Doubly Nonlinear Degenerate Parabolic Systems with Coupled Nonlinear Boundary Conditions

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In this paper, we study the global existence and the global nonexistence of doubly nonlinear degenerate parabolic systems with nonlinear boundary conditions. We first prove a local existence result by the regularization method. Next, we construct a weak comparison principle. Then we discuss the large time behavior of solutions by using a modified upper and lower solution methods and constructing various upper and lower solutions. Necessary and sufficient conditions on the global existence of all positive (weak) solutions are obtained. © 2002 Elsevier Science (USA)

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1. INTRODUCTION AND MAIN RESULTS

In this paper, we study the existence and nonexistence of global solutions of the following problem:

\[
\begin{align*}
(|u|^{m_1-1}u)_x &= (|u_x|^{p_1-1}u_x)_x, & 0 < x < 1, & t > 0, \\
(|v|^{m_2-1}v)_x &= (|v_x|^{p_2-1}v_x)_x, & 0 < x < 1, & t > 0, \\
ux|_{x=0} &= 0, & u_x|_{x=1} &= \lambda u^{l_{11}}v^{l_{12}}|_{x=1}, & t > 0, \\
vx|_{x=0} &= 0, & v_x|_{x=1} &= \lambda u^{l_{21}}v^{l_{22}}|_{x=1}, & t > 0, \\
u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & 0 \leq x \leq 1,
\end{align*}
\]

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where $m_i, p_i > 0$, $i = 1, 2, l_{12}, l_{21} > 0, l_{11}, l_{22} \geq 0$ and $\lambda > 0$ are all constants. According to physical settings, $\phi_i(s) = |s|^{m_i-1}s$ stands for the mass concentration, the term $A_i(s) = |s_x|^{p_i-1}s_x$ reflects the diffusion effect which do not conform to Darcy’s law [15, 30], $u^{l_{11}+l_{22}}$ the forced flux at the boundary $x = 1$. The differential equations in (1.1), which is the typical example of the non-Newtonian filtration equations, have been suggested as some models, see [1, 7–9, 15, 17, 36, 37, 39] and the references therein. For example, the gas flow equation through a porous medium and the completely turbulent flow fall in the class of equations we consider. The nonlinear boundary conditions in (1.1) can be physically interpreted as a nonlinear radiation law, which here is actually an absorption law, see [3, 19, 34].

In the recent years, the questions like blow-up and global solvability for semilinear parabolic equation or systems with nonlinear boundary condition have been intensively studied, see [3–8, 14, 16, 18–22, 24, 27–29, 34, 37, 40–48] and references therein. The Dirichlet or Cauchy Problem for the $p$-Laplacian has also been studied in that extent, see [2, 25, 26, 33] and references therein.

For problem (1.1) with $p_1 = p_2 = 1$ or more general form of (1.1) with the Darcy’s law diffusion, there were many results on the local existence, global existence and blow-up in finite time, see [3, 4, 16, 23, 37, 38, 42, 44, 46]. Amann [3, 4] considered the (classical) local solvability and the geometric theory of the following quasilinear problem:

$$
\begin{align*}
    u_t + \mathcal{A}(t,u)u &= f(t,u), \quad x \in \Omega, \quad t > 0, \\
    \mathcal{B}(t,u)u &= g(t,u), \quad x \in \partial\Omega, \\
    u(x,0) &= u_0(x), \quad x \in \tilde{\Omega}
\end{align*}
$$

