Global well-posedness of the Cauchy problem of the fifth-order shallow water equation

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Abstract

By using the \( I \)-method, we prove that the Cauchy problem of the fifth-order shallow water equation

\[
\partial_t u - \partial_x^2 \partial_t u + \partial_x^3 u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u \partial_x^3 u - \partial_x^5 u = 0
\]

is globally well-posed in the Sobolev space \( H^s(\mathbb{R}) \) provided \( s > (5\sqrt{7} - 10)/4 \).

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1. Introduction

This paper is concerned with the Cauchy problem of the following fifth-order equation:

\[
\partial_t u - \partial_x^2 \partial_t u + \partial_x^3 u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u \partial_x^3 u - \partial_x^5 u = 0. \tag{1.1}
\]

This equation is a higher-order modification of the following shallow water equation or Camassa–Holm (CH) equation:

\[
\partial_t u - \partial_x^2 \partial_t u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u \partial_x^3 u = 0. \tag{1.2}
\]

The modification is for the purpose to gain global well-posedness in suitable Sobolev spaces which Eq. (1.2) does not possess (see the succeeding section). A more immediate way to mo-
tivate (1.1) is that it connects (1.2) with the KdV equation in a natural way. Indeed, by acting (1.2) with the Bessel potential \( (1 - \partial_x^2)^{-1} \) we obtain the following form of it which is commonly adopted by authors on this equation:

\[
\partial_t u + \frac{1}{2} \partial_x (u^2) + (1 - \partial_x^2)^{-1} \partial_x \left[ u^2 + \frac{1}{2} (\partial_x u)^2 \right] = 0. \tag{1.3}
\]

Using the same technique to treat (1.1), we get the following equation:

\[
\partial_t u + \partial_x^3 u + \frac{1}{2} \partial_x (u^2) + (1 - \partial_x^2)^{-1} \partial_x \left[ u^2 + \frac{1}{2} (\partial_x u)^2 \right] = 0. \tag{1.4}
\]

Clearly, omitting the fourth term on the left-hand side of this equation yields the KdV equation, while omitting the second term gives us the CH equation.

Equation (1.2) has been intensively studied by many authors, cf., for instance, [3,4,9–14, 17–20,26–29]. It has been proved that the Cauchy problem of (1.2) is locally well-posed in \( H^s(\mathbb{R}) \) (see [24,27]) and \( H^s(T) \) (see [20,26]) for \( s > \frac{1}{2} \). Global existence of solutions of (1.2) in \( \mathbb{R} \times T \) and \( \mathbb{R} \times \mathbb{R} \) was studied by Constantin and Escher [10–12], Constantin and Molinet [13], and Xin and Zhang [28,29]. We note that these global existence results do not imply global well-posedness; the latter does not hold in either \( H^s(\mathbb{R}) \) or \( H^s(T) \) for any \( s \in \mathbb{R} \), see [9,24,27]. As we have mentioned before, this lack of global well-posedness for (1.2) is one major motivation of using (1.1) to modify (1.2), see [17–19].

Well-posedness of the Cauchy problem of (1.1) in Sobolev spaces has been investigated by a few authors, see, for instance, [2,17–19,25]. In [17,18] Himonas and Misiolek considered the periodic initial value problem, and in [19] they investigated the nonperiodic case. In particular, they proved in [19] that the Cauchy problem of (1.1) is locally well-posed in \( H^s(\mathbb{R}) \) for \( s > \frac{1}{2} \). This local result combined with energy conservation law naturally yields that (1.1) is globally well-posed in \( H^1(\mathbb{R}) \). Recently, using the bilinear estimate method initiated by Bourgain [1] and developed by Kenig, Ponce and Vega [21,22], Byers [2] and Liu and Jin [25] lowered down the minimal value of the index \( s \) and proved local well-posedness in \( H^s(\mathbb{R}) \) for \( s > \frac{1}{4} \). However, this improvement on local well-posedness does not lead to progress on global well-posedness by merely using the standard conservation law argument.

In this paper we want to apply the \( I \)-method introduced by Colliander, Kell, Staffilani, Takaoka and Tao [5–8] to improve the above-mentioned global well-posedness result for Eq. (1.1). To precisely state our main result, we first introduce some notation.

We use the notation \( a^+ \) and \( a^- \) to denote respectively expressions of the forms \( a + \varepsilon \) and \( a - \varepsilon \), where \( 0 < \varepsilon \ll 1 \). We denote by \( D^s_x \) the Riesz potential of order \( -s \), or the Fourier multiplier with symbol \( |\xi|^s \) \((s > 0)\). Recall that the Sobolev space \( H^s(\mathbb{R}) \) is defined by

\[
f \in H^s(\mathbb{R}) \iff \| f \|_{H^s(\mathbb{R})} := \| \langle \xi \rangle^s \check{f}(\xi) \|_{L^2(\mathbb{R})} < \infty,
\]

where \( \langle \xi \rangle^s := (1 + |\xi|^2)^{s/2} \), and \( \check{f} \) represents the Fourier transformation in one variable. We define the space \( X_{s,\alpha}(\mathbb{R}^2) \) (as in [1,21,22]) by

\[
u \in X_{s,\alpha}(\mathbb{R}^2) \iff \| \nu \|_{X_{s,\alpha}(\mathbb{R}^2)} := \| \langle \xi \rangle^s (\tau - |\xi|^3)\alpha \check{u}(\xi, \tau) \|_{L^2_x L^2_\xi} < \infty,
\]
where \( \sim \) represents the Fourier transformation in two variables. For any given interval \( L \), we define the space \( X_{s,\alpha}(L \times \mathbb{R}) \) to be the restriction of \( X_{s,\alpha}(\mathbb{R}^2) \) on \( L \times \mathbb{R} \), with norm

\[
\|u\|_{X_{s,\alpha}(L \times \mathbb{R})} = \inf \{ \|U\|_{X_{s,\alpha}(\mathbb{R}^2)} : U|_{L \times \mathbb{R}} = u \}.
\]

If \( L = [0, T] \) (respectively \([0, \delta] \)), we use \( X_{T,s,\alpha} \) (respectively \( X_{\delta,s,\alpha} \)) to abbreviate \( X_{s,\alpha}(L \times \mathbb{R}) \).

