Journal of Functional Analysis **169**, 148–163 (1999)
Article ID jfan.1999.3490, available online at http://www.idealibrary.com on IDEAL®

Carleson Measures and Multipliers for Dirichlet Spaces

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Communicated by D. Sarason

Received January 16, 1998; revised October 22, 1998; accepted June 6, 1999

We characterize Carleson measures on the Dirichlet spaces. Our result leads to necessary and sufficient conditions for multipliers of the Dirichlet spaces. © 1999 Academic Press

Let $\mathbb{D}$ be the unit disk in the complex plane and dA be the normalized area measure on $\mathbb{D}$. For $\alpha > -1$ and $p > 0$ the Bergman space $\mathscr{A}^p_\alpha$ consists of all analytic functions f defined on $\mathbb{D}$ such that

$$\|f\|^p_{\mathscr{A}^p_\alpha} = \int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z) < \infty.$$

Here $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \, dA(z)$ has total mass 1 on $\mathbb{D}$. The space $\mathscr{D}^p_\alpha$ consists of all analytic functions g defined on $\mathbb{D}$ such that

$$\|g\|^p_{p,\alpha} = |g(0)|^p + \|g'\|^p_{\mathscr{A}^p_\alpha} < \infty.$$

If $p < \alpha + 1$, it is standard that $\mathscr{D}^p_\alpha = \mathscr{A}^p_{\alpha - p}$ with equivalence of norms. It is also trivial that the Hardy space $\mathscr{H}^2$ can be identified with $\mathscr{D}^2_1$ with equivalence of norms. The space $\mathscr{D}^p_\alpha$ is called a Dirichlet space if $p \geqslant \alpha + 1$. Particularly, $\mathscr{D}^2_0$ is the classical Dirichlet space.

A nonnegative measure μ on $\mathbb{D}$ is called a Carleson measure for $\mathscr{D}^p_\alpha$ if there is a constant $C > 0$ such that

$$\int_{\mathbb{D}} |g(z)|^p \, d\mu(z) \leqslant C \, \|g\|^p_{p,\alpha}, \qquad \forall g \in \mathscr{D}^p_\alpha. \tag{1.1}$$

In this paper we obtain the following characterization of Carleson measures for the Dirichlet spaces.

[1] Research was supported by National Science Foundation DMS 9622890.

0022-1236/99 $30.00

THEOREM 1. *Suppose $\alpha > -1$ and $p \geqslant \alpha + 1$. A nonnegative measure μ on $\mathbb{D}$ is a Carleson measure for $\mathscr{D}_\alpha^p$ if and only if there is a constant $C > 0$ such that*

(a) *for* $p > \alpha + 2$

$$\mu(\mathbb{D}) \leqslant C;$$

(b) *for* $\min(1, \alpha+1) < p \leqslant \alpha + 2$

$$\mu(T(O)) \leqslant C\, cap(O; \mathscr{B}^p_{1-(\alpha+1)/p}), \qquad \text{for any open set} \quad O \subset \partial\mathbb{D};$$

(c) *for* $\alpha + 1 \leqslant p \leqslant 1$

$$\mu(T(I)) \leqslant C\, |I|^{\alpha+2-p}, \qquad \text{for any arc} \quad I \subset \partial\mathbb{D};$$

(d) *for* $1 < p = \alpha + 1 \leqslant 2$

$$\mu(T(I)) \leqslant C\, |I|, \qquad \text{for any arc} \quad I \subset \partial\mathbb{D}.$$

It can be shown (see later) that if $p > \alpha + 2$ then $\mathscr{D}_\alpha^p \subset \mathscr{H}^\infty$, the space of bounded analytic functions on $\mathbb{D}$. Therefore the interesting ranges for (p, α) are $\alpha + 1 \leqslant p \leqslant \alpha + 2$ and $\alpha > -1$. Part (b) of Theorem 1 involves capacity for Besov space (defined later). A capacitary strong type estimate on $\partial\mathbb{D}$ (the boundary of $\mathbb{D}$) is established in our approach. We conjecture that the result in part (d) of Theorem 1 is also true for the missed case $p = \alpha + 1 > 2$. As an application, we obtain necessary and sufficient conditions, in Section 4, for multipliers of the Dirichlet spaces.

Carleson measures play an important role in harmonic analysis and operator theory in holomorphic function spaces. For example, it is well-known that if f is harmonic on $\mathbb{D}$ then the measure $|\nabla f|^2\, dA_1$ is a Carleson measure for $\mathscr{H}^2$ $(= \mathscr{D}_1^2)$ if and only if $f(e^{i\theta}) \in \mathrm{BMO}$; and the Hankel operator with analytic symbol f is bounded on $\mathscr{D}_\alpha^2$ $(0 \leqslant \alpha \leqslant 1)$ if and only if the measure $|f'|^2\, dA_\alpha$ is a Carleson measure for $\mathscr{D}_\alpha^2$ (see [RW] and [W1, W2] for $0 \leqslant \alpha < 1$). Carleson measures have been characterized by Carleson in [C] for the Hardy spaces; Luecking in [L] for the Bergman spaces; Stegenga in [S] for the Dirichlet space $\mathscr{D}_\alpha^2$ with $0 \leqslant \alpha < 1$ (Kerman and Sawyer in [KS] obtained also a different characterization); and Maźya and Shaposhnikova in [MS, p. 179] for $\mathscr{D}_\alpha^1$ with $-1 < \alpha < 0$.

We will see later that in general the Dirichlet space should be viewed as an analytic Besov space, though Stegenga in [S] viewed $\mathscr{D}_\alpha^2$ as an analytic Sobolev space in order to employ the capacitary strong type estimate for Sobolev space established first by Adams in [A]. It worth mentioning also that strong type estimates for more general Sobolev-type spaces have been studied by Hansson in [H], and a characterization of Carleson measures for harmonic function spaces on $\mathbb{R}^{n+1}_{+}$ of certain Sobolev-type spaces has been obtained by Nagel, Rudin and Shapiro in [NRS] (see also a different approach on $\mathbb{D}$ in [KS]). However the methods in these papers cannot be used effectively to deal with Carleson measures for Dirichlet spaces (excepts for $p=2$, and in this case Besov-type and Sobolev-type are essentially same!). Different treatments appear in our approach for different ranges of p. This reflects another different but interesting feature of Dirichlet spaces compared to Sobolev-type spaces.

Throughout this paper the letter C denotes a positive constant that may change from one step to the next. For two positive functions a and b, we say a dominates by b, denoted by $a=O(b)$, if there is a constant $C>0$ such that $a\leqslant Cb$; and we say a and b are equivalent, denoted by $a\asymp b$, if both $a=O(b)$ and $b=O(a)$ hold.

2. BESOV SPACES AND A CAPACITARY STRONG TYPE ESTIMATE

For $p>0$ and $0<\sigma<1$, the Besov space $\mathscr{B}^p_\sigma$ consists of all real functions f in $\mathscr{L}^p(\partial\mathbb{D})$ such that

$$\|f\|^p_{\mathscr{B}^p_\sigma}=\|f\|^p_{\mathscr{L}^p(\partial\mathbb{D})}+\int_0^{2\pi}\int_0^{2\pi}\frac{|f(e^{i(\theta+t)})-f(e^{i\theta})|^p}{|1-e^{it}|^{1+p\sigma}}\,\frac{d\theta\,dt}{4\pi^2}<\infty. \qquad (2.1)$$

Let φ be a nondecreasing function in $C_0^\infty(\mathbb{R})$ which satisfies

$$\varphi(t)=\begin{cases}0, & \text{if}\quad t\leqslant 1/2;\\ 1, & \text{if}\quad t\geqslant 1.\end{cases}$$

Consider the smooth truncation $\{F_j\}^{\infty}_{-\infty}$:

$$F_j(f)=2^j\varphi\left(\frac{|f|}{2^j}\right),\qquad j=0,\ \pm 1,\ \pm 2,\ \ldots .$$

D. R. Adams first used this smooth truncation in [A] to prove a strong type estimate for Sobolev space on $\mathbb{R}^n$ (see also [S]). The key properties of this smooth truncation are:

$$0\leqslant F_j(f)\leqslant 2^j,\qquad \{F_j(f)>0\}\subset\{|f|>2^{j-1}\}$$

and if $2^{j-1} \leqslant |f| < 2^j$, then

$$F_k(f) = \begin{cases} 2^k, & \text{for} \quad k < j; \\ 0, & \text{for} \quad k > j. \end{cases}$$

LEMMA 2.1. *For $0 < \sigma < 1$ and $p > 0$, the smooth truncation $\{F_j\}_{-\infty}^{\infty}$ maps $\mathscr{B}_\sigma^p$ to itself and there is a constant $C > 0$ such that*

$$\sum_{j=-\infty}^{\infty} \|F_j(f)\|_{\mathscr{B}_\sigma^p}^p \leqslant C \|f\|_{\mathscr{B}_\sigma^p}^p$$

holds for all $f \in \mathscr{B}_\sigma^p$.

Proof. For any set $E \subset \partial\mathbb{D}$, denote by $|E|$ the Lebesgue measure of E on $\partial\mathbb{D}$. It is easy to check that $\|F_j(f)\|_{\mathscr{L}^p(\partial\mathbb{D})}^p \leqslant 2^{jp} |\{|f| > 2^{j-1}\}|$. Therefore we have

$$\begin{aligned} \sum_{j=-\infty}^{\infty} \|F_j(f)\|_{\mathscr{L}^p(\partial\mathbb{D})}^p &\leqslant \sum_{j=-\infty}^{\infty} 2^{jp} |\{|f| > 2^{j-1}\}| \\ &\leqslant C \int_0^\infty |\{|f| > t\}| \, dt^p \\ &= C \|f\|_{\mathscr{L}^p(\partial\mathbb{D})}^p. \end{aligned}$$

By Definition (2.1), it suffices to show that for any $t, s \in \mathbb{R}$

$$\sum_{l=-\infty}^{\infty} |F_l(f(e^{it})) - F_l(f(e^{is}))|^p \leqslant C |f(e^{it}) - f(e^{is})|^p.$$

Without loss of generality, assume $j \geqslant k$,

$$2^{j-1} \leqslant |f(e^{it})| < 2^j \qquad \text{and} \qquad 2^{k-1} \leqslant |f(e^{is})| < 2^k.$$

If $j = k$, by the mean value theorem, there is a $\xi \in (0, 1)$ such that

$$\begin{aligned} \sum_{l=-\infty}^{\infty} |F_l(f(e^{it})) - F_l(f(e^{is}))|^p &= |F_j(f(e^{it})) - F_j(f(e^{is}))|^p \\ &= |\varphi'(\xi)|^p \, ||f(e^{it})| - |f(e^{is})||^p \\ &\leqslant C |f(e^{it}) - f(e^{is})|^p. \end{aligned}$$

If $j \geqslant k+1$, there are $\xi, \eta \in (0,1)$ such that

$$\begin{aligned}
\sum_{l=-\infty}^{\infty} & |F_l(f(e^{it})) - F_l(f(e^{is}))|^p \\
&= |2^k - F_k(f(e^{is}))|^p + |F_j(f(e^{it}))|^p \\
&= 2^{kp} \, |\varphi(1) - \varphi(2^{-k} \, |f(e^{is})|)|^p + 2^{jp} \, |\varphi(2^{-j} \, |f(e^{it})|) - \varphi(1/2)|^p \\
&= |\varphi'(\xi)|^p (2^k - |f(e^is)|)^p + |\varphi'(\eta)|^p \, (|f(e^{it})| - 2^{j-1})^p \\
&\leqslant C(|f(e^{it})| - |f(e^{is})|)^p \\
&\leqslant C \, |f(e^{it}) - f(e^{is})|^p.
\end{aligned}$$

The proof is complete. ∎

For any open set $O \subset \partial \mathbb{D}$, the $\mathscr{B}^p_\sigma$-capacity of O is defined by

$$\operatorname{cap}_{\mathscr{B}^p_\sigma}(O) = \operatorname{cap}(O; \mathscr{B}^p_\sigma) = \inf\{ \|f\|^p_{\mathscr{B}^p_\sigma} : f \geqslant 1 \text{ on } O\}.$$

THEOREM 2.2. *For $0 < \sigma < 1$ and $p > 0$ there is a constant $C > 0$ such that the following strong type estimate*

$$\int_0^\infty cap_{\mathscr{B}^p_\sigma}(\{|f| > t\}) \, dt^p \leqslant C \, \|f\|^p_{\mathscr{B}^p_\sigma}$$

holds for all $f \in \mathscr{B}^p_\sigma$.

Proof. It is standard that

$$\int_0^\infty \operatorname{cap}_{\mathscr{B}^p_\sigma}(\{|f| > t\}) \, dt^p \asymp \sum_{k=-\infty}^{\infty} 2^{kp} \operatorname{cap}_{\mathscr{B}^p_\sigma}(\{|f| > 2^k\}).$$

Since $2^{-k} F_k(f) \geqslant 1$ on the set $\{|f| > 2^k\}$, and hence

$$\operatorname{cap}_{\mathscr{B}^p_\sigma}(\{|f| > 2^k\}) \leqslant 2^{-kp} \, \|F_k(f)\|^p_{\mathscr{B}^p_\sigma},$$

we have therefore

$$\sum_{k=-\infty}^{\infty} 2^{kp} \operatorname{cap}_{\mathscr{B}^p_\sigma}(\{|f| > 2^k\}) \leqslant \sum_{k=-\infty}^{\infty} \|F_k(f)\|^p_{\mathscr{B}^p_\sigma} \leqslant C \, \|f\|^p_{\mathscr{B}^p_\sigma}.$$

This last inequality follows from Lemma 2.1. ∎

Assume $p \geqslant 1$. Given $f \in \mathscr{L}^p(\partial\mathbb{D})$, the harmonic extension of f onto $\mathbb{D}$, denoted also by f, is

$$f(z) = \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{1-|z|^2}{|e^{it}-z|^2} f(e^{it})\, dt, \qquad \forall z \in \mathbb{D}.$$

It is standard that

$$\lim_{r \to 1_-} f(re^{i\theta}) = f(e^{i\theta}), \qquad \text{a.e. on} \quad \partial\mathbb{D}.$$

Without causing any confusion, we identify a function in $\mathscr{L}^p(\partial\mathbb{D})$ with its harmonic extension on $\mathbb{D}$.

For $p \geqslant 1$, we view $\mathscr{B}^p_\sigma$ as a space of harmonic functions f on $\mathbb{D}$ with boundary function $f(e^{i\theta})$ satisfying (2.1). The following standard relation (see, for example, [St. p. 151]) is a "harmonic" way to understand the norm of $\mathscr{B}^p_\sigma$.

$$\|f\|^p_{\mathscr{B}^p_\sigma} \asymp \|f\|^p_{\mathscr{L}^p(\partial\mathbb{D})} + \int_{\mathbb{D}} |\nabla f(z)|^p\, (1-|z|^2)^{p-p\sigma-1}\, dA(z). \tag{2.3}$$

Here $\nabla = (\partial/\partial x, \partial/\partial y)$ is the gradiant operator.

Let $\Gamma(e^{i\theta})$ be the convex hull of the disk $\{|z| < 1/2\}$ and the point $e^{i\theta}$. Given a function f defined in $\mathbb{D}$, the nontangential maximal function of f is the function on $\partial\mathbb{D}$ defined by

$$N(f)(e^{i\theta}) = \sup_{z \in \Gamma(e^{i\theta})} |f(z)|.$$

Lemma 2.3. *For $0 < \sigma < 1$ and $p > 1$, the nontangential maximal operator N is bounded on $\mathscr{B}^p_\sigma$. More precisely, there is a constant $C > 0$ such that*

$$\|N(f)\|_{\mathscr{B}^p_\sigma} \leqslant C\, \|f\|_{\mathscr{B}^p_\sigma}$$

holds for all $f \in \mathscr{B}^p_\sigma$.

Proof. It is well-known that $\|N(f)\|_{\mathscr{L}^q(\partial\mathbb{D})} \leqslant C\, \|f\|_{\mathscr{L}^q(\partial\mathbb{D})}$ if $q > 1$ and f is a harmonic function on $\mathbb{D}$. Let R_t be the operator of rotation by e^{it}, i.e. $R_t f(z) = f(e^{it}z)$. It is easy to check that R_t commutes with the nontangential maximal operator. Therefore we have $R_t N(f) = N(R_t f)$. It is easy to verify that for any f, g defined on $\mathbb{D}$ we have $|N(f) - N(g)| \leqslant N(|f-g|)$, hence

$$|R_t N(f) - N(f)| = |N(R_t f) - N(f)| \leqslant |N(|R_t f - f|)|.$$

Note that for a harmonic f on $\mathbb{D}$ with boundary function in $\mathscr{L}^q(\partial\mathbb{D})(q \geqslant 1)$, the boundary function of $R_t f$ is

$$\lim_{r \to 1_-} R_t f(re^{i\theta}) = f(e^{i(\theta+t)}) = R_t f(e^{i\theta}).$$

Therefore $\|R_t N(f) - N(f)\|_{\mathscr{L}^q(\partial\mathbb{D})} \leqslant C\, \|R_t f - f\|_{\mathscr{L}^q(\partial\mathbb{D})}$, if $q > 1$.

Finally for $f \in \mathscr{B}^p_\sigma$, we have

$$\begin{aligned}\|N(f)\|^p_{\mathscr{B}^p_\sigma} &= \|N(f)\|^p_{\mathscr{L}^p(\partial\mathbb{D})} + \int_0^{2\pi} \frac{\|R_t N(f) - N(f)\|^p_{\mathscr{L}^p(\partial\mathbb{D})}}{|1-e^{it}|^{1+p\sigma}} \frac{dt}{2\pi} \\ &\leqslant C\, \|f\|^p_{\mathscr{L}^p(\partial\mathbb{D})} + C \int_0^{2\pi} \frac{\|R_t f - f\|^p_{\mathscr{L}^p(\partial\mathbb{D})}}{|1-e^{it}|^{1+p\sigma}} \frac{dt}{2\pi} \\ &\leqslant C\, \|f\|^p_{\mathscr{B}^p_\sigma}.\end{aligned}$$

The proof is complete. ∎

LEMMA 2.4. *Suppose $p \geqslant 1$ and $0 < \sigma < 1$. There is a constant $C > 0$ such that for any harmonic f on $\mathbb{D}$ the following estimate holds*:

$$\|f - f(0)\|^p_{\mathscr{L}^p(\partial\mathbb{D})} \leqslant C \int_{\mathbb{D}} |\nabla f(z)|^p\, (1-|z|^2)^{p-p\sigma-1}\, dA(z).$$

Proof. Without loss of generality we assume $f(e^{i\theta})$ exists (or consider instead $f(re^{i\theta})$ and then let $r \to 1_-$). We start with the trivial estimates

$$|f(e^{i\theta}) - f(0)| = \left| \int_0^1 \frac{\partial f(r^2 e^{i\theta})}{\partial r}\, dr \right| \leqslant 2 \int_0^1 |\nabla f(r^2 e^{i\theta})|\, r\, dr. \tag{2.4}$$

If $p = 1$, then $1 \leqslant (1-|z|^2)^{p-p\sigma-1}$ and hence

$$\begin{aligned}\|f - f(0)\|_{\mathscr{L}^1(\partial\mathbb{D})} &\leqslant 2 \int_{\mathbb{D}} |\nabla f(|z|\, z)|\, (1-|z|^2)^{p-p\sigma-1}\, dA(z) \\ &\leqslant 2 \int_{\mathbb{D}} |\nabla f(z)|\, (1-|z|^2)^{p-p\sigma-1}\, dA(z).\end{aligned}$$

This last inequality holds because $|\nabla f|$ is subharmonic on $\mathbb{D}$, and for any subharmonic function F on $\mathbb{D}$ and $0 \leqslant r_1 < r_2 < 1$ the following estimate holds:

$$\int_0^{2\pi} F(r_1 e^{i\theta})\, d\theta \leqslant \int_0^{2\pi} F(r_2 e^{i\theta})\, d\theta.$$

If $p>1$, applying Hölder's inequality to the last integral in estimate (2.4), we have

$$|f(e^{i\theta})-f(0)|^p \leqslant 2^p \int_0^1 |\nabla f(r^2 e^{i\theta})|^p\,(1-r^2)^{p-p\sigma_1}\,r\,dr$$
$$\times\left(\int_0^1 (1-r^2)^{p\sigma/(p-1)-1}\,r\,dr\right)^{p-1}$$
$$\leqslant C\int_0^1 |\nabla f(r^2 e^{i\theta})|^p\,(1-r^2)^{p-p\sigma-1}\,r\,dr.$$

Since $|\nabla f|^p$ is subharmonic on $\mathbb{D}$, the above estimate yields the desired result. ▮

As a consequence of Lemma 2.4, we have for $p\geqslant 1$

$$\|f\|^p_{\mathscr{B}^p_\sigma} \asymp |f(0)|^p + \int_{\mathbb{D}} |\nabla f(z)|^p\,(1-|z|^2)^{p-p\sigma-1}\,dA(z). \tag{2.5}$$

We can view the Besov space $\mathscr{B}^p_\sigma$ (for $p\geqslant 1$) as the real version of the Dirichlet space.

3. CARLESON MEASURES

Theorem 1(a) is a consequence of the fact that $\mathscr{D}^p_\alpha \subset \mathscr{H}^\infty$ if $p>\alpha+2$, which is proved in the proof of Theorem 4.2. We prove the other parts of Theorem 1 separately.

Assume $\alpha>-1$. For any $a\in\mathbb{D}$ the following estimate is standard:

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{|1-\bar{a}z|^\lambda}\,dA(z) \asymp \begin{cases} (1-|a|^2)^{\alpha+2-\lambda}, & \text{if } \lambda>\alpha+2;\\ \log\dfrac{2}{1-|a|^2}, & \text{if } \lambda=\alpha+2;\\ 1, & \text{if } \lambda<\alpha+2.\end{cases} \tag{3.1}$$

LEMMA 3.1. *Suppose $\alpha>-1$ and $p>0$. Let I be an arc on $\partial\mathbb{D}$ and $0<|I|<1$. If the nonnegative measure μ is a Carleson measure for $\mathscr{D}^p_\alpha$ then*

$$\mu(T(I)) = O\begin{cases} 1, & \textit{if } p>\alpha+2;\\ \left(\log\dfrac{2}{|I|}\right)^{1-p}, & \textit{if } p=\alpha+2;\\ |I|^{\alpha+2-p}, & \textit{if } p<\alpha+2.\end{cases}$$

Proof. The first estimate is trivial, because $g \equiv 1$ is a function in $\mathscr{D}^p_\alpha$.

Let $e^{i\theta}$ be the center point of the arc I, $a = (1 - |I|)^{1/2} e^{i\theta}$ and $\lambda = 2 - 2(\alpha + 2)/p$. Consider the test function

$$g_I(z) = \begin{cases} \log \dfrac{2}{1 - \bar{a}z}, & \text{if } \lambda = 0; \\ (1 - \bar{a}z)^\lambda, & \text{if } \lambda < 0. \end{cases}$$

By estimate (3.1), we have

$$\| g_I \|^p_{p,\alpha} \asymp \begin{cases} \log \dfrac{2}{|I|}, & \text{if } \lambda = 0; \\ |I|^{\lambda p/2}, & \text{if } \lambda < 0. \end{cases}$$

It is easy to verify that

$$\inf_{z \in T(I)} |g_I(z)| \asymp |g_I(a)| = \begin{cases} \log \dfrac{2}{|I|}, & \text{if } \lambda = 0; \\ |I|^\lambda, & \text{if } \lambda < 0. \end{cases}$$

We have therefore

$$\mu(T(I)) \leqslant C\, |g_I(a)|^{-p} \int_{\mathbb{D}} |g_a(z)|^p \, d\mu(z) \leqslant C\, |g_I(a)|^{-p} \, \| g_a \|^p_{p,\alpha}.$$

This is enough. ∎

Denote the tent of an open set $O \subset \partial\mathbb{D}$ in $\mathbb{D}$ by

$$T(O) = \{ z \in \mathbb{D} : \{ e^{i\theta} : |e^{i\theta} - z/|z|| < 1 - |z| \} \subset O \}.$$

Lemma 3.2. *Let μ be a nonnegative measure on $\mathbb{D}$. Then for any μ measurable function f on $\mathbb{D}$ and any $t > 0$*

$$\mu(\{ z \in \mathbb{D} : |f(z)| > t \}) \leqslant \mu(T(\{ e^{i\theta} \in \partial\mathbb{D} : N(f)(e^{i\theta}) > t \})).$$

Lemma 3.2 was proved implicitly in [S] (see also [W3]).

Proof of Theorem 1(b). The approach in the following, which fails to cover parts (c) and (d) of Theorem 1, is similar to the approach in [S] (there the case $p = 2$ is proved).

Assume $\sigma = 1 - (\alpha+1)/p$. Clearly $0 < \sigma < 1$ is equivalent to $p > \alpha + 1$. For any harmonic function f on $\mathbb{D}$ it is standard that there is a conjugate harmonic function $\tilde{f}$ on $\mathbb{D}$ such that $g = f + i\tilde{f}$ is analytic on $\mathbb{D}$. By Cauchy–Riemann equations, we have

$$|\nabla f(z)| = |\nabla \tilde{f}(z)| = |g'(z)|.$$

These, together with (2.5), imply that for $p \geqslant 1$

$$\mathscr{B}^p_\sigma = \{\operatorname{Re}(g)\colon g \in \mathscr{D}^p_\alpha\}, \qquad \text{and} \qquad \|\operatorname{Re}(g)\|_{\mathscr{B}^p_\sigma} \asymp \|g\|_{p,\alpha}.$$

Therefore estimate (1.1) is equivalent to

$$\int_{\mathbb{D}} |f(z)|^p \, d\mu(z) \leqslant C \, \|f\|^p_{\mathscr{B}^p_\sigma}, \qquad \forall f \in \mathscr{B}^p_\sigma. \tag{3.2}$$

We prove the "only if" part first. By definition, there is a function $f \in \mathscr{B}^p_\sigma$ such that $f \geqslant 1$ on the set O and $\|f\|^p_{\mathscr{B}^p_\sigma} \leqslant 2\operatorname{cap}(O; \mathscr{B}^p_\sigma)$. We can assume further that $f \geqslant 0$ on $\partial\mathbb{D}$, because it is easy to show (similar to the proof of Lemma 2.3) that $\||f|\|_{\mathscr{B}^p_\sigma} \leqslant \|f\|_{\mathscr{B}^p_\sigma}$. Let $O = \bigcup_j O_j$, where $\{O_j\}$ are disjoint arcs on $\partial\mathbb{D}$. For any $z \in T(O_j)$, we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{1-|z|^2}{|e^{it}-z|^2} f(e^{it}) \, dt \\ &\geqslant \frac{1}{2\pi} \int_{O_j} \frac{1-|z|^2}{|e^{it}-z|^2} f(e^{it}) \, dt \\ &\geqslant \frac{1}{2\pi} \int_{|e^{it}-1| < 1-|z|} \frac{1-|z|^2}{|e^{it}-|z||^2} \, dt \\ &\geqslant \frac{1}{4\pi}. \end{aligned}$$

Therefore by the fact that $T(O) = \bigcup_j T(O_j)$, we have

$$\mu(T(O)) \leqslant (4\pi)^p \int_{\mathbb{D}} |f(z)|^p \, d\mu(z) \leqslant C \, \|f\|^p_{\mathscr{B}^p_\sigma} \leqslant C \operatorname{cap}(O; \mathscr{B}^p_\sigma).$$

To prove the "iff" part, we note that by Lemma 3.2 and the assumption we have

$$\mu(\{z \in \mathbb{D}\colon |f(z)| > t\}) \leqslant \mu(T(\{N(f) > t\})) \leqslant C \operatorname{cap}_{\mathscr{B}^p_\sigma}(\{N(f) > t\}).$$

Hence the desired result follows from the estimate

$$\int_{\mathbb{D}} |f(z)|^p \, d\mu(z) = \int_0^\infty \mu(\{z \in \mathbb{D} : |f(z)| > t\}) \, dt^p$$
$$\leqslant C \int_0^\infty \operatorname{cap}_{\mathscr{B}^p_\sigma}(\{N(f) > t\}) \, dt^p,$$

and Theorem 2.2 and Lemma 2.3. ∎

Let $\beta > -1$ and $q > 0$. One can show that there is a constant $C > 0$ such that

$$|h(z)| \leqslant C(1 - |z|^2)^{-(\beta+2)/q} \, \|h\|_{\mathscr{A}^q_\beta}, \qquad \forall z \in \mathbb{D}, \tag{3.3}$$

for all analytic functions h on $\mathbb{D}$. In fact, let $D(z)$ be the disk centered at z with radius $(1-|z|^2)/4$. Then the area of $D(z)$ is $(\pi/16)(1-|z|^2)^2$ and $(1-|w|^2) \asymp (1-|z|^2)$ for $w \in D(z)$. Applying the mean value inequality to the subharmonic function $|h|^p$ on the disk $D(z)$, we have

$$|h(z)|^q \leqslant \frac{16}{\pi(1-|z|^2)^2} \int_{D(z)} |h(w)|^q \, dA(w)$$
$$\leqslant C(1-|z|^2)^{-2-\beta} \int_{D(z)} |h(w)|^q \, dA_\beta(w)$$
$$\leqslant C(1-|z|^2)^{-2-\beta} \, \|h\|^q_{\mathscr{A}^q_\beta}.$$

As a consequence of estimate (3.3), we have

$$\int_{\mathbb{D}} |h(z)|^2 \, (1-|z|^2)^{2(\beta+2)/q} \, dA(z)$$
$$\leqslant C \left(\int_{\mathbb{D}} |h(z)|^q \, (1-|z|^2)^\beta \, dA(z) \right)^{2/q}. \tag{3.4}$$

Given $\tau > 0$, the following reproducing formula for $\mathscr{A}^2_\tau$ is standard:

$$h(z) = \int_{\mathbb{D}} \frac{h(w)}{(1-\bar{w}z)^{\tau+2}} \, dA_\tau(w), \qquad \forall z \in \mathbb{D}, \quad \forall h \in \mathscr{A}^2_\tau. \tag{3.5}$$

THEOREM 3.3. *Suppose* $\alpha > -1$ *and* $\alpha + 1 \leqslant p \leqslant 1$. *A nonnegative measure* μ *on* $\mathbb{D}$ *is a Carleson measure for* $\mathscr{D}^p_\alpha$ *if and only if*

$$\int_{\mathbb{D}} \frac{d\mu(w)}{|1-\bar{w}z|^2} = O((1-|z|^2)^{\alpha-p}). \tag{3.6}$$

Remark. Theorem 1(c) is a consequence of Theorem 3.3, because condition (3.6) is equivalent to the estimate

$$\mu(T(I)) \leqslant C\,|I|^{\alpha+2-p}, \qquad \text{for any arc} \quad I \subset \partial\mathbb{D}.$$

In fact this result is standard for $\alpha+2-p=1$ (for a proof see, for example, [G, p. 239]). For $\alpha+2-p<1$ the proof is similar.

Proof of Theorem 3.3. For $a \in \mathbb{D}$, consider the test function

$$g_a(z) = \frac{1}{(1-\bar{a}z)^{2/p}}.$$

It is easy to check by estimate (3.1) that $\|g_a\|_{p,\alpha}^p \asymp (1-|a|^2)^{\alpha-p}$. Hence the "only if" part follows.

Let $g \in \mathscr{D}_\alpha^p$ and $\tau = 2(\alpha+2)/p$. By (3.4) we have $g' \in \mathscr{A}_\alpha^p \subset \mathscr{A}_\tau^2$. Hence by (3.5)

$$g'(z) = \int_{\mathbb{D}} \frac{g'(w)}{(1-\bar{w}z)^{\tau+2}}\, dA_\tau(w), \qquad \forall z \in \mathbb{D}.$$

Therefore $g(z)$ can be reproduced from g' by

$$g(z) = g(0) + \int_{\mathbb{D}} \frac{(1-|w|^2)^\tau}{\bar{w}(1-\bar{w}z)^{\tau+1}}\, g'(w)\, dA(w), \qquad \forall z \in \mathbb{D}. \tag{3.7}$$

Let $\beta = 2\alpha + 2p + 2$. For fixed $z \in \mathbb{D}$ the function $g'(w)/(1-\bar{z}w)^{\tau+1}$ is clearly in $\mathscr{A}_\alpha^p \subset \mathscr{A}_\beta^p$. Using estimate (3.3), we have

$$\left|\frac{g'(w)}{(1-\bar{z}w)^{\tau+1}}\right| \leqslant C(1-|w|^2)^{-(\beta+2)/p} \times \left(\int_{\mathbb{D}} \left|\frac{g'(\zeta)}{(1-\bar{z}\zeta)^{\tau+1}}\right|^p (1-|\zeta|^2)^\beta\, dA(\zeta)\right)^{1/p}.$$

Since $p \leqslant 1$, we write $|g'(w)/(1-\bar{z}w)^{\tau+1}| = |g'(w)/(1-\bar{z}w)^{\tau+1}|^p = |g'(w)/(1-\bar{z}w)^{\tau+1}|^{1-p}$. Using the above estimate for the second factor, with the fact that $\tau - (\beta+2)(1-p)/p = \beta$, we have

$$\begin{aligned}\left|\frac{g'(w)}{(1-\bar{z}w)^{\tau+1}}\right| (1-|w|^2)^\tau \\ &\leqslant C\left|\frac{g'(w)}{(1-\bar{z}w)^{\tau+1}}\right|^p (1-|w|^2)^\beta \\ &\quad \times \left(\int_{\mathbb{D}} \left|\frac{g'(\zeta)}{(1-\bar{z}\zeta)^{\tau+1}}\right|^p (1-|\zeta|^2)^\beta\, dA(\zeta)\right)^{(1-p)/p}.\end{aligned}$$

Hence by formula (3.7) and the trivial estimate $(1-|w|^2)/|1-\bar{w}z| \leqslant C$, we obtain

$$|g(z)|^p \leqslant |g(0)|^p + C\int_{\mathbb{D}} \left|\frac{g'(w)}{(1-\bar{w}z)^{\tau+1}}\right|^p (1-|w|^2)^{\beta}\, dA(w)$$

$$\leqslant |g(0)|^p + C\int_{\mathbb{D}} \frac{|g'(w)|^p}{|1-\bar{w}z|^2} (1-|w|^2)^p\, dA(w).$$

Finally by Fubini's theorem and the assumption we have

$$\int_{\mathbb{D}} |g(z)|^p\, d\mu(z)$$

$$\leqslant |g(0)|^p\, \mu(\mathbb{D}) + C\int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{(1-|w|^2)^p}{|1-\bar{w}z|^2}\, d\mu(z)\right) |g'(w)|^p\, dA(w)$$

$$\leqslant |g(0)|^p\, \mu(\mathbb{D}) + C\int_{\mathbb{D}} |g'(w)|^p (1-|w|^2)^{\alpha}\, dA(w)$$

$$\leqslant C\, \|g\|^p_{p,\alpha}.$$

This proof is complete. ∎

Proof of Theorem 1(d). The "only if" part is a consequence of Lemma 3.1.

Denote by $\mathscr{H}^p$ (Hardy space) the space of all analytic functions f on $\mathbb{D}$ with $f(e^{i\theta}) \in \mathscr{L}^p(\partial\mathbb{D})$. To prove the "iff" part, we first note that, as in the proof of Lemma 2.4 for $p=1$, one can show that $\mathscr{D}^1_0 \subset \mathscr{H}^1$. It is known that $\mathscr{D}^2_1 = \mathscr{H}^2$. Therefore by interpolation theory, we have for $1 \leqslant p \leqslant 2$

$$\mathscr{D}^p_{p-1} \subset \mathscr{H}^p \qquad \text{and} \qquad \|g\|_{\mathscr{H}^p} \leqslant C\, \|g\|_{p,\,p-1}. \tag{3.8}$$

We recall that the condition $\mu(T(I)) \leqslant C\,|I|$ for any are $I \subset \partial\mathbb{D}$ is equivalent to the condition that the measure μ is a Carleson measure for $\mathscr{H}^p$ with $p \geqslant 1$. Therefore for $1 < p \leqslant 2$ and $g \in \mathscr{D}^2_{p-1}$, we have by (3.8)

$$\int_{\mathbb{D}} |g(z)|^p\, d\mu(z) \leqslant C\, \|g\|^p_{\mathscr{H}^p} \leqslant C\, \|g\|^p_{p,\,p-1}.$$

The proof is complete. ∎

4. MULTIPLIERS

An analytic function f defined on $\mathbb{D}$ is a multiplier for $\mathcal{D}^p_\alpha$ if $f\mathcal{D}^p_\alpha \subset \mathcal{D}^p_\alpha$, i.e. $fg \in \mathcal{D}^p_\alpha$, for all $g \in \mathcal{D}^p_\alpha$. It is standard that, by the closed-graph theorem, f is a multiplier for $\mathcal{D}^p_\alpha$ if and only if there is a constant $C > 0$ such that

$$\|fg\|_{p,\alpha} \leqslant C \|g\|_{p,\alpha}, \qquad \forall g \in \mathcal{D}^p_\alpha.$$

Multipliers for Bergman spaces or Hardy spaces are just functions in $\mathcal{H}^\infty$. However, the story of multipliers for Dirichlet spaces is different. Multipliers for $\mathcal{D}^2_\alpha$ $(0 \leqslant \alpha)$ and $\mathcal{D}^1_\alpha$ $(-1 < \alpha < 0)$ have been characterized in [S] and [MS], respectively. There are other, incomplete, characterizations for multipliers of the Dirichlet spaces. An interesting one is due to Verbitskii in [V] (see also [MS, p. 182]), who found a necessary and sufficient condition for an inner function to be a multiplier for $\mathcal{D}^p_\alpha$ with $\alpha > -1$ and $\min(1, \alpha+1) < p < \alpha+2$. Conditions on multipliers from one Dirichlet space to another are studied in [WY].

In this section we show that $f\mathcal{D}^p_\alpha \subset \mathcal{D}^p_\alpha$ is equivalent to $f \in \mathcal{H}^\infty$ and $f'\mathcal{A}^p_\alpha \subset \mathcal{D}^p_\alpha$. This later condition is equivalent to the condition that the measure $|f'|^p\, dA_\alpha$ is a Carleson measure for $\mathcal{D}^p_\alpha$. Therefore we can apply Theorem 1.

Lemma 4.1. *Suppose $\alpha > -1$, $p > 0$ and f is analytic on $\mathbb{D}$. Then $f\mathcal{D}^p_\alpha \subset \mathcal{D}^p_\alpha$ if and only if $f \in \mathcal{H}^\infty$ and $f'\mathcal{A}^p_\alpha \subset \mathcal{D}^p_\alpha$.*

Proof. The "if" part is trivial, because for any $g \in \mathcal{D}^p_\alpha$, $fg \in \mathcal{D}^p_\alpha$ is equivalent to $fg' + f'g \in \mathcal{A}^p_\alpha$, but by assumption $\|fg'\|_{\mathcal{A}^p_\alpha} \leqslant \|f\|_{\mathcal{H}^\infty} \|g'\|_{\mathcal{A}^p_\alpha}$. To prove the "only if" part, it suffices to show $f \in \mathcal{H}^\infty$.

For fixed $a \in \mathbb{D}$ and $m > -\alpha + (\alpha+2)/p$, consider the test function

$$g_a(z) = (1-|a|^2)^{m+\alpha-(\alpha+2)/p} \frac{z-a}{(1-\bar{a}z)^{m+\alpha}}.$$

It is easy to verify (by (3.3)) that $g_a \in \mathcal{D}^p_\alpha$ and there is a constant $C > 0$ such that

$$\|g_a\|_{p,\alpha} \leqslant C, \qquad \text{for any} \quad a \in \mathbb{D}.$$

Since f is a multiplier, we have $(fg_a)'(z) \in \mathcal{A}^p_\alpha$. Clearly

$$(fg_a)'(a) = f'(a)\, g_a(a) + f(a)\, g_a'(a) = (1-|a|^2)^{-(\alpha+2)/p}\, f(a).$$

Applying the mean value inequality to the subharmonic function $|(fg_a)'|^p$ on $D(a)$, the disk centered at a with radius $(1-|a|^2)/4$, we have

$$\begin{aligned}(1-|a|^2)^{-(\alpha+2)}\,|f(a)|^p &= |(fg_a)'\,(a)|^p\\ &\leqslant \frac{16}{\pi(1-|a|^2)^2}\int_{D(a)} |(fg_a)'\,(z)|^p\,dA(z)\\ &\leqslant C(1-|a|^2)^{-\alpha-2}\int_{D(a)} |(fg_a)'\,(z)|^p\,dA_\alpha(z)\\ &\leqslant C(1-|a|^2)^{-(\alpha+2)}\,\|g_a\|_{p,\alpha}^p.\end{aligned}$$

This implies $|f(a)|\leqslant C$, for all $a\in\mathbb{D}$. Therefore $f\in\mathcal{H}^\infty$. ∎

THEOREM 4.2. *Suppose $\alpha>-1$, $p>0$ and f is analytic on $\mathbb{D}$.*

(a) *For $p>\alpha+2$, $f\mathcal{D}_\alpha^p\subset\mathcal{D}_\alpha^p$ if and only if $f\in\mathcal{D}_\alpha^p$;*

(b) *For $p\leqslant\alpha+2$, $f\mathcal{D}_\alpha^p\subset\mathcal{D}_\alpha^p$ if and only if $f\in\mathcal{H}^\infty$ and the measure $|f'|^p\,dA_\alpha$ is a Carleson measure for $\mathcal{D}_\alpha^p$.*

Proof. Part (b) is a consequence of Lemma 4.1. For part (a), the "only if" part is trivial. To prove the "if" part, we show first that $\mathcal{D}_\alpha^p\subset\mathcal{H}^\infty$. Given $g\in\mathcal{D}_\alpha^p$, we have $g'\in\mathcal{A}_\alpha^p$. Note that $p>\alpha+2>1$. The reproducing formula of $\mathcal{A}_\alpha^p$ gives

$$g'(z)=\int_{\mathbb{D}}\frac{g'(w)}{(1-\bar{w}z)^{\alpha+2}}\,dA_\alpha(w),\qquad \forall z\in\mathbb{D}.$$

Therefore $g(z)$ can be recovered from g' by

$$g(z)=g(0)+\frac{1}{\alpha+1}\int_{\mathbb{D}}\frac{g'(w)}{\bar{w}(1-\bar{w}z)^{\alpha+1}}\,dA_\alpha(w),\qquad \forall z\in\mathbb{D}.$$

Applying Hölder's inequality to the above integral, we get

$$\begin{aligned}|g(z)|\leqslant |g(0)|+C\left(\int_{\mathbb{D}}\frac{|g'(w)|^p}{|w|}\,dA_\alpha(w)\right)^{1/p}&\\ \times\left(\int_{\mathbb{D}}\frac{dA_\alpha(w)}{|w|\,|1-\bar{w}z|^{p(\alpha+1)/(p-1)}}\right)^{(p-1)/p}&\leqslant C\,\|g\|_{p,\alpha}.\end{aligned}$$

This last inequality is obtained by using the last estimate of (3.3).

Now for any $g \in \mathscr{D}_\alpha^p$, we have

$$\begin{aligned}\|fg\|_{p,\alpha} &\leqslant |f(0)\, g(0)| + \|f'g + fg'\|_{\mathscr{A}_\alpha^p} \\ &\leqslant |f(0)\, g(0)| + \|g\|_{\mathscr{H}^\infty} \|f'\|_{\mathscr{A}_\alpha^p} + \|f\|_{\mathscr{H}^\infty} \|g'\|_{\mathscr{A}_\alpha^p} \\ &\leqslant C\, \|f\|_{p,\alpha}\, \|g\|_{p,\alpha}.\end{aligned}$$

The proof is complete. ∎

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