(1.2)

together with certain results concerning the continuous dependence of the solutions upon the data of the problem, by means of semigroups methods, and proved that the semilinear parabolic problem (1.2) possesses a unique maximal classical solution. Escher [16] considered the global existence and global nonexistence of problem (1.2) and obtained some sufficient conditions on which the solution of (1.2) blows up in finite time by means of “concavity method” of abstract Cauchy problem. Pao [37] established the upper and lower solutions method, and gave some sufficient conditions on the global solutions and blow-up in finite time for a more general problem. Samarskii et al. [38] considered global solvability and blow-up of solutions to semilinear parabolic systems with Dirichlet boundary conditions and gave optimal conditions of global existence of arbitrary solutions to the initial boundary value problem of the porous medium flows. Galaktionov et al. [23] studied blow-up phenomena for nonlinear parabolic equations.
and systems with nonlinear diffusivity, source terms and boundary conditions. Wang et al. [46] and M.X. Wang et al. [42, 44] considered the following porous medium flow problem:

\[
\begin{align*}
    u_t &= \nabla (u^m \nabla u) + \lambda u^{l_{11}} v^{l_{12}}, \quad x \in \Omega, \quad t > 0, \\
    v_t &= \nabla (u^n \nabla u) + \lambda u^{l_{21}} v^{l_{22}}, \quad x \in \Omega, \quad t > 0, \\
    \frac{\partial u}{\partial \eta} &= u^p v^p, \quad \frac{\partial v}{\partial \eta} = v^q v^q, \quad x \in \partial \Omega, \quad t > 0, \\
    u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega,
\end{align*}
\]

respectively, and obtained necessary and sufficient conditions on the global existence of all positive (classical) solutions.

Recently, S. Wang et al. [47] considered the single doubly nonlinear parabolic equation with nonlinear boundary conditions, namely,

\[
\begin{align*}
    (|u|^{m-1}u)_t &= (|u_x|^{p-1}u_x)_x, \quad 0 < x < 1, \quad t > 0, \\
    u_x|_{x=0} &= 0, \quad u_x|_{x=1} = \lambda u^a|_{x=1}, \quad t > 0, \quad (1.3) \\
    u(x, 0) &= u_0(x), \quad 0 \leq x \leq 1
\end{align*}
\]

and obtained necessary and sufficient conditions on the global existence of all positive (weak) solutions by using upper and lower methods. Filo [20] considered the multidimensional variant of (1.3) and a local existence result of weak solution was obtained by using the semidiscretization method. Filo and Kalur [21] also considered the local existence of a more general scalar case of (1.3) and gave some sufficient conditions on the global existence of weak solutions under some assumptions of general nonlinear term by using \(L^\infty\)-estimate methods.

In this paper we will consider large time behaviors of weak solutions to (1.1).

Throughout the paper we assume that (A)

(i) \(u_0(x), v_0(x) \in C^{2+\mu}([0, 1])\) for some \(0 < \mu < 1\), \(u_0(x), v_0(x) > 0\);  
(ii) \(|u_0x|^{p-1}u_0x, (|v_0x|^{p-1}v_0x) \in L^2([0, 1])\) on \([0, 1]\);  
(iii) \(u_0(x), v_0(x)\) satisfy the compatibility conditions:

\[
\begin{align*}
    u_0x(0) &= 0, \quad u_0x(1) = \lambda u_0^{l_{11}}(1)v_0^{l_{12}}(1), \\
    v_0x(0) &= 0, \quad v_0x(1) = \lambda u_0^{l_{21}}(1)v_0^{l_{22}}(1).
\end{align*}
\]

Our main result is
Theorem. All positive (weak) solutions of (1.1) exist globally if and only if
\[
l_{11} < \min \left\{ \frac{m_1}{p_1}, \frac{m_1 + 1}{p_1 + 1} \right\}, \quad l_{22} < \min \left\{ \frac{m_2}{p_2}, \frac{m_2 + 1}{p_2 + 1} \right\},
\]
\[
l_{12} l_{21} \leq \left( \min \left\{ \frac{m_1}{p_1}, \frac{m_1 + 1}{p_1 + 1} \right\} - l_{11} \right) \left( \min \left\{ \frac{m_2}{p_2}, \frac{m_2 + 1}{p_2 + 1} \right\} - l_{22} \right). \tag{1.4}
\]

The rest of the paper is organized as follows: In Section 2, the definition of weak solution is given and a weak comparison principle is established to serve as the basis of the study. Local existence and continuation results are proved by the regularization method in Section 3. In Sections 4 and 5, our main theorem is proved by constructing various upper and lower solutions.