The main result of this paper is as follows.

\textbf{Theorem 1.1.} The Cauchy problem of Eq. (1.1) is globally well-posed in \( H^s(\mathbb{R}) \) for \( s > \frac{5\sqrt{7}-10}{4} \).

More precisely, let \( \varphi \in H^s(\mathbb{R}) \) with \( s > \frac{5\sqrt{7}-10}{4} \). Then for any \( T > 0 \) Eq. (1.1) has a unique solution \( u \in X_{T,s,\alpha} \) satisfying the initial condition \( u(0, \cdot) = \varphi \), and the mapping \( \varphi \rightarrow u(t, \cdot) \) belongs to \( C(H^s(\mathbb{R}), X_{T,s,\alpha}) \). We note that the above result is weaker than the corresponding result for the KdV equation, for which the best known global well-posedness result is in \( H^{s}(\mathbb{R}) \) for \( s > -\frac{3}{4} \) (see [8]). This is due to the fact that (1.1) is much more complex than the KdV equation.

Throughout this paper we assume that \( s < 1 \) and shall not repeat this assumption, since for the case \( s \geq 1 \) global well-posedness is known [2,19]. The next section is devoted to presenting a variant local well-posedness result, in which we obtain a precise lower bound for the lifespan of the solution and an upper bound of the norm of the solution in terms of the norm of the initial data. In Section 3 we show that certain modified energy related to the \( H^s(\mathbb{R}) \) norm of the solution is almost conserved, which permits us to iterate the local solution up to a sufficiently large step. In the last section we show precisely how this iteration is performed to get the global-in-time result, i.e., Theorem 1.1 above.

\section{A variant local well-posedness result}

In this section we establish a local well-posedness result. This local result is a variant of that of [2,25], with precise estimates on the lifespan and the norm of the solution which are not considered in these references. We first introduce some more notation.

For given \( N \gg 1 \) and \( s < 1 \), we define the multiplier operator \( I_N^s : H^s(\mathbb{R}) \rightarrow H^1(\mathbb{R}) \) by

\[
(I_N^s u)(\xi) := m_{s,N}(\xi) \hat{u}(\xi), \quad u \in H^s(\mathbb{R}),
\]

where \( m_{s,N}(\xi) \) is an even \( C^\infty \) function, nonincreasing in \( |\xi| \), and

\[
m_{s,N}(\xi) = \begin{cases} 
1, & |\xi| \leq N, \\
\left(\frac{|\xi|}{N}\right)^{s-1}, & |\xi| > 2N.
\end{cases}
\]

It is obvious that for some positive constant \( C \) there hold the inequalities

\[
C^{-1} \|u\|_{H^s(\mathbb{R})} \leq \|I_N^s u\|_{H^1(\mathbb{R})} \leq C N^{1-s} \|u\|_{H^s(\mathbb{R})}.
\]

We denote by \( \|\cdot\|_{X_{s,\alpha,N}(\mathbb{R}^2)} \) the equivalent norm in \( X_{s,\alpha}(\mathbb{R}^2) \) defined by

\[
\|u\|_{X_{s,\alpha,N}(\mathbb{R}^2)} := \|I_N^s u\|_{X_{1,\alpha}(\mathbb{R}^2)}.
\]
The space $X_{s,\alpha}(\mathbb{R}^2)$ endowed with this norm will be redenoted as $X_{s,\alpha,N}(\mathbb{R}^2)$. Clearly, there also hold the inequalities
\[ C^{-1} \|u\|_{X_{s,\alpha}(\mathbb{R}^2)} \leq \|I_N^s u\|_{X_{1,\alpha}(\mathbb{R}^2)} \leq C N^{1-s} \|u\|_{X_{s,\alpha}(\mathbb{R}^2)}. \]

The notation $X_{s,\alpha,N}^\delta$ denotes the restriction of $X_{s,\alpha,N}(\mathbb{R}^2)$ on $\mathbb{R} \times [0, \delta]$.

We shall need the following basic embedding result: if either $\frac{1}{2} > b' > b \geq 0$ or $0 \geq b' > b > -\frac{1}{2}$, $s \in \mathbb{R}$, then
\[ \|u\|_{X_{s,b}^\delta} \leq C \delta^{b-b'} \|u\|_{X_{s,b'}^\delta}, \quad (2.1) \]
(see, e.g., [15]).

