A Necessary Condition for Local Asymptotic Stability of Discrete-Time Nonlinear Systems with Exogenous Disturbance

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Abstract—Local asymptotic stability of nonlinear systems with real-parametric uncertainty or disturbance is one of the important problems in the control systems literature. In this paper, we investigate the problem of asymptotic stability for discrete-time nonlinear systems with time-varying disturbance. We assume that the disturbance vector is generated by an exosystem, which is neutrally stable. Thus, the disturbances that we consider include both constant and periodic signals. For this class of nonlinear systems with time-varying disturbance, we derive a necessary condition for local asymptotic stability of equilibria. As corollaries of our general result, we deduce the necessary condition obtained by Sundarapandian [1] for discrete-time nonlinear systems with constant real parametric uncertainty, and the necessary condition obtained by Lin and Byrnes [2] for discrete-time nonlinear autonomous systems. We illustrate our result with several examples. © 2006 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Local asymptotic stabilization of nonlinear control systems is one of the central problems in control systems literature. It is one of the primary design criteria in physical, engineering, and industrial problems.

The problem of stabilizability for nonlinear control systems can be viewed as a modern version of the classical stability problem for equilibria of ordinary differential equations. Some important works on stabilizability of nonlinear works are the works of Jurdjevic and Quinn [3], Brockett [4], Aeyels [5], Byrnes and Isidori [6-9], Sussmann [10], and Sontag [11].

In a recent work [12], Byrnes and Sundarapandian derived a necessary condition for local asymptotic stability of continuous-time nonlinear systems with constant real parametric uncertainty, which is an enhancement of Brockett's necessary condition [4] for nonlinear autonomous systems. In [1], we derived the discrete-time version of the necessary condition obtained by
This paper is a discrete-time analog of our recent work [13] on continuous-time nonlinear systems with time-varying exogenous disturbance. In this paper, we derive a necessary condition for local asymptotic stability of equilibria of continuous-time nonlinear systems with time-varying disturbance. As a corollary of our general result, we deduce our recent necessary condition [1] for discrete-time nonlinear systems with constant real-parametric uncertainty and the necessary condition derived by Lin and Byrnes [2] for local asymptotic stability of equilibria of nonlinear discrete-time autonomous systems.

In this paper, we consider a nonlinear system described by

$$x_{k+1} = f(x_k, \mu_k) = f_{\mu_k}(x_k),$$  \hspace{1cm} (1)

where $x \in \mathbb{R}^n$ is the state and $\mu \in \mathbb{R}^m$ is the parametric uncertainty or disturbance for nonlinear system (2). The disturbance vector $\mu$ is assumed to satisfy the exosystem dynamics given by

$$\mu_{k+1} = s(\mu_k).$$  \hspace{1cm} (2)

We assume that $x \in U$ where $U$ is an open neighborhood of the origin of $\mathbb{R}^n$, $\mu \in \Omega$, where $\Omega$ is an open neighborhood of $\mu^* \in \mathbb{R}^m$, $f$ is $C^1$ in $x$ and jointly continuous in $x$ and $\mu$, $s$ is $C^1$ in $\mu$, and also that

$$f(0, \mu^*) = 0 \quad \text{and} \quad s(\mu^*) = 0.$$

We assume that the exosystem (2) is assumed to be neutrally stable (Lyapunov stable in both forward and backward directions of time). Thus, the exogenous disturbance that we consider in this paper includes the important special cases of constant real-parametric uncertainty considered in [1], and periodic signals.

In this paper, we investigate the problem of finding a suitable necessary condition for the asymptotic stability of discrete-time nonlinear systems with time-varying disturbance described by dynamics (2) for the value $\mu^*$. Explicitly, we are interested in finding a necessary condition for the asymptotic stability of the nonlinear system,

$$x_{k+1} = f(x_k, \mu^*) = f_{\mu^*}(x_k).$$ \hspace{1cm} (3)

This problem is related to the problem of robustly stabilizing a control system of the form,

$$x_{k+1} = g(x_k, \mu^*, u_k).$$

The local solvability of the equation,

$$x - f_{\mu}(x) = y,$$ \hspace{1cm} (4)

for all $\mu$ near $\mu^*$ is a prerequisite for questions such as asymptotic stability of, or stabilization about, an equilibrium. In this paper, we study this basic question.