2. PRELIMINARIES. COMPARISON PRINCIPLE

Due to the gradient degeneration or singularity of the equation, we cannot expect problem (1.1) to be solvable in the classical sense even for smooth data. Therefore, a notion of weak solution to problem (1.1) is needed.

Definition. For any \( T > 0 \), denote \( Q_T = (0,1) \times (0,T) \). A pair of nonnegative function \( (u(x,t), v(x,t)) \in C(\bar{Q}_T) \times C(\bar{Q}_T) \) is called a weak upper (or lower) solution of problem (1.1) in \( Q_T \) with \( \lambda \) if

(i) \( u, v \in L^\infty(0,T; W^{1,\infty}((0,1))) \cap W^{1,2}(0,T; L^2((0,1))), \quad (u(x,0), v(x,0)) \geq (\leq) (u_0(x), v_0(x)) \);

(ii) for any nonnegative functions \( w_1(x,t), w_2(x,t) \in L^1(0,T; W^{1,2}((0,1))) \cap L^2(Q_T) \), we have
\[
\int_Q \left[ w_1(u^{m_1})_t + w_1 u_1 |u^{p_1-1}u_2| dx \right] dt \geq (\leq) \int_0^T w_1(\lambda u^{l_{11}} v^{l_{12}})_{x=1} dx dt
\]
and
\[
\int_Q \left[ w_2(v^{m_2})_t + w_2 v_2 |v^{p_2-1}v_1| dx \right] dt \geq (\leq) \int_0^T w_2(\lambda u^{l_{21}} v^{l_{22}})_{x=1} dx dt.
\]

\((u(x,t), v(x,t))\) is called a (weak) solution of problem (1.1) if it is both a (weak) upper solution and lower solution of problem (1.1) in \( Q_T \) with \( \lambda \).
The main purpose of this paper is to give the necessary and sufficient conditions on the existence of global positive (weak) solutions. The basic technique used in the present paper is constructing various upper and lower solutions and comparing them with the solutions of problem (1.1). To this aim, we first prove a weak comparison principle.

**Proposition 2.1 (Comparison Principle).** Assume that \( \lambda < \tilde{\lambda} < \tilde{\lambda} \). Let \((u, v) \geq (\tilde{u}, \tilde{v})\) be a lower and upper solution of (1.1) in \( \tilde{Q}_T \) with \( \lambda \) and \( \tilde{\lambda} \), respectively. Then \((u, v) \leq (\tilde{u}, \tilde{v})\) on \( \tilde{Q}_T \).

**Proof.** For small \( \delta > 0 \). Let

\[
H_\delta(z) = \min\left(1, \max\left(\frac{z}{\delta}, 0\right)\right)
\]

and set \( w_1 = H_\delta(u - u) \), then according to the definitions of solution and lower solution we have

\[
\int_{Q_T} \left\{ H_\delta(u - u)(u^{m_1} - u^{m_1})_t + (H_\delta(u - u))_x\left|u_x\right|^{p-1}u_x - \left|u_x\right|^{p-1}u_x \right\} dx dt
\]

\[
\leq \int_0^\tau H_\delta(u - u)[(\lambda u^{l_{11}} v^{l_{12}})^{p_1} - (\lambda u^{l_{11}} v^{l_{12}})^{p_1}]_{x=1} dt, \quad \tau \in [0, T].
\]