**Theorem 2.1.** For any $\frac{1}{4} < s < 1$, the initial value problem of (1.1) is locally well-posed in $H^s(\mathbb{R})$. More precisely, for given $\varphi \in H^s(\mathbb{R})$ and $N \gg 1$, there exists a corresponding $\delta > 0$ such that (1.1) has a unique solution $u \in X_{s,1/2}^\delta, N \subseteq C([0, \delta], H^s(\mathbb{R}))$ satisfying the condition $u(0, \cdot) = \varphi$. Moreover, the lifespan satisfies the estimate
\[ \delta \sim \|I_N^s \varphi\|_{H^{1}(\mathbb{R})}^{-\frac{1}{2}}, \quad \beta = \frac{s}{9} - \frac{1}{36} \quad (2.2) \]
and the solution satisfies the estimate
\[ \|I_N^s u\|_{X_{s,1/2}^\delta} \leq C \|I_N^s \varphi\|_{H^{1}(\mathbb{R})}. \quad (2.3) \]

**Proof.** Since (1.1) can be rewritten as (1.4), the initial value problem of (1.1) can be written in the form:
\[ \begin{cases} \partial_t u + \partial_3^2 u + \frac{1}{2} \partial_x (u^2) + (1 - \partial_x^2)^{-1} \partial_x [u^2 + \frac{1}{2} (\partial_x u)^2] = 0 \quad \text{in } \mathbb{R}^2, \\
 u(x, 0) = \varphi(x) \quad \text{for } x \in \mathbb{R}. \end{cases} \quad (2.4) \]

We denote
\[ W(t)\varphi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(x\xi + t\xi^3)} \hat{\varphi}(\xi) \, d\xi, \quad \varphi \in S(\mathbb{R}), \]
and for every $\delta \in (0, 1)$ we introduce a mapping $S_\delta : S(\mathbb{R}^2) \to L^1_{\text{loc}}(\mathbb{R}^2)$ as follows:
\[ S_\delta u(x, t) = \psi(t)W(t)\varphi(x) - \psi(t) \int_0^t W(t-\tau)\psi_\delta(\tau)w(x, \tau) \, d\tau, \quad u \in S(\mathbb{R}^2), \]
where $\psi$ is a function in $C^\infty_0(\mathbb{R})$, satisfying $\psi \geq 0$, $\psi = 1$ in $[0, 1]$, and supp $\psi \subset [-1, 2]$, $\psi_\delta(t) = \psi(\frac{t}{\delta})$, and
\[ w(x, t) = w_1(u, u)(x, t) + w_2(u, u)(x, t) = \frac{1}{2} \partial_x(u^2) + (1 - \partial_x^2)^{-1} \partial_x \left[ u^2 + \frac{1}{2} (\partial_x u)^2 \right]. \]
Let \( \alpha \) be a positive number slightly larger than \( \frac{1}{2} \). We know that (cf. [21, 22])

\[
\| \psi(t) W(t) \phi(x) \|_{X_{s,\alpha,N}} \leq C \| I_N^s \phi \|_{H^1},
\]

(2.5)

\[
\left\| \psi(t) \int_0^t W(t - \tau) w(x, \tau) d\tau \right\|_{X_{s,\alpha,N}} \leq C \| w \|_{X_{s,\alpha-1,N}}.
\]

(2.6)

Thus

\[
\| S_\delta u \|_{X_{s,\alpha,N}} \leq C \| I_N^s \phi \|_{H^1} + C \| \psi \|_{X_{s,\alpha-1,N}} + C \| \psi \|_{X_{s,\alpha-1,N}}.
\]

(2.7)

Let \( \alpha' = \alpha + 2\beta \). Clearly \( \alpha < \alpha' < 1 \), so that, by (2.1) and [21, Corollary 2.7], we have

\[
\| \psi \|_{X_{s,\alpha-1,N}} \leq C \delta^{\alpha' - \alpha} \| \psi \|_{X_{s,\alpha',\alpha}} \leq C \delta^{2\beta} \| u \|_{X_{s,\alpha,N}}.
\]

(2.8)

Besides, from the proof of [2, Proposition 3.7] we see that

\[
\| \psi \|_{X_{s,\alpha-1,N}} \leq C \delta^{2\beta} \| u \|_{X_{s,\alpha,N}}.
\]

(2.9)

Substituting (2.8), (2.9) into (2.7) we get

\[
\| S_\delta u \|_{X_{s,\alpha,N}} \leq C \| I_N^s \phi \|_{H^1} + C \delta^{2\beta} \| u \|_{X_{s,\alpha,N}}.
\]

A similar argument shows that also

\[
\| S_\delta u - S_\delta v \|_{X_{s,\alpha,N}} \leq C \delta^{2\beta} \| u \|_{X_{s,\alpha,N}} + \| v \|_{X_{s,\alpha,N}} \| u - v \|_{X_{s,\alpha,N}}.
\]

Thus if we take

\[
\delta = (4Cr)^{-\frac{1}{\beta}}, \quad r = 4C \| I_N^s \phi \|_{H^1},
\]

then \( S_\delta \) can be extended into the closed ball in \( X_{s,\alpha,N} : B(0, r) = \{ u \in X_{s,\alpha,N} : \| u \|_{X_{s,\alpha,N}} \leq r \} \), such that it maps this ball into itself and, clearly, it is a contraction mapping. It follows from the Banach fixed point theorem that \( S_\delta \) has a unique fixed point in \( B(0, r) \). Since \( \psi(t) = 1 \) for \( t \in [0, 1] \), it is easy to see that the restriction of this fixed point on \( \mathbb{R} \times [0, \delta] \) is a solution of problem (2.4) in the time interval \([0, \delta]\). Relations (2.2), (2.3) are immediate consequences of the above deduction. \( \square \)

