^2 - W_s^1 = 0$ which is the minimum value, and our claim follows.

From (7.41) and (7.42) we get

$$\mathbb{E}_{\mu^\delta}\left[W_t^\delta\right] - \mathbb{E}_\mu\left[W_t\right] \geq \mathbb{P}_\mu\left(\eta_1(2) = 0, S \leq t\right). \tag{7.43}$$

Consider the event $\{\eta_1(2) = 0\}$. There are two possible situations: either at time S the site W_S+2 was not yet visited by W, in which case $\eta_1(2)$ is still in equilibrium, or it was. In the latter case, let s be the time of the last visit to this site before S. By geometrical constraints, at time s only a hole could have been observed at this site, so the probability that its state at time S is a hole is larger than at equilibrium, which is $d_0/(d_0+d_1)$. In other words,

$$\mathbb{P}_\mu\left(\eta_1(2) = 0 \mid S, W_{[0,S]}\right) \geq \frac{d_0}{d_0+d_1}, \tag{7.44}$$

which, together with (7.43) and the fact that S has distribution Exp(δ), gives us

$$\mathbb{E}_{\mu^\delta}\left[W_t^\delta\right] - \mathbb{E}_\mu\left[W_t\right] \geq \frac{d_0}{d_0+d_1}\left(1-e^{\delta t}\right). \tag{7.45}$$

Since δ is arbitrary, we may repeat the argument for systems with rates $d_1 + (k/n)\delta$, $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n$, to obtain

$$\mathbb{E}_{\mu^\delta}\left[W_t^\delta\right] - \mathbb{E}_\mu\left[W_t\right] \geq \tfrac{d_0}{d_0+d_1} n\left(1-e^{\delta t/n}\right), \tag{7.46}$$

and we get (7.40) by letting $n \to \infty$. $\square$

7.5.2. Sign of the speed

If $d_0 = d_1$, then $w = 0$, since by symmetry $W_t = -W_t$ in distribution. Hence we can summarize as follows.

Corollary 7.9. *For an independent spin-flip system with rates d_0 and d_1,*

$$\begin{aligned} &w \geq \frac{d_0}{d_0+d_1}(d_1-d_0) \quad \textit{if } d_1 > d_0,\\ &w = 0 \quad \textit{if } d_1 = d_0,\\ &w \leq -\frac{d_1}{d_0+d_1}(d_0-d_1) \quad \textit{if } d_1 < d_0. \end{aligned} \tag{7.47}$$

Applying this result to the systems $\xi^\pm$ of Section 7.2, we obtain the following.

Proposition 7.10. *Let W be the random walk for the $(\infty, 0)$-model in a spin-flip system with rates given by (7.9). Then, $\mathbb{P}_\mu$-a.s.,*

$$\begin{aligned} &\liminf_{t\to\infty} t^{-1} W_t \geq \frac{c_0+\lambda_0}{c_1+c_0+\lambda_0}(c_1-c_0-\lambda_0) \quad \textit{if } c_1 \geq c_0+\lambda_0,\\ &\limsup_{t\to\infty} t^{-1} W_t \leq -\frac{c_1+\lambda_1}{c_0+c_1+\lambda_1}(c_0-c_1-\lambda_1) \quad \textit{if } c_0 \geq c_1+\lambda_1. \end{aligned} \tag{7.48}$$

This concludes the proof of Theorem 4.2 and the discussion of our two classes of IPSs for the $(\infty, 0)$-model. In Section 8 we give additional examples and discuss some limitations of our setting.

8. Other examples

We describe here three types of examples that satisfy our hypotheses: generalizations of the (α, β)-model and of the $(\infty, 0)$-model, and mixed models. We also discuss an example that is beyond the reach of our setting.

1. Internal noise models. For $x \in \mathbb{Z} \setminus \{0\}$ and $\eta \in E$, let $\pi_x(\eta)$ be functions with a finite range of dependence R. These are the rates to jump x from the position of the walk. Let $\pi_x := \sup_\eta \pi_x(\eta)$ and suppose that, for some $u > 0$,

$$\sum_{x \in \mathbb{Z}\setminus\{0\}} e^{u|x|} \pi_x < \infty. \tag{8.1}$$

This implies that also

$$\Pi := \sum_{x \in \mathbb{Z}\setminus\{0\}} \pi_x < \infty. \tag{8.2}$$

The walk starts at the origin, and waits an independent Exponential(Π) time τ until it jumps to x with probability $\pi_x(\xi_\tau)/\Pi$. These probabilities do not necessarily sum up to one, so the walk may well stay at the origin. The subsequent jumps are obtained analogously, with ξ_τ substituted by the environment around the walk at the time of the attempted jump. It is clear that (A1)–(A3) hold. The walk has a bounded probability of standing still independently of the environment, and its jumps have an exponential tail. We take