If the origin is a locally exponentially stable equilibrium of system (3), then it follows immediately from Lyapunov stability theory that

$$A = \frac{\partial f}{\partial x}(0, \mu^*)$$

is a convergent matrix, i.e., all the eigenvalues of $A$ lie inside the open unit disc of the complex plane, and so rank of $I - A$ is $n$. In particular, from the inverse function theorem [14], it follows that for all $\mu$ near $\mu^*$, system (4) is locally solvable. We contend that this is the case for critically asymptotically stable systems as well, as illustrated in the following examples.
EXAMPLE 1. Consider the nonlinear system

\[ x_{k+1} = x_k (1 + \mu_k) - x_k^3, \]
\[ \mu_{k+1} = -\mu_k, \]  \hspace{1cm} (5)

where \( \mu \) is the exogenous disturbance.

The exosystem dynamics is clearly neutrally stable. In fact, the general solution of the exosystem has the form,

\[ \mu_k = (-1)^k \mu_0. \]

Note that \( x = 0 \) is a critically asymptotically stable equilibrium of the zero-parameter system,

\[ x_{k+1} = x_k - x_k^3. \]

Note especially that for all values of \( \mu \), near \( \mu^* = 0 \), the equation,

\[ x - f_{\mu}(x) = -\mu x + x^3 = y, \]

is locally solvable.

EXAMPLE 2. Consider the nonlinear system,

\[ x_{k+1} = x_k + \mu_k + x_k^2, \]
\[ \mu_{k+1} = -\mu_k, \]  \hspace{1cm} (6)

where \( \mu \) is the exogenous disturbance.

As pointed out in Example 1, the exosystem dynamics described by \( \mu_{k+1} = -\mu_k \) is clearly neutrally stable.

Clearly, \( x = 0 \) is an unstable equilibrium of the zero-parameter system,

\[ x_{k+1} = x_k + x_k^2. \]  \hspace{1cm} (7)

Note that the equation,

\[ x - f_{\mu}(x) = -\mu - x^2 = y, \]

is not locally solvable for any \( \mu > 0 \).

2. MAIN RESULT

In this section, using degree theory, we derive a necessary condition for \( x = 0 \) to be a locally asymptotically stable equilibrium of the discrete-time nonlinear system,

\[ x_{k+1} = f(x_k, \mu^*) = f_{\mu^*}(x_k), \]

which is a particular case of the general discrete-time nonlinear system,

\[ x_{k+1} = f(x_k, \mu_k) = f_{\mu_k}(x_k), \]

subject to time-varying parameter vector \( \mu \) satisfying the neutrally stable exosystem described by

\[ \mu_{k+1} = s(\mu_k). \]

Our main result is an enhancement of the necessary condition obtained by the author [1] for discrete-time nonlinear systems with constant real-parametric uncertainty. Our result is applicable to a wider class of nonlinear systems subject to time-varying disturbance, which includes constant and periodic parametric uncertainties as special cases. Our result asserts, in particular, that any asymptotically stable equilibrium of a discrete-time nonlinear system described by a \( C^1 \) map persists as an equilibrium in a robust way.
Theorem 1. Consider the nonlinear system described by

\[ x_{k+1} = f(x_k, \mu_k) \equiv f_{\mu_k}(x_k), \]
\[ \mu_{k+1} = s(\mu_k), \] (8)

where \( x \in \mathbb{R}^n \) is the state and \( \mu \in \mathbb{R}^m \) is the exogenous disturbance. The state \( x \) is defined in an open neighborhood of the origin of \( \mathbb{R}^n \), and the disturbance \( \mu \) is defined in an open neighborhood of \( \mu^* \in \mathbb{R}^m \). We assume the following.

(A1) \( f \) is locally \( C^1 \) in \( x \), jointly continuous in \( x \) and \( \mu \), and \( s \) is locally \( C^1 \) in \( x \).

(A2) \( f_{\mu^*}(0) = 0 \) and \( s(\mu^*) = 0 \).

(A3) The exosystem dynamics \( \mu_{k+1} = s(\mu_k) \) is neutrally stable.

A necessary condition for \( x = 0 \) to be a locally asymptotically stable equilibrium of the system,

\[ x_{k+1} = f(x_k, \mu^*) \equiv f_{\mu^*}(x_k), \] (9)

is that for all \( \mu \) near \( \mu^* \in \mathbb{R}^m \), the map \( I - f_{\mu} \) is locally onto, i.e., the equation

\[ x - f_{\mu}(x) = y, \]

is locally solvable.