As in [2], by letting \( \delta \to 0 \) we get

\[
\int_{Q_T} (u^{m_1} - u^{m_1})_t [u > u] dx dt
\]

\[
\leq \int_0^\tau [(\lambda u^{l_{11}} v^{l_{12}})^{p_1} - (\lambda u^{l_{11}} v^{l_{12}})^{p_1}] [u > u]_{x=1} dt
\]

\[
\leq \int_0^\tau \left\{ v^{p_1 l_{11}} [\lambda u^{l_{11}}]^{p_1} - (\lambda u^{l_{11}})^{p_1} \right\}
\]

\[
+ \lambda^{p_1} u^{p_1 l_{11}} v^{l_{12}} \theta^l_{1 l_{12}} p_{1 l_{12}} [v - v] \right\} [u > u]_{x=1} dt
\]

for some \( \theta_1 > 0 \) lying between \( v(1, t) \) and \( \delta(1, t) \). If

\[
f(x, t) = v^{p_1 l_{11}} [\lambda u^{l_{11}}]^{p_1} - (\lambda u^{l_{11}})^{p_1} + \lambda^{p_1} u^{p_1 l_{11}} p_{1 l_{12}} \theta^l_{1 l_{12}} p_{1 l_{12}} [v - v],
\]

then

\[
\int_{Q_T} (u^{m_1} - u^{m_1})_t [u > u] dx dt \leq \int_0^\tau f(x, t) [u > u]_{x=1} dt, \quad \tau \in [0, T].
\]  

(2.1)
Similarly, we have
\[
\iint_{Q_T} (v^{m_1} - v^{m_2}) \chi[v > v] \, dx \, dt \leq \int_0^T g(x, t) \chi[v > v]_{x=1} \, dt, \quad \tau \in [0, T],
\]
where
\[
g(x, t) = u^{p_2l_2}[(\tilde{\lambda} v^{l_2})^{p_2} - (\tilde{\lambda} v^{l_2})^{p_2}] + \tilde{\lambda}^{p_2} v^{p_2l_2} p_2 l_2 (\theta_2^{p_2l_2-1})(u - u)
\]
for some \(\theta_2 > 0\) lying between \(u(1, t)\) and \(u(1, t)\).

Since \((0, 0) < (\tilde{\delta}, \tilde{\delta}) \leq (u(1, 0), v(1, 0)) \leq (u(1, 0), v(1, 0)) \) and \(\tilde{\lambda} < \lambda\), we follow from the continuity of \(u, v, u\) and \(v\) that there exists a time \(\tau_1 > 0\) such that \(f(1, t) < 0\) and \(g(1, t) < 0\) for all \(t \in [0, \tau_1]\). Therefore, we have that \((u, v) \leq (u, v)\) on \(\tilde{Q}_{\tau_1}\). Define
\[
\tau^* = \sup\{\tau \in [0, T]: u(1, t) \leq u(1, t) \text{ and } v(1, t) \leq v(1, t) \text{ for all } t \in [0, \tau]\}.
\]
We claim that \(\tau^* = T\). Otherwise, from the continuity of \(u, v, u\) and \(v\) there exists \(\varepsilon > 0\) such that \(\tau^* + \varepsilon < T\), \(f(1, t) < 0\) and \(g(1, t) < 0\) for all \(t \in [0, \tau^* + \varepsilon]\).

By (2.1) and (2.2) we have that \((u, v) \leq (u, v)\) on \(\tilde{Q}_{\tau^* + \varepsilon}\), which contradicts the definition of \(\tau^*\). Hence, \((u, v) \leq (u, v)\) on \(\tilde{Q}_{T}\).

Obviously, \((\tilde{\delta}, \tilde{\delta})\) is a lower solution of (1.1) on \(\tilde{Q}_{T}\) with \(\tilde{\lambda}\). Therefore, \((u, v) \geq (\tilde{\delta}, \tilde{\delta}) > (0, 0)\) on \(\tilde{Q}_{T}\). Using this fact, as in the above proof we can prove that \((u, v) \leq (\tilde{u}, \tilde{v})\) on \(\tilde{Q}_{T}\).

The proof of Proposition 2.1 is completed. \(\blacksquare\)

Remark 2.1. The above comparison principle do not imply the uniqueness. The uniqueness of solutions to problem (1.1) is open.

3. LOCAL EXISTENCE

In this section, we prove the following local existence result.

Proposition 3.1. Let \((A)\) be satisfied. Then there exists a time \(T > 0 < T \leq + \infty\) such that (1.1) has a weak solution \((u, v)\) satisfying \((u, v) \geq (\tilde{\delta}, \tilde{\delta}) > 0\) on \(\tilde{Q}_T\). Moreover, as a solution \((u, v)\) can be extended to \(Q_T\) with a maximum \(T > 0\), that is, either \(T = + \infty\), or \(T < + \infty\) so that
\[
\lim_{t \to T} \sup_{x \in [0, 1]} (||u(\cdot, t)||_\infty + ||v(\cdot, t)||_\infty) = + \infty.
\]
To prove the above local existence result, we need the following lemmas from Sobolev’s space.

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ of boundary $\partial \Omega$ and for $0 < T < \infty$ let $Q_T$ denote the cylindrical domain $\Omega \times (0, T]$.

Let $h$ be any real number and for a function $u$ consider the truncations of $u$ given by

$$(u - h)^+ = \max\{u - h, 0\}.$$  

**Lemma 3.2** (Ladyzenskaja et al. [31]; Ladyzenskaja and Ural’ceva [32]). Let $u \in L^1(Q_T)$ satisfy

$$\int_0^T \int_{\Omega} (u - h)^+ \, dx \, dt \leq F|Q_T \cap \{u > h\}|^{1+\beta}, \quad \forall h \geq k_0 > 0,$$

where $F, \beta$ are positive constants. Then

$$\operatorname{esssup}_{Q_T} u^+ \leq k_0 + \left(1 + \frac{1}{\beta}\right)F|\Omega|^\beta T^\beta.$$

**Lemma 3.3** (Dibenedetto [15]). Let $m, p \geq 1$ and $u \in V^{m,p}(Q_T) = L^\infty(0, T; L^m(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$. Then there exists a constant $c$ depending only upon $N, p, m$ and the structure of $\partial \Omega$ such that

$$\|u\|_{L^1(Q_T)} \leq c\left(1 + \frac{T}{|\Omega|^{\frac{N(p-m)+mp}{(Nm)}}}\right)^{1/l} \|u\|_{V^{m,p}(Q_T)},$$

where $l = \frac{N+m}{N}$ and $\|u\|_{V^{m,p}(Q_T)} = \operatorname{esssup}_{0 \leq t \leq T} \|u(\cdot, t)\|_{m, \Omega} + \|Du\|_{p, Q_T}$.

We shall prove Proposition 3.1 in two steps: First, we consider a regularized problem and derive some necessary estimates; then, by an approximation process and passing to limit a solution to the original problem is obtained. To this end, put

$$M = \|u_0(x)\|_\infty + \|v_0(x)\|_\infty, \quad K = \|u_{0x}(x)\|_\infty + \|v_{0x}(x)\|_\infty$$

and consider the problem

$$\Phi_1(u_\varepsilon)u_{\varepsilon t} = (u_{\varepsilon xx}^2 + \varepsilon)\frac{p-1}{2} u_{\varepsilon x}, \quad 0 < x < 1, \quad 0 < t < \tau,$$

$$\Phi_2(v_\varepsilon)v_{\varepsilon t} = (v_{\varepsilon xx}^2 + \varepsilon)\frac{p-1}{2} v_{\varepsilon x}, \quad 0 < x < 1, \quad 0 < t < \tau,$$

$$u_{\varepsilon x}|_{x=0} = 0, \quad u_{\varepsilon x}|_{x=1} = G_1(u_\varepsilon, v_\varepsilon)|_{x=1}, \quad 0 < t < \tau,$$

$$v_{\varepsilon x}|_{x=0} = 0, \quad v_{\varepsilon x}|_{x=1} = G_2(u_\varepsilon, v_\varepsilon)|_{x=1}, \quad 0 < t < \tau,$$

$$u_\varepsilon(x, 0) = u_0(x), \quad v_\varepsilon(x, 0) = v_0(x), \quad 0 \leq x \leq 1 \quad (3.1)$$
for any $\tau \in (0, +\infty)$ and any fixed $\varepsilon \in (0, 1)$. We define $\Phi_i(w)$ and $G_i(w, z)$ as follows: $\Phi_i(w) \in C^\infty(R), G_i(w, z) \in C^\infty(R^2)$, $\Phi_i(w) = m_i w^{m_i - 1}$, $G_i(w, z) = \lambda w^{l_i}$ for $\delta \leq w, z \leq M + 1$, there exist positive constants $l_i, L$ such that

$$0 < l_i \leq \Phi_i(w), G_i(w, z) \leq L < +\infty,$$

for any $w, z \in R$ and $\frac{\partial G_i(w, z)}{\partial z} \geq 0$ and $\frac{\partial G_i(w, z)}{\partial w} \geq 0$ for any $w, z \in R$. Moreover, let $a_{i\varepsilon}(z) \in C^\infty(R), 0 < \rho_\varepsilon \leq a_{i\varepsilon}(z) \leq \rho_\varepsilon^{-1}$ on $R$ for some $0 < \rho_\varepsilon < 1$ and

$$a_{i\varepsilon}(z) = (z^2 + \varepsilon) \frac{\rho_\varepsilon - 1}{\varepsilon} z$$

for $|z| \leq K + L + 1$

and consider the following problem:

$$\begin{align*}
\Phi_1(u_\varepsilon)u_{\varepsilon t} &= (a_{1\varepsilon}(u_{\varepsilon x}))_x, \\
\Phi_2(v_\varepsilon)v_{\varepsilon t} &= (a_{2\varepsilon}(v_{\varepsilon x}))_x, \\
u_{\varepsilon x}|_{x=0} &= 0, \\
v_{\varepsilon x}|_{x=0} &= 0, \\
u_\varepsilon(x, 0) &= u_0(x), \\
v_\varepsilon(x, 0) &= v_0(x),
\end{align*}$$

for $0 < \varepsilon < 1, \quad 0 < t < \tau, \quad 0 < t < \tau, \quad 0 < t < \tau, \quad 0 < t < \tau, \quad 0 \leq x \leq 1.$

Now, the standard parabolic theory (see [31, 37]) guarantees the existence of a unique solution $(u_\varepsilon, v_\varepsilon)$ to (3.2) in the class $H^{2+\beta, 1+\beta/2}(\tilde{Q}_\tau)$ for some $\beta \in (0, 1)$. Obviously, Comparison principle holds for (3.2). Therefore,

$$u_\varepsilon(x, t) \geq \delta > 0, \quad v_\varepsilon(x, t) \geq \delta > 0 \quad \text{and} \quad u_{\varepsilon t}, v_{\varepsilon t} \geq 0 \quad \text{on} \quad \tilde{Q}_\tau.$$

Firstly, we have

$$\|u_{\varepsilon x}\|_\infty \leq K + L, \quad \|v_{\varepsilon x}\|_\infty \leq K + L$$

on $\tilde{Q}_\tau$.

In fact, let $w = u_{\varepsilon x}, z = v_{\varepsilon x}$ then $(w, z)$ satisfies

$$\begin{align*}
\Phi_1(u_\varepsilon)w_t - \alpha_{1\varepsilon}(u_{\varepsilon x})w_{xx} - \alpha_{1\varepsilon}(u_{\varepsilon x})w_x^2 + \Phi_1'(u_\varepsilon)u_{\varepsilon t}w &= 0, \\
\Phi_2(v_\varepsilon)z_t - \alpha_{2\varepsilon}(v_{\varepsilon x})z_{xx} - \alpha_{2\varepsilon}(v_{\varepsilon x})z_x^2 + \Phi_2'(v_\varepsilon)v_{\varepsilon t}z &= 0, \\
w|_{x=0} &= 0, \\
z|_{x=0} &= 0, \\
w(x, 0) &= u_{0\varepsilon}(x), \\
z(x, 0) &= v_{0\varepsilon}(x),
\end{align*}$$

for $0 < t < \tau, \quad 0 < t < \tau, \quad 0 \leq x \leq 1.$

The maximum principle yields that

$$\|w\|_\infty \leq \max\{||u_{0\varepsilon}\|_\infty, ||v_{0\varepsilon}\|_\infty, L\}$$
and

\[ \|z\|_\infty \leq \max\{\|u_0\|_\infty, \|v_0\|_\infty, L\}. \]

Therefore, we have

\[ \|u_{tx}\|_\infty \leq K + L, \|v_{tx}\|_\infty \leq K + L \quad \text{on } \bar{Q}_T. \]

Thus,

\[ a_1(u_{tx}) = (u_{tx}^2 + \epsilon) \frac{p_1-1}{2} u_{tx}, \quad a_2(v_{tx}) = (v_{tx}^2 + \epsilon) \frac{p_2-1}{2} v_{tx} \]
on \bar{Q}_T. And hence \((u_\varepsilon, v_\varepsilon)\) is a solution of (3.1) in \(\bar{Q}_T\).

Secondly, we shall show that there exists \(T \in (0, \tau)\), independent of \(\varepsilon\), such that

\[ u_\varepsilon(x, t) \leq M + 1 \quad \text{and} \quad v_\varepsilon(x, t) \leq M + 1 \quad \text{on } \bar{Q}_T. \]

Let \(s \in (0, 1)\) and fix \(t \in [0, s]\). Denote \(k_0 = \|u_0\|_\infty + \|v_0\|_\infty, z^+ = \max\{z, 0\}\), \(A_h(t) = \{x \in (0, 1): u_\varepsilon(x, t) > h\}\) and \(\mu(h) = \int_0^t \int_{A_h(s)} dx \, ds\).

We may think that \(\mu(h) \leq 1\).

From (3.1) we easily derive

\[ \int_0^t \int_0^1 \Phi_1(u_\varepsilon)u_{tx}(u_\varepsilon - h)^+ \, dx \, dt + \int_0^t \int_0^1 \left( u_{tx}^2 + \epsilon \right) \frac{p_1-1}{2} u_{tx}(u_\varepsilon - h)^+_x \, dx \, dt \]

\[ = \int_0^t \left( G_1^2(u_\varepsilon, v_\varepsilon) + \epsilon \right) \frac{p_1-1}{2} G_1(u_\varepsilon, v_\varepsilon)(u_\varepsilon - h)^+ |_{x=1} \, dt. \]

Owing to the fact that \(0 < l \leq \Phi_1(u_\varepsilon), G_1(u_\varepsilon, v_\varepsilon) \leq L\), we see that

\[ \int_0^1 (u_\varepsilon - h)^+ dx + \int_0^t \int_0^1 |u_{tx}|^{p_1-1}(u_\varepsilon - h)^+_x \, dx \, dt \leq c \int_0^t (u - h)^+ |_{x=1} \, dt \]

for some positive constant \(c\) independent of \(\varepsilon\). By using trace embedding theorem we have for any \(\tau > 1\) and a.e. \(t \in [0, s]\)

\[ (u_\varepsilon - h)^+ |_{x=1} \leq c \|\{(u_\varepsilon - h)^+(\cdot, t)\}|_{W^{1, \frac{p_1}{\tau}}((0, 1))}. \]

Therefore, the above inequality and Holder’s inequality yield