$$\Gamma_L := \{\tau > L\}. \tag{8.3}$$

By defining an auxiliary walk $(H_t)_{t \geq 0}$ that also tries to jump at time τ, but only to sites $x > 0$ with probability π_x/Π, we see that $W_t \leq H_t$ and that H_t has properties analogous to the process defined in the proof of Lemma 7.2. Therefore, (H1)–(H3) are always satisfied for this model. Since Γ_L is independent of ξ, (H4) is the (unconditional) cone-mixing property. Observe that $W_0 = 0$, so that $\bar{\mu} = \mu$. Therefore the LLN for this model holds in both examples discussed in Section 7, and also for the IPSs for which cone-mixing was shown in Avena et al. [2]. The (α, β)-model is an internal noise model with $R = 0$ (the rates depend only on the state of the site where the walker is) and $\pi_x(\eta) = 0$, except for $x = \pm 1$, for which $\pi_1(1) = \alpha = \pi_{-1}(0)$ and $\pi_1(0) = \beta = \pi_{-1}(1)$.

2. Pattern models. Take $\aleph$ to be a finite sequence of 0's and 1's, which we call a *pattern*, and let R be the length of this sequence. Take the environment ξ to be of the same type used to define the $(\infty, 0)$-walk. Let $q : \{0, 1\}^R \setminus \{\aleph\} \to [0, 1]$. The pattern walk is defined similarly as the $(\infty, 0)$-walk, with the trap being substituted by the pattern, and a Bernoulli(q) random variable being used to decide whether the walk jumps to the right or to the left. More precisely, let $\vartheta = (\xi_0(0), \ldots, \xi_0(R-1))$. If $\vartheta = \aleph$, then we set $W_0 = 0$, otherwise we sample b_0 as an independent Bernoulli($q(\vartheta)$) trial. If $b_0 = 1$, then W_0 is set to be the starting position of the first occurrence of $\aleph$ in ξ_0 to the right of the origin, while if $b_0 = 0$, then the first occurrence of $\aleph$ to the left of the origin is taken instead. Then the walk waits at this position until the configuration of one of the R states to its right changes, at which time the procedure to find the jump is repeated with the environment as seen from W_0. Subsequent jumps are obtained analogously. The $(\infty, 0)$-model is a pattern model with $\aleph := (1, 0)$, $q(1, 1) := 1$, $q(0, 0) := 0$ and $q(0, 1) := 1/2$.

For spin-flip systems given by (7.9), the pattern walk is defined and finite for all times, no matter what $\aleph$ is, the reasoning being exactly the same as for the $(\infty, 0)$-walk. Also, it may be analogously defined so as to satisfy assumptions (A1)–(A3). Defining the events Γ_L as

$$\Gamma_L := \left\{\xi_s^{\pm}(j) = \xi_0^{\pm}(j)\ \forall s \in [0, L] \text{ and } j \in \{0, \ldots, R-1\}\right\}, \tag{8.4}$$

we may indeed, by completely analogous arguments, reobtain all the results of Section 7, so that hypotheses (H1)–(H4) hold and, therefore, the LLN as well.

3. Pattern models with extra jumps. Examples of models that fall into our setting and for which the events Γ_L depend non-trivially both on ξ and Y can be constructed by taking a pattern model and adding noise in the form of non-zero jump rates while sitting on the pattern. More precisely, add to Y an independent Poisson process N with positive rate and let W jump also at events of N but with the same jump mechanism, i.e., choosing the sign of the jump according to the result of a Bernoulli(q) random variable, and the displacement using the pattern. Taking $\Gamma_L := \Gamma_L^{\aleph} \cap \{N_L = 0\}$, where $\Gamma_L^{\aleph}$ is the corresponding event for the pattern model, we may check that, for the two examples of dynamic random environments considered in Theorem 4.2, (A1)–(A3) and (H1)–(H4) are all verified.

4. Mixtures of pattern and internal noise models. Another class of models with nontrivial dependence structure for the regeneration-inducing events can be constructed as follows. Let W^0 be an internal noise model and W^1 a pattern model (with or without extra jumps) on the same random environment ξ and let Y^i, $i \in \{0, 1\}$, be the corresponding random elements associated to each model. Let $X = (X)_{n\in\mathbb{N}}$ be a sequence of i.i.d. Bernoulli(p) random variables independent of all the rest, where $p \in (0, 1)$. Then the mixture is the model for which the dynamics associated to $i \in \{0, 1\}$ are applied in the time interval $[n - 1, n)$ when $X_n = i$. Note that this model will have deterministic jumps.

Letting $Y := \left(Y^0, Y^1, X\right)$ where Y^i is the corresponding random element associated to the model i, it is easily checked that this model indeed falls into our setting.