Proof. The proof of this result is analogous to the proof given in [1] for a similar result established for discrete-time nonlinear systems with constant real-parametric uncertainty. Our proof essentially follows from the Kransnoselski’s necessary condition [14] for locally asymptotically stable nonlinear systems.

First, we shall establish that

\[ (x, \mu) = (0, \mu^*) \]

is a Lyapunov stable equilibrium of composite system (8). Note that \( x = 0 \) is an asymptotically stable equilibrium of the system,

\[ x_{k+1} = f(x_k, \mu^*), \]

and that \( \mu^* = 0 \) is a Lyapunov stable equilibrium of the system,

\[ \mu_{k+1} = s(\mu_k). \]

Hence, it follows that \( (x, \mu) = (0, \mu^*) \) is a Lyapunov stable equilibrium of composite system (8) in view of its triangular structure. Thus, given any neighborhood \( \mathcal{N} \) of \( (x, \mu) = (0, \mu^*) \), we can find a neighborhood \( \mathcal{N}_0 \subset \mathcal{N} \) such that all solutions \( (x(t), \mu(t)) \) initialized in \( \mathcal{N}_0 \) stay inside \( \mathcal{N} \) for all values of \( t \geq 0 \).

Now, consider a general, discrete-time, autonomous system,

\[ x_{k+1} = F(x_k), \quad F(0) = 0, \quad x \in \mathbb{R}^n \text{ and } F \text{ is } C^1. \] (10)

If \( x = 0 \) is a locally asymptotically stable equilibrium of the dynamics (10), then, by Kransnoselski’s theorem [14, Theorem 39.1, p. 235], it is necessary that

\[ \text{ind}(I - F, 0) = 1, \]

where \( \text{ind}(I - F, 0) \) denotes the index of the map \( I - F \) at the critical point \( x = 0 \).

Now, suppose that \( x = 0 \) is a locally asymptotically stable equilibrium of the system,

\[ x_{k+1} = f_{\mu^*}(x_k), \quad x \in \mathbb{R}^n. \]
By Krasnoselski's necessary condition, it follows that
\[
\text{ind} (I - f_{\mu^*}, 0) = 1.
\]

Since the index operator is robust with respect to small variations in the parameter, it follows that for all values of \( \mu \) near \( \mu^* \), we have
\[
\text{ind} (I - f_{\mu}, 0) \neq 0.
\]

Now, we can apply degree theory [16] to conclude that for all values of \( \mu \) near \( \mu^* \), the map \( I - f_{\mu} \) is locally onto, i.e., the equation
\[
x - f_{\mu} (x) = y,
\]
is locally solvable.

**Corollary 1.** (See [1].) Under the same assumptions of Theorem 1 with the exosystem dynamics replaced by
\[
\mu (k + 1) = \mu_k,
\]
(note that (A3) holds trivially for the above exosystem dynamics), a necessary condition for \( x = 0 \) to be a locally asymptotically stable equilibrium of the system (9) is that for all \( \mu \) near \( \mu^* \in \mathbb{R}^m \), the map \( I - f_{\mu} \) is locally onto, i.e., the equation \( x - f_{\mu} (x) = y \) is locally solvable.

**Corollary 2.** (See [2].) A necessary condition for the origin to be a locally asymptotically stable equilibrium of the discrete-time nonlinear system,
\[
x_{k+1} = F (x_k), \quad [F (0) = 0, \ x \in \mathbb{R}^n, \ F \text{ is } C^1],
\]
is that the map \( I - F \) is locally onto, i.e., the equation,
\[
x - F (x) = y,
\]
is locally solvable.

**Proof.** See the proof of Corollary 1 in [1].

Next, we point out that the necessary condition given in Theorem 1 is not sufficient.

**Example 3.** Consider the nonlinear system described by
\[
x_{k+1} = x_k (1 + \mu_k) + x^3,
\]
\[
\mu_{k+1} = \mu_k.
\]

It is easy to verify that all the Assumptions (A1)–(A3) stated in the hypotheses of Theorem 1 are satisfied by the above system with \( \mu^* = 0 \). We note also that the necessary condition stated in Theorem 1 holds, i.e., the map,
\[
x - f_{\mu} (x) = -\mu x - x^3 = y,
\]
is locally solvable for all values of \( \mu \) near \( \mu^* = 0 \).

However, the zero-parameter system,
\[
x_{k+1} = x_k + x^3_k,
\]
is unstable at \( x = 0 \). This shows that necessary condition given in Theorem 1 is not sufficient.
REFERENCES