3. The almost conserved energy

It can be easily shown that for the solution \( u \) of (1.1), the energy \( \| u(t) \|_{H^1(\mathbb{R})}^2 \) is conserved (see [2, 11]). However, since we are searching solutions in \( C(\mathbb{R}, H^s(\mathbb{R})) \) with \( s < 1 \), this conserved quantity cannot be directly used to meet our requirement. As in [5–8], we shall alteratively consider the modified energy \( \| I_N^s u(t) \|_{H^1(\mathbb{R})}^2 \). In general, the modified energy \( \| I_N^s u(t) \|_{H^1(\mathbb{R})}^2 \) is not conserved, but similarly as for the KdV equation (see [8]), we can prove that it has a very slow increment in time in terms of \( N \) if it is sufficiently large. First we give the precise expression of the increment of \( \| I_N^s u(t) \|_{H^1(\mathbb{R})}^2 \) in the following lemma.
Lemma 3.1. If \( u \) is a solution of (1.1) on \([0, \delta]\) in the sense of Theorem 2.1, then we have

\[
\left\| I_N^s u(\delta) \right\|_{H^1(\mathbb{R})}^2 - \left\| I_N^s \varphi \right\|_{H^1(\mathbb{R})}^2 = \int_0^\delta \int_\mathbb{R} \partial_x (I_N^s u) \left[ I_N^s (u^2) - (I_N^s u)^2 \right] \, dx \, dt - \int_0^\delta \int_\mathbb{R} \partial_x^3 (I_N^s u) \left[ I_N^s (u^2) - (I_N^s u)^2 \right] \, dx \, dt \\
+ 2 \int_0^\delta \int_\mathbb{R} \left( 1 - \partial_x^2 \right)^{-1} \partial_x (I_N^s u) \left[ I_N^s (u^2) - (I_N^s u)^2 \right] \, dx \, dt \\
+ \int_0^\delta \int_\mathbb{R} \left( 1 - \partial_x^2 \right)^{-1} \partial_x (I_N^s u) \left[ I_N^s (\partial_x u)^2 - \left( \partial_x I_N^s u \right)^2 \right] \, dx \, dt \\
- 2 \int_0^\delta \int_\mathbb{R} \left( 1 - \partial_x^2 \right)^{-1} \partial_x^3 (I_N^s u) \left[ I_N^s (\partial_x u)^2 - \left( \partial_x I_N^s u \right)^2 \right] \, dx \, dt \\
- \int_0^\delta \int_\mathbb{R} \partial_x I_N^s u \left[ I_N^s \left( 1 - \partial_x^2 \right)^{-1} \partial_x I_N^s (\partial_x u)^2 \right] \, dx \, dt.
\]  

(3.1)

**Proof.** First applying the \( I_N^s \)-operator to the first line equation in (2.4) and then using the operator \( \partial_x \) we obtain, respectively,

\[
\partial_t I_N^s u + \partial_x^3 I_N^s u + \frac{1}{2} \partial_x I_N^s (u^2) + \left( 1 - \partial_x^2 \right)^{-1} \partial_x \left[ I_N^s (u^2) + \frac{1}{2} I_N^s (\partial_x u)^2 \right] = 0, \\
\partial_t \partial_x I_N^s u + \partial_x^4 I_N^s u + \frac{1}{2} \partial_x^2 I_N^s (u^2) + \left( 1 - \partial_x^2 \right)^{-1} \partial_x^2 \left[ I_N^s (u^2) + \frac{1}{2} I_N^s (\partial_x u)^2 \right] = 0.
\]

From these relations it follows that

\[
\frac{d}{dt} \left\| I_N^s u \right\|_{H^1(\mathbb{R})}^2 = 2 \int (I_N^s u) \partial_t (I_N^s u) + 2 \int \partial_x (I_N^s u) \partial_t \partial_x (I_N^s u) \\
= - \int I_N^s u \partial_x I_N^s (u^2) - 2 \int I_N^s u \left( 1 - \partial_x^2 \right)^{-1} \partial_x I_N^s (u^2) \\
- \int I_N^s u \left( 1 - \partial_x^2 \right)^{-1} \partial_x I_N^s (\partial_x u)^2 - \int \partial_x I_N^s u \partial_x^2 I_N^s (u^2) \\
- 2 \int \partial_x I_N^s u \left( 1 - \partial_x^2 \right)^{-1} \partial_x^2 I_N^s (u^2) - \int \partial_x I_N^s u \left( 1 - \partial_x^2 \right)^{-1} \partial_x^2 I_N^s (\partial_x u)^2.
\]  

(3.2)
We claim that

\[
\int I_N^s u \partial_x (I_N^s u)^2 + 2 \int I_N^s u (1 - \partial_x^2)^{-1} \partial_x (I_N^s u)^2 + \int I_N^s u (1 - \partial_x^2)^{-1} \partial_x (I_N^s u)^2 \\
+ \int \partial_x I_N^s u \partial_x^3 (I_N^s u)^2 + 2 \int \partial_x I_N^s u (1 - \partial_x^2)^{-1} \partial_x^2 (I_N^s u)^2 \\
+ \int \partial_x I_N^s u (1 - \partial_x^2)^{-1} \partial_x^2 (I_N^s u)^2 = 0. \tag{3.3}
\]

Indeed, denoting by LHS the left side of (3.3), we have

\[
\text{LHS} = i \int \left( \xi_1 - \frac{2\xi_1}{1 + \xi_2} + \frac{\xi_1 \xi_2 \xi_3}{1 + \xi^2} - \xi_3 - \frac{2\xi_3}{1 + \xi^2} + \frac{\xi_1^3 \xi_2 \xi_3}{1 + \xi^2} \right) \\
\times (I_N^s u_1)^* (I_N^s u_2)^* (I_N^s u_3)^* \xi_3 \xi_3 \\
= i \int \left( \xi_1 - 2\xi_2 + \xi_1 \xi_2 \xi_3 - \xi_1^3 \right) (I_N^s u_1)^* (I_N^s u_2)^* (I_N^s u_3)^* \xi_3 \xi_3 \\
= -i \int \left( \xi_1 - \xi_1 \xi_2 \xi_3 + \xi_1^3 \right) (I_N^s u_1)^* (I_N^s u_2)^* (I_N^s u_3)^* \xi_3 \xi_3 \\
= -i \int \left( \frac{\xi_1 + \xi_2 + \xi_3}{3} - \xi_1 \xi_2 \xi_3 + \frac{\xi_1^3 + \xi_2^3 + \xi_3^3}{3} \right) \\
\times (I_N^s u_1)^* (I_N^s u_2)^* (I_N^s u_3)^* \xi_3 \xi_3 = 0;
\]

here and in what follows \( \int_* \) represents integration over the plane \( \xi_1 + \xi_2 + \xi_3 = 0 \), and to get the last equality we used the fact that on this plane \( \xi_1^3 + \xi_2^3 + \xi_3^3 = 3\xi_1\xi_2\xi_3 \).