Choosing

$$\Gamma_L := \Gamma_L^1 \cap \{X_k = 1, k = 1, \dots, L\} \tag{8.5}$$

where Γ_L^1 is the corresponding event for the pattern model, it is not hard to verify, using the results of Section 7, that, for the two classes of random environments considered in Theorem 4.2, the mixed model satisfies (A1)–(A3) and (H1)–(H4).

An open example. We will close with an example of a model that does not satisfy the hypotheses of our LLN (in dynamic random environments given by spin-flip systems). When $\xi(0) = j$, let C^j be the cluster of j's around the origin. Define jump rates for the walk as follows:

$$\begin{aligned} \pi_1(\eta) &= \begin{cases} |C^1| & \text{if } \eta(0) = 1, \\ |C^0|^{-1} & \text{if } \eta(0) = 0, \end{cases} \\ \pi_{-1}(\eta) &= \begin{cases} |C^0| & \text{if } \eta(0) = 0, \\ |C^1|^{-1} & \text{if } \eta(0) = 1. \end{cases} \end{aligned} \tag{8.6}$$

Even though this looks like a fairly natural model, it does not satisfy (A2). It also will not satisfy (H1) and (H2) together for any reasonable random environment, which is actually the hardest obstacle. The problem is that, while we are able to transport a.s. properties of the equilibrium measure to the measure of the environment as seen from the walk, we cannot control the distortion in events of positive measure. Thus, even if Γ_L has positive probability at time zero, there is no a priori guarantee that it will have an appreciable probability from the point of view of the walk at later times. Because of this, we cannot implement our regeneration strategy, and our proof of the LLN breaks down.

Appendix. Coupling rates

Here we give the rates for a coupling between Ξ and Ξ', mentioned in Section 7.3.2, such that corresponding pairs of coordinates are distributed according to the Vasershtein coupling. Let

η, η' be the state of the middle coordinates ξ and ξ'; the states outside the origin of the other coordinates play no role. Then the flip rates at the origin are given schematically by

$$
\begin{aligned}
(000)(000) &\quad\to\quad \begin{cases} (111)(111) & c_1, \\ (011)(011) & c(\eta) \wedge c(\eta') - c_1, \\ (011)(001) & c(\eta) - c(\eta) \wedge c(\eta'), \\ (001)(011) & c(\eta') - c(\eta) \wedge c(\eta'), \\ (001)(001) & c_1 + \lambda_1 - c(\eta) \vee c(\eta'), \end{cases} \\
(001)(001) &\quad\to\quad \begin{cases} (111)(111) & c_1, \\ (011)(011) & c(\eta) \wedge c(\eta') - c_1, \\ (011)(001) & c(\eta) - c(\eta) \wedge c(\eta'), \\ (001)(011) & c(\eta') - c(\eta) \wedge c(\eta'), \\ (000)(000) & c_0, \end{cases} \\
(001)(011) &\quad\to\quad \begin{cases} (111)(111) & c_1, \\ (011)(011) & c(\eta) - c_1, \\ (001)(001) & c(\eta') - c_0, \\ (000)(000) & c_0, \end{cases} \\
(000)(001) &\quad\to\quad \begin{cases} (111)(111) & c_1, \\ (011)(011) & c(\eta) \wedge c(\eta') - c_1, \\ (011)(001) & c(\eta) - c(\eta) \wedge c(\eta'), \\ (001)(011) & c(\eta') - c(\eta) \wedge c(\eta'), \\ (001)(001) & c_1 + \lambda_1 - c(\eta) \vee c(\eta'), \\ (000)(000) & c_0, \end{cases} \\
(000)(011) &\quad\to\quad \begin{cases} (111)(111) & c_1, \\ (011)(011) & c(\eta) - c_1, \\ (001)(011) & c_1 + \lambda_1 - c(\eta), \\ (000)(000) & c_0, \\ (000)(001) & c(\eta') - c_0, \end{cases} \\
(000)(111) &\quad\to\quad \begin{cases} (111)(111) & c_1, \\ (011)(111) & c(\eta) - c_1, \\ (001)(111) & c_1 + \lambda_1 - c(\eta), \\ (000)(000) & c_0, \\ (000)(001) & c(\eta') - c_0, \\ (000)(011) & c_0 + \lambda_0 - c(\eta'). \end{cases}
\end{aligned}
\tag{A.1}
$$

The other transitions, starting from

$$
\begin{aligned}
&(111)(111), \quad (011)(011), \quad (011)(001), \quad (111)(011), \\
&(111)(001) \quad \text{and} \quad (111)(000),
\end{aligned}
\tag{A.2}
$$

can be obtained from the ones in (A.1) by symmetry, by exchanging the roles of η/η' or of particles/holes.

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