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The Lusternik–Schnirelmann category of certain infinite *CW*-complexes

Joseph Roitberg[a,b,*,1]

[a]*Department of Mathematics and Statistics, Hunter College, CUNY, 695 Park Avenue, New York, NY 10021, U.S.A.*
[b]*Ph.D. Program in Mathematics, The Graduate School and University Center, CUNY 33 West 42nd Street, New York, NY 10036-8099, U.S.A.*

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Abstract

Examples are constructed to illustrate: (i) The *LS* category of a 1-connected, finite type *CW*-complex X which is the homotopy colimit of a sequence $X_1 \to X_2 \to X_3 \to \cdots$ of 1-connected, finite *CW*-complexes may exceed the *LS* category of each X_i; and (ii) *LS* category is *not* an invariant of the localization genus of a 1-connected, finite type *CW*-complex.

1. Introduction

In this note, we settle in the negative various questions which have been raised about $\mathrm{cat}(X)$, the Lusternik–Schnirelmann category of a (pointed) *CW*-complex X. [We follow the convention which yields cat(point) $= 0$. Thus, $\mathrm{cat}(X) \leqslant 1$ is equivalent to X admitting the structure of a co-H-space.]

In [4], Ganea discusses the problem of finding an upper bound for $\mathrm{cat}(X)$ when X is a homotopy colimit of the form

$$X = \mathrm{hocolim}\,(X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to \cdots),$$

assuming that the set of integers $\{\mathrm{cat}(X_i)\}$, $i \geqslant 1$, is bounded. His example

$$K(\mathbb{Q}, 1) = \mathrm{hocolim}(S^1 \xrightarrow{f_1} S^1 \xrightarrow{f_2} S^1 \to \cdots), \tag{1.1}$$

* E-mail: roitberg@xena.hunter.cuny.edu.

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where f_i is a map of degree i, shows that $\mathrm{cat}(X)$ may exceed $\sup_{i \geqslant 1}\{\mathrm{cat}(X_i)\}$. Note the non-simple connectivity of the spaces in (1.1); indeed, the invariant which proves $\mathrm{cat}(K(\mathbb{Q}, 1)) > 1$ is precisely the fundamental group $\pi_1(K(\mathbb{Q}, 1))$, which is not a free group. The same argument shows that for any proper subset P of primes, the P-localization $K(\mathbb{Z}_{(P)}, 1)$ of S^1 satisfies $\mathrm{cat}(K(\mathbb{Z}_{(P)}, 1)) > 1$. By way of contrast, Toomer [13; Theorem 4] proves that if X is 1-connected, then

$$\mathrm{cat}(X_{(P)}) \leqslant \mathrm{cat}(X)$$

for all P. Thus, one might ask

Question 1. *If X is a homotopy colimit of* 1-*connected CW-complexes X_i, is* $\mathrm{cat}(X) \leqslant \sup_{i \geqslant 1}\{\mathrm{cat}(X_i)\}$?

Our first example answers this question negatively.

Example 1. There exists a 1-connected, finite type (over $\mathbb{Z}$) CW-complex X which is a homotopy colimit of 1-connected, finite CW-complexes X_i such that

$$\mathrm{cat}(X_i) = 1, i \geqslant 1 \quad \text{and} \quad \mathrm{cat}(X) = 2.$$

Another question, raised by Toomer in [13] following his Theorem 4 (and attributed to Peter Hilton), concerns relating $\mathrm{cat}(X)$ with $\mathrm{cat}(X_{(p)})$, p a prime. Here, X is assumed to be a nilpotent space.

Question 2. *When is* $\mathrm{cat}(X) = \sup_p\{\mathrm{cat}(X_{(p)})\}$?

Cornea shows in [3] that for X 1-connected, of finite type (over $\mathbb{Z}$),

$$\mathrm{cat}(X) \leqslant 2\,\sup_p\{\mathrm{cat}(X_{(p)})\} + 1. \tag{1.2}$$

He furthermore expresses the belief that the stronger inequality

$$\mathrm{cat}(X) \leqslant 2\cdot\sup_p\{\mathrm{cat}(X_{(p)})\} \tag{1.3}$$

holds, and proves (1.3) under the additional assumption that X be a *finite CW*-complex.

A question somewhat related to Question 2 was posed by McGibbon in his survey paper [8]; see Problem 2.2.

Question 3. *Is* $\mathrm{cat}(\,)$ *a generic property, that is if X, Y are* (*nilpotent*) *of finite type and $X_{(p)} \simeq Y_{(p)}$ for all primes p, is* $\mathrm{cat}(X) = \mathrm{cat}(Y)$?

Our second example, which is actually a refinement of Example 1, shows that: (i) equality in Question 2 does not necessarily hold, even if X is 1-connected; (ii) the conjectured inequality (1.3), if true, is sharp; and (iii) Question 3 has a negative answer.

Example 2. There exist 1-connected, finite type (over $\mathbb{Z}$) CW-complexes X, Y in the same genus such that

$$\text{cat}(X_{(p)}) = 1 = \text{cat}(Y_{(p)}), \text{ all primes } p,$$
$$\text{cat}(X) = 2, \text{cat}(Y) = 1. \tag{1.4}$$

Example 2 is also relevant to a result and a problem in [10]. Namely, Corollary 5.1 of that paper implies the statement that "$\text{cat}(X) = 1$ is a generic property provided X is a 1-connected, *finite CW*-complex". Furthermore, in the paragraph following Corollary 5.1, McGibbon essentially reiterates Question 3 in the case $\text{cat}(Y) \leqslant 1$, which our Example 2 answers negatively. Incidentally, [10; Corollary 5.1] suggests that there may be a positive answer to Question 3 if X is a 1-connected, *finite CW*-complex.

The following theorem leads to both Examples 1 and 2.

Theorem. *Let* $\varphi : \Sigma K(\mathbb{Z}, 5) \to S^4$ *be an essential phantom map and let* X *be the mapping cone of* φ. *Then* $\text{cat}(X) = 2$.

The proof of this theorem, and the deductions of Examples 1 and 2 from it, will be carried out in the next section. The invariant we use to prove $\text{cat}(X) = 2$ is a "Hopf invariant" of the type employed by Berstein and Hilton in their study of *LS* category [1] and recently generalized and exploited by Iwase [6] in his construction of counterexamples to the "$\text{cat}(X \times S^n) = \text{cat}(X) + 1$" problem of Ganea. We rely heavily on some results in [6].

We would be remiss in not saying a word about the Eckmann–Hilton duals of our examples. Particularly interesting is the outstanding question of whether admitting an H-space structure is a generic property; see [8; Problem 1.3]. Our examples suggest that we should upgrade this question to a

Conjecture. *There exist* 1-*connected, finite type* (*over* $\mathbb{Z}$) *CW-complexes* V, W *in the same genus such that* V *admits an* H-*space structure while* W *admits no* H-*space structure.*

A more precise form of this conjecture will be given at the end of the paper.?

2. Proofs

Proof of Theorem. We abbreviate $K = K(\mathbb{Z}, 5)$ and take the obvious (suspension) co-H-structures on ΣK, S^4 (the latter being unique); thus

$$\text{cat}(\Sigma K) = 1 = \text{cat}(S^4).$$

For $\varphi : \Sigma K \to S^4$, it is classical [1; Theorem 2.6 (i)] that

$$\text{cat}(X) \leqslant 2.$$

We aim to prove that

$$\text{cat}(X) = 2,$$

if (and only if) φ is essential. To this end, consider the diagram

$$\begin{array}{ccc} \Sigma\Omega(\Sigma K) & \xrightarrow{\Sigma\Omega\varphi} & \Sigma\Omega S^4 \\ {\scriptstyle\sigma_1}\uparrow\downarrow{\scriptstyle e_1} & & {\scriptstyle\sigma_2}\uparrow\downarrow{\scriptstyle e_2} \\ \Sigma K & \xrightarrow{\varphi} & S^4, \end{array}$$

where e_1, e_2 denote the respective evaluation maps and σ_1, σ_2 are induced by the given co-H-structures on $\Sigma K, S^4$. We have

$$e_1 \circ \sigma_1 \simeq 1, \quad e_2 \circ \sigma_2 \simeq 1, \quad e_2 \circ \Sigma\Omega\varphi \simeq \varphi \circ e_1 \tag{2.1}$$

where "1" generically denotes the identity map. However, $\Sigma\Omega\varphi \circ \sigma_1$ need not be homotopic to $\sigma_2 \circ \varphi$. By (2.1), the difference

$$-\sigma_2 \circ \varphi + \Sigma\Omega\varphi \circ \sigma_1 \tag{2.2}$$

lifts to the homotopy fiber $E^2(\Omega S^4) = \Omega S^4 * \Omega S^4$ of e_2. Moreover, the lift is unique since, in the long fibration sequence

$$\cdots \to \Omega S^4 \to E^2(\Omega S^4) \to \Sigma\Omega S^4 \xrightarrow{e_2} S^4,$$

the fiber inclusion $\Omega S^4 \to E^2(\Omega S^4)$ is inessential. Thus, (2.2) gives rise to a well-defined element

$$H_1(\varphi) \in [\Sigma K, E^2(\Omega S^4)],$$

which may be called the Iwase–Berstein–Hilton–Hopf invariant of φ, and which clearly measures the failure of φ to be a co-H-map with respect to the given co-H-structures on $\Sigma K, S^4$. The map

$$H_1 : [\Sigma K, S^4] \to [\Sigma K, E^2(\Omega S^4)] \tag{2.3}$$

is easily checked to be a homomorphism [6; Definition 2.4]. A special case of [6; Theorem 3.8 (1)] asserts that if X is the mapping cone of a map $\varphi : A \to B$ of co-H-spaces, then $\mathrm{cat}(X) = 2$ if the following conditions are met:

(i) A is $(e-1)$-connected, B is $(d-1)$-connected, $e \geqslant d \geqslant 2$;
(ii) $\dim(B) \leqslant 2(d-1)$;
(iii) $E^2(\Omega i) \circ H_1(\varphi)$ is essential, where

$$E^2(\Omega i) : E^2(\Omega B) \to E^2(\Omega X) \tag{2.4}$$

is induced by the inclusion $i : B \to X$.

In our situation, with $A = \Sigma K$, $B = S^4$, conditions (i) and (ii) plainly hold, so we concentrate on verifying (iii).

First we show that (2.3) is an isomorphism of non-0 groups. Let

$$r : \Sigma K \to (\Sigma K)_{(0)} = S^6_{(0)}$$

be a rationalization map, where $S^6_{(0)}$ is the rationalized 6-sphere; r is a co-H-map with respect to the given co-H-structure on ΣK and the (unique) co-H-structure on $S^6_{(0)}$. Consider the diagram

$$\begin{array}{ccc} [\Sigma K, S^4] & \xrightarrow{H_1} & [\Sigma K, E^2(\Omega S^4)] \\ \uparrow r_1^* & & \uparrow r_2^* \\ [S^6_{(0)}, S^4] & \xrightarrow{H_1'} & [S^6_{(0)}, E^2(\Omega S^4)], \end{array} \tag{2.5}$$

where the vertical arrows are induced by r and where H_1' is defined in the same way as H_1; by appealing to [6; Proposition 2.9 (1)], we see that (2.5) is commutative. By phantom map theory (see, e.g., [9; Theorem 5.4] or [12; Theorem 4.2]), r_1^*, r_2^* are isomorphisms and each element of $[\Sigma K, S^4]$, $[\Sigma K, E^2(\Omega S^4)]$ is represented by a phantom map. [In the case of $[\Sigma K, E^2(\Omega S^4)]$, we need to observe that although $E^2(\Omega S^4)$ is not a finite CW-complex, it admits a decomposition into a bouquet of spheres, with a single S^7 as its bottom cell, hence its mod p cohomology ring is locally finite as a module over the Steenrod algebra for each prime p. A theorem of Lannes and Schwartz [7] allows us to apply [9; Theorem 5.4] to infer the results stated above]. So to prove H_1 is an isomorphism, it suffices to prove H_1' is an isomorphism.

Now H_1' is certainly *not* induced by a map $S^4 \to E^2(\Omega S^4)$, but

$$\mathrm{adj}(H_1'): [S^5_{(0)}, \Omega S^4] \to [S^5_{(0)}, \Omega E^2(\Omega S^4)],$$

the adjoint of H_1', factors as

$$[S^5_{(0)}, \Omega S^4] \xrightarrow{\alpha} [S^5_{(0)}, \Omega S^7] \xrightarrow{\beta} [S^5_{(0)}, \Omega E^2(\Omega S^4)].$$

Here α is induced by a left homotopy inverse $\Omega S^4 \to \Omega S^7$ of Ω(Hopf map from S^7 to S^4) – α is the adjoint of the James–Hopf invariant γ_2: $[S^6_{(0)}, S^4] \to [S^6_{(0)}, S^7]$ [2; Definition 3.10]; see also [5; Example 4.2] – and β is induced by Ω(inclusion of the bottom S^7 into $E^2(\Omega S^4)$). It is now clear that α, β are both isomorphisms of groups, each of which is isomorphic to $\mathbb{R}$, viewed as a vector space over $\mathbb{Q}$ of uncountable dimension.

Next we show that the map

$$[S^6_{(0)}, E^2(\Omega S^4)] \to [S^6_{(0)}, E^2(\Omega X)] \tag{2.6}$$

induced by (2.4) (in the case $B = S^4$) is a monomorphism. For this purpose, consider the commutative diagram

$$\begin{array}{ccccc} [S^6_{(0)}, E^2(\Omega S^4)] & & \longrightarrow & & [S^6_{(0)}, E^2(\Omega X)] \\ & \nwarrow \beta_1 & & \beta_2 \nearrow & \\ & & [S^6_{(0)}, S^7], & & \end{array}$$

where β_1, β_2 are induced by the inclusions of the bottom S^7 into $E^2(\Omega S^4)$, $E^2(\Omega X)$ respectively. Note that β_1 is precisely the adjoint of β, defined above, hence is an isomorphism. To prove that (2.6) is a monomorphism, it therefore suffices to prove that β_2 is a monomorphism. We will achieve this by finding a retraction from $E^2(\Omega X)$ to S^7.

Consider $\mathrm{adj}(i): S^3 \to \Omega X$. We claim that this map admits a left homotopy inverse, that is there is a retraction $\Omega X \to S^3$. If X were homotopy equivalent to $S^4 \vee \Sigma^2 K$, then ΩX would be homotopy equivalent to $\Omega(S^4 \vee \Sigma^2 K)$ and the latter retracts to ΩS^4 (obvious), which in turn retracts to S^3 (using the H-structure on S^3). Of course, ΩX may not be homotopy equivalent to $\Omega(S^4 \vee \Sigma^2 K)$, but since φ is phantom, the n-skeleton $(\Omega X)_n$ of ΩX is homotopy equivalent to the n-skeleton of $\Omega(S^4 \vee \Sigma^2 K)$ for every n. Thus, for each $n \geqslant 3$, there is a retraction $(\Omega X)_n \to S^3$. Moreover, since the higher homotopy groups of S^3 are finite, the number of (homotopy classes of) such retractions is finite. A classical argument (compare with [11; Lemma 2] then allows us to conclude the existence of a retraction $\Omega X \to S^3$.

Observe now that a retraction $\Omega X \to S^3$ induces a retraction $E^2(\Omega X) = \Omega X * \Omega X \to S^3 * S^3 = S^7$, as desired.

We have now verified (iii) and the proof of the Theorem is complete. □

Deduction of Example 1 from the Theorem: Let X be as in the Theorem. Choosing a particular cell decomposition of K, set

$$X_i = S^4 \cup_{\varphi_i} \mathrm{cone}(\Sigma K_i),$$

where K_i is the i-skeleton of K and $\varphi_i = \varphi | K_i$. Of course, φ_i is inessential for all i, so that

$$X_i \simeq S^4 \vee \Sigma^2 K_i$$

and

$$\mathrm{cat}(X_i) = 1.$$

On the other hand,

$$X = \mathrm{hocolim}(X_1 \to X_2 \to X_3 \to \cdots)$$

satisfies

$$\mathrm{cat}(X) = 2$$

by the Theorem.

Deduction of Example 2 from the Theorem: We again take X as in the Theorem but we must choose φ more carefully than before. Namely, we choose an essential φ so that for each prime p, the composite

$$\Sigma K \xrightarrow{\varphi} S^4 \to S^4_{(p)}$$

is inessential, where $S^4 \to S^4_{(p)}$ is a p-localization map. According to [4; Theorem 2.2], there are uncountably many phantom maps of this type in the situation at hand. [Such phantom maps have been designated as *special* phantom maps in [11; Section 5] and as *clones* of the constant map by McGibbon and Møller; see e.g. [8; Section 6].] We take $Y = S^4 \vee \Sigma^2 K$.

Clearly,

$$\mathrm{cat}(Y) = 1 = \mathrm{cat}(Y_{(p)}) \quad \text{for all } p.$$

By the way we have chosen φ, $X_{(p)} \simeq Y_{(p)}$ for all p, hence

$$\mathrm{cat}(X_{(p)}) = 1 \quad \text{for all } p.$$

As $\mathrm{cat}(X) = 2$ by the Theorem, we have succeeded in producing the desired example.

Finally, we return to the Eckmann–Hilton duals of our examples. Focusing specifically on Example 2, we consider an essential, special phantom map

$$\psi : K(\mathbb{Z}, 2) \to \Omega S^6$$

and set

$$Z = \text{homotopy fiber of } \psi.$$

Z seems to be a reasonable candidate for a dual of the space X of Example 2. As φ is not a suspension class, ψ is not a loop class; as $\Sigma\varphi$ is inessential, $\Omega\psi$ is inessential; as the i-skeleta of X, Y are homotopy equivalent for all i, the ith Postnikov approximations of $Z, \Omega^2 S^6 \times K(\mathbb{Z}, 2)$ are homotopy equivalent for all i; and as X, Y are p-equivalent for all p, $Z, \Omega^2 S^6 \times K(\mathbb{Z}, 2)$ are p-equivalent for all p.

Conjecture. *Although* $\mathbb{Z}$ *and* $\Omega^2 S^6 \times K(\mathbb{Z}, 2)$ *are in the same genus*, Z *admits no H-structure*; *equivalently*, $\mathrm{cocat}(Z) > 1$.

Dualizing the proof of the Theorem in order to verify this conjecture is another matter altogether…

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