By (3.3) and integration by part, we get

\[
\frac{d}{dt} \| I_N^s u \|_{H^1(\mathbb{R})}^2 = \int \partial_x (I_N^s u) [I_N^s (u^2) - (I_N^s u)^2] dx - \int \partial_x^3 (I_N^s u) [I_N^s (u^2) - (I_N^s u)^2] dx \\
+ 2 \int (1 - \partial_x^2)^{-1} \partial_x (I_N^s u) [I_N^s (u^2) - (I_N^s u)^2] dx \\
+ \int (1 - \partial_x^2)^{-1} \partial_x (I_N^s u) [I_N^s (\partial_x u)^2 - (\partial_x I_N^s u)^2] dx \\
- 2 \int (1 - \partial_x^2)^{-1} \partial_x^3 (I_N^s u) [I_N^s (u^2) - (I_N^s u)^2] dx \\
- \int (1 - \partial_x^2)^{-1} \partial_x^3 (I_N^s u) [I_N^s (\partial_x u)^2 - (\partial_x I_N^s u)^2] dx. \tag{3.4}
\]

Integrating both sides of (3.4) over the interval \([0, \delta]\), we obtain (3.1). \(\square\)
Next we apply Lemma 3.1 to deduce an exact estimate on the increment of the modified energy \( \| I_N u \|_{H^1(R)}^2 \) in terms of the norm \( \| I_N u \|_{X_{1,1/2}^\delta} \). We first deduce a few simple preliminary estimates.

The following embedding inequality is established in [21]:

\[
\| u \|_{L^6_t} \leq C \| u \|_{X_{0,1/2}^\delta}.
\]
(3.5)

From the well-known embedding \( X_{0,1/2}^\delta \subset C([0, \delta], L^2(R)) \) we have

\[
\| u \|_{L^2_t L^2(R \times [0, \delta])} = \| u \|_{L^2_t L^2([0, \delta], L^2)} \leq C \delta^{\frac{1}{2}} \| u \|_{X_{0,1/2}^\delta}.
\]
(3.6)

Interpolating (3.6) with (3.5) we get

\[
\| u \|_{L^2_t L^2(R \times [0, \delta])} \leq C \delta^{\frac{1}{2}} \| u \|_{X_{0,1/2}^\delta}.
\]
(3.7)

Next, from [16] we know that the following bilinear estimate holds:

\[
\| D_x^{\frac{1}{2}} T_x^{\frac{1}{2}} (u_1 u_2) \|_{L^2_t} \leq C \| u_1 \|_{X_{0,1/2}^\delta} \| u_2 \|_{X_{0,1/2}^\delta},
\]
(3.8)

where \( (I_a^u(u_1 u_2)) (\xi, \tau) = \int_{\xi = \xi_1 + \xi_2, \tau = \tau_1 + \tau_2} |\xi_1 - \xi_2|^a \tilde{u}_1(\xi_1, \tau_1) \tilde{u}_2(\xi_2, \tau_2) d\xi_1 d\tau_1 \). Besides, from [6] we know that there also holds the following bilinear estimate:

\[
\| \partial_x (u_1 u_2) \|_{X_{0,-1/2}^\delta} \leq C \| u_1 \|_{X_{-3/8+,-1/2}^\delta} \| u_2 \|_{X_{-3/8+,1/2}^\delta},
\]
(3.9)

provided \( \tilde{u}_1, \tilde{u}_2 \) are supported outside \( \{|\xi| < 1\} \).

**Lemma 3.2.** *If* \( u \) *is the solution of* (1.1) *on* \([0, \delta]\) *in the sense of Theorem 2.1, then*

\[
\| I_N u(\delta) \|_{H^1(R)}^2 - \| I_N \varphi \|_{H^1(R)}^2 \leq C \left( N^{-1+\frac{1}{2}} + N^{-2+\frac{1}{2} \delta} + N^{-\frac{3}{2} + \frac{1}{2} \delta} \right) \| I_N u \|_{X_{1,1/2}^\delta}^3.
\]
(3.10)

**Proof.** We denote the six terms on the right-hand side of (3.1) in their appearing order by \( J_1, J_2, \ldots, J_6 \), respectively. In the sequel we consider each \( J_i \) separately. For simplicity of notation we omit the subscripts \( s, N \) of the multiplier \( m_{s,N}(\xi) \).

1. *Estimate of* \( J_1 \). *We only need to prove that for any triple \((u_1, u_2, u_3) \in C([0, \delta], S(R))^3\) such that the frequency support of each \( u_j \) is located in a dyadic band \(|\xi| \sim N_j \) (i.e., \( C_1 N_j \leq |\xi| \leq C_2 N_j \)) for some positive numbers \( N_j \) \((j = 1, 2, 3)\), there holds

\[
\int_0^{\delta} \left( \left| \frac{m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)} \right| |\xi_1 + \xi_2| \left| \tilde{u}_1(\xi_1, \tau) \tilde{u}_2(\xi_2, \tau) \tilde{u}_3(\xi_3, \tau) \right| \right) d\tau \leq C \left( N^{-3+\frac{1}{2}} + N^{-2+\frac{1}{2} \delta} \right) \prod_{j=1}^3 N_j^0 \| u_j \|_{X_{1,1/2}^\delta}^3.
\]
(3.11)
Indeed, if this estimate is proved, then by using the Littlewood–Paley decomposition we see readily that

$$|J_1| \leq C \left( N^{-3+\delta} + N^{-2+\delta^5} \right) \left\| f_N u \right\|_{X_{1,(1/2)^+}^3}. \tag{3.12}$$

To prove (3.11), we note that since on both sides of this inequality the positions of the subscripts 1 and 2 are symmetric, and moreover the multiplier vanishes if both $|\xi_1|$ and $|\xi_2|$ are less than $\frac{N}{2}$, without loss of generality we only consider the part of the integral $\int_{\ast}$ over the region $|\xi_2| \geq \max\{|\xi_1|, \frac{N}{4}\}$. We decompose this region into three subregions:

$$\{ |\xi_2| \gg |\xi_1|, |\xi_1| < N \}, \quad \{ |\xi_2| \gg |\xi_1|, |\xi_1| \geq N \}, \quad \text{and} \quad \{ |\xi_1| \sim |\xi_2| \}.$$ 

The integrals over these three subregions will be respectively denoted as $J_{11}, J_{12},$ and $J_{13}$.

1.1. The subregion $|\xi_2| \gg |\xi_1|, |\xi_1| < N$. In this subregion there holds $|\theta \xi_1 + \xi_2| \sim |\xi_2|$ for any $0 < \theta < 1$. Thus by the mean value theorem we have

$$|m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)| = |m(\xi_1 + \xi_2) - m(\xi_2)| = |m'(\theta \xi_1 + \xi_2)| |\xi_1| \leq C \frac{m(\xi_2)}{|\xi_2|} |\xi_1|. $$

Therefore, the multiplier is dominated by $\frac{C|\xi_1|}{|\xi_2|} \sim \frac{CN_1}{N_2}$. Thus by using (3.6) and (3.8) we see that

$$J_{11} \leq C \frac{N_1}{N_2} \int_0^\delta \left( \int |\xi_1 + \xi_2|^\frac{1}{2} |\xi_1 - \xi_2|^\frac{1}{2} |u_1(\xi_1, t)u_2(\xi_2, t)u_3(\xi, t)| \right) dt$$

$$\leq C \frac{N_1}{N_2} \left\| u_1 \right\|_{X_{0,(1/2)^+}^3} \left\| u_2 \right\|_{X_{0,(1/2)^+}^3} \left\| u_3 \right\|_{L_{xt}^2} \leq C \delta^\frac{1}{2} N_2^{-3} \prod_{j=1}^3 \left\| u_j \right\|_{X_{1,(1/2)^+}^3}$$

$$\leq C \delta^\frac{1}{2} N^{-3+3} \prod_{j=1}^3 N_j^{0-} \left\| u_j \right\|_{X_{1,(1/2)^+}^3}. $$

1.2. The subregion $|\xi_2| \gg |\xi_1|, |\xi_1| \geq N$. In this subregion we have $m(\xi_1 + \xi_2) \sim m(\xi_2)$. Since $0 < m(\xi_1) \leq 1$, we see that

$$\frac{|m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)|}{m(\xi_1)m(\xi_2)} \leq \frac{\max\{m(\xi_1 + \xi_2), m(\xi_2)\}}{m(\xi_1)m(\xi_2)} \leq \frac{C}{m(\xi_1)} \leq C \left( \frac{|\xi_1|}{N} \right)^{1-s}. $$

It follows that

$$J_{12} \leq C \left( \frac{N_1}{N} \right)^{1-s} \int_0^\delta \left( \int |\xi_1 + \xi_2|^\frac{1}{2} |\xi_1 - \xi_2|^\frac{1}{2} |u_1(\xi_1, t)u_2(\xi_2, t)u_3(\xi, t)| \right) dt$$

$$\leq C \left( \frac{N_1}{N} \right)^{1-s} \left\| u_1 \right\|_{X_{0,(1/2)^+}^3} \left\| u_2 \right\|_{X_{0,(1/2)^+}^3} \left\| u_3 \right\|_{L_{xt}^2}$$

$$\leq CN^{s-1} N_2^{-s} \delta^\frac{1}{2} \prod_{j=1}^3 \left\| u_j \right\|_{X_{1,(1/2)^+}^3}$$

$$\leq C \delta^\frac{1}{2} N^{-3+3} \prod_{j=1}^3 N_j^{0-} \left\| u_j \right\|_{X_{1,(1/2)^+}^3}. $$
1.3. The subregion $|\xi_2| \sim |\xi_1|$. In this subregion we have $m(\xi_2) \sim m(\xi_1)$, so that

$$
\left| \frac{m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)} \right| \leq \frac{1}{m(\xi_1)m(\xi_2)} \leq C\left( \frac{|\xi_1|}{N} \right)^{2(1-s)}.
$$

Hence, using (3.6) and (3.7) we obtain

$$
J_{13} \leq C\left( \frac{N_1}{N} \right)^{2(1-s)} \int_0^\delta \left( \int_\ast \left| \hat{u}_1(\xi_1, t)\hat{u}_2(\xi_2, t) \cdot |\xi_3|\hat{u}_3(\xi_3, t) \right| \right) dt
\leq C\left( \frac{N_1}{N} \right)^{2(1-s)} \|u_1\|_{L^4_{xt}} \|u_2\|_{L^4_{xt}} \|D_xu_3\|_{L^2_{xt}}
\leq CN^{2(1-s)}N_1^{-2s}\delta^\frac{5}{6} \prod_{j=1}^3 \|u_j\|_{X^j_{1,(1/2)+}}
\leq C\delta^\frac{5}{6}N^{-2+} N_1^{-2} \prod_{j=1}^3 N_j^{0-}\|u_j\|_{X^j_{1,(1/2)+}}.
$$

2. Estimate of $J_2$. Similarly as before we only need to prove that for any triple $(u_1, u_2, u_3)$ similar as before there holds

$$
\int_0^\delta \left( \int_\ast \left| \frac{m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)} \right| \cdot |\hat{u}_1(\xi_1, t)\hat{u}_2(\xi_2, t) \cdot |\xi_3|^3\hat{u}_3(\xi_3, t) \right) dt
\leq C(N^{-1+\delta\frac{1}{2}} + N^{-\frac{7}{2}+}) \prod_{j=1}^3 N_j^{0-}\|u_j\|_{X^j_{1,(1/2)+}}.
$$

(3.13)

Again, without loss of generality we only consider the part of the integral $\int_\ast$ over the region $|\xi_2| \geq \max\{|\xi_1|, \frac{N}{2}\}$, and we decompose this region into three subregions:

$$
\{ |\xi_2| \gg |\xi_1|, |\xi_1| < N \}, \quad \{ |\xi_2| \gg |\xi_1|, |\xi_1| \geq N \}, \text{ and } \{ |\xi_1| \sim |\xi_2| \}.
$$

The integrals over these three subregions will be respectively denoted as $J_{21}, J_{22}$, and $J_{23}$.

2.1. The subregion $|\xi_2| \gg |\xi_1|, |\xi_1| < N$. We have

$$
J_{21} \leq C\frac{N_1}{N_2} \int_0^\delta \left( \int_\ast \left| \frac{1}{|\xi_1 + \xi_2|^2}|\xi_1 - \xi_2|\frac{1}{2}|\hat{u}_1(\xi_1, t)||\xi_2|\hat{u}_2(\xi_2, t)||\xi_3\hat{u}_3(\xi_3, t) \right| \right) dt
\leq C\frac{N_1}{N_2} \|u_1\|_{X^\delta_{1,1,(1/2)+}} \|u_2\|_{X^\delta_{1,1,(1/2)+}} \|D_xu_3\|_{L^2_{xt}}
\leq C\delta^\frac{1}{2}N_2^{-1} \prod_{j=1}^3 \|u_j\|_{X^j_{1,(1/2)+}}
\leq C\delta^\frac{1}{2}N^{-1+} \prod_{j=1}^3 N_j^{0-}\|u_j\|_{X^j_{1,(1/2)+}}.
$$
2.2. The subregion $|\xi_2| > |\xi_1|, |\xi_1| \geq N$. Similarly as in 2.1 we have
\[
J_{22} \lesssim C\left(\frac{N_1}{N}\right)^{1-s} \|u_1\|_{X_{0,(1/2)+}^\delta} \|u_2\|_{X_{1,(1/2)+}^\delta} \|D_x u_3\|_{L^2_{xt}}
\]
\[
\lesssim C N^{s-1} N_1^{-s} \delta^3 \prod_{j=1}^{3} \|u_j\|_{X_{1,(1/2)+}^\delta} \lesssim C \delta^{1/2} N^{-1} N_1^0 \prod_{j=1}^{3} \|u_j\|_{X_{1,(1/2)+}^\delta}.
\]

2.3. The subregion $|\xi_2| \sim |\xi_1|$. Since $\xi_3 = -(\xi_1 + \xi_2)$, the present assumption implies that $|\xi_3| \leq C |\xi_2|$. Hence, by using (3.9) we have
\[
J_{23} \lesssim C \left(\frac{|N_1|}{N}\right)^{2(1-s)} \int_0^t \left( \left\| \left(\xi_1 + \xi_2\right) \hat{u}_1(\xi_1, t) \cdot \xi_2 \hat{u}_2(\xi_2, t) \right\| \right) dt
\]
\[
\lesssim C \left(\frac{|N_1|}{N}\right)^{2(1-s)} \|D_x(u_1 D_x u_2)\|_{X_{0,(1/2)-}^\delta} \|D_x u_3\|_{X_{0,(1/2)+}^\delta}
\]
\[
\lesssim C \left(\frac{|N_1|}{N}\right)^{2(1-s)} \|u_1\|_{X_{-(3/8)+,(1/2)+}^\delta} \|D_x u_2\|_{X_{-(3/8)+,(1/2)+}^\delta} \|u_3\|_{X_{1,(1/2)+}^\delta}
\]
\[
\lesssim C \left(\frac{|N_1|}{N}\right)^{2(1-s)} N_1^{-1/3} N_2^{-3/8} \prod_{j=1}^{3} \|u_j\|_{X_{1,(1/2)+}^\delta}
\]
\[
\lesssim N^{-2/3} \prod_{j=1}^{3} N_j^{0-\delta} \|u_j\|_{X_{1,(1/2)+}^\delta}.
\]

Having established (3.13), as before we can use the Littlewood–Paley decomposition to get
\[
|J_2| \lesssim C (N^{-1} \delta^{1/2} + N^{-2} \delta^{3/8}) \|I_N^\delta u\|_{X_{1,(1/2)+}^\delta}.
\]  
(3.14)

3. Estimates of the rest terms. Estimate of $J_3$ may be dealt with similarly as for $J_1$, and estimates of $J_4, J_5$ and $J_6$ can be obtained similarly as for $J_2$ by using the following facts:
\[
(1 + |\xi_3|^2)^{-1} \leq \min(1, N_3^{-2}), \quad \frac{|\xi_3|^2}{1 + |\xi_3|^2} \leq 1.
\]

We omit the details and only give the results:
\[
|J_3| \lesssim C (N^{-5} + \delta^{1/2} + N^{-2} + \delta^{3/8}) \|I_N^\delta u\|_{X_{1,(1/2)+}^\delta},
\]  
(3.15)

\[
|J_4|, |J_5| \lesssim C (N^{-3} + \delta^{1/2} + N^{-3/2}) \|I_N^\delta u\|_{X_{1,(1/2)+}^\delta},
\]  
(3.16)

\[
|J_6| \lesssim C (N^{-1} + \delta^{1/2} + N^{-2} + \delta^{3/8}) \|I_N^\delta u\|_{X_{1,(1/2)+}^\delta}.
\]  
(3.17)

By (3.1), (3.12) and (3.14)–(3.17), we immediately obtain (3.10). 
\[ \square \]
4. Proof of Theorem 1.1

The goal of this section is to construct the solution of the initial value (1.1) on an arbitrary fixed time interval $[0, T]$.

Since $\| I_N^{s} \varphi \|^2_{H^1(\mathbb{R})} \leq C N^{2(1-s)} \leq 2 C N^{2(1-s)}$, it follows from Theorem 2.1 that the solution $u$ exists on $[0, \delta]$ with

$$\delta \geq C' \left\| I_N^{s} \varphi \right\|_{H^1(\mathbb{R})}^{-\frac{1}{2}} \geq C'' \left( \sqrt{2 C} N^{1-s} \right)^{-\frac{1}{2}} = C_0 N^{-\frac{1-s}{2}}, \quad C_0 = C' (2C')^{-\frac{1}{2}},$$

and

$$\left\| I_N^{s} u(t) \right\|_{H^1(\mathbb{R})}^{1/2} \leq C \left\| I_N^{s} \varphi \right\|_{H^1(\mathbb{R})} \leq \sqrt{2 C'} C N^{1-s} \quad \text{for } 0 \leq t \leq \delta.$$  

By Lemma 3.2, we have

$$\left\| I_N^{s} u(\delta) \right\|_{H^1(\mathbb{R})}^2 - \left\| I_N^{s} \varphi \right\|_{H^1(\mathbb{R})}^2 \leq C''' \left( N^{-1+\frac{1}{2}} + N^{-2+\frac{5}{6}} + N^{-\frac{7}{4}} \right) N^{3(1-s)},$$

where $C'''$ depends only on $\sqrt{2C'}$. As long as

$$C''' \left( N^{-1+\frac{1}{2}} + N^{-2+\frac{5}{6}} + N^{-\frac{7}{4}} \right) N^{3(1-s)} \leq C' N^{2(1-s)},$$

we have

$$\left\| I_N^{s} u(\delta) \right\|_{H^1(\mathbb{R})}^2 \leq \left\| I_N^{s} \varphi \right\|_{H^1(\mathbb{R})}^2 + C''' \left( N^{-1+\frac{1}{2}} + N^{-2+\frac{5}{6}} + N^{-\frac{7}{4}} \right) N^{3(1-s)} \leq 2 C' N^{2(1-s)}.$$

It follows, by considering $\delta$ as the initial time, using $I_N^{s} u(\delta)$ as the initial value, and applying Theorem 2.1, that the problem (1.1) has a solution on $\mathbb{R} \times [\delta, 2\delta]$. In this way we succeed to extend the solution of (1.1) to the time interval $[0, 2\delta]$.

The above argument can be repeated for $K$ steps as long as the following condition on $K$ is satisfied:

$$C''' \left( N^{-1+\frac{1}{2}} + N^{-2+\frac{5}{6}} + N^{-\frac{7}{4}} \right) N^{3(1-s)} K \leq C' N^{2(1-s)}.$$

In order to extend the solution to the time interval $[0, T]$; we must have $K \delta \geq T$, or $K \geq T \delta^{-1}$. Since the minimum of all such $K$ satisfies $K \sim T \delta^{-1}$, to arrive at this goal we only need to have

$$C C''' C_0^{-1} \left( N^{-1+\frac{1}{2}} + N^{-2+\frac{5}{6}} + N^{-\frac{7}{4}} \right) N^{3(1-s)} T \delta^{-1} \leq C' N^{2(1-s)}.$$

Since $\delta \geq C_0 N^{-\frac{1-s}{2}}$, this can be fulfilled if we can choose a sufficiently large number $N$ so that

$$C C''' C_0^{-1} \left( N^{-1+\frac{1}{2}} + N^{-2+\frac{5}{6}} + N^{-\frac{7}{4}} \right) N^{3(1-s)} T \leq C' N^{2(1-s)}.$$

The above condition is satisfied, provided the following conditions hold:

$$-1 + \frac{1-s}{4\beta} + 3(1-s) < 2(1-s) \iff s > \frac{\sqrt{13} - 2}{2}.$$
\[-2 + \frac{1-s}{12\beta} + 3(1-s) < 2(1-s) \iff s > \frac{1}{2},\]
\[-\frac{7}{4} + \frac{1-s}{2\beta} + 3(1-s) < 2(1-s) \iff s > \frac{5\sqrt{7} - 10}{4},\]

which is true because we have assumed that \(s > \frac{5\sqrt{7} - 10}{4}\).

Hence, the solution exists on \(\mathbb{R} \times [0, T]\) for any \(T > 0\), and it belongs to and is unique in \(X^T_{s,(1/2)+}\). The continuous dependence of the solution on the initial data follows from a standard argument, cf., for instance, \([5–8,21–23]\). We omit the details.

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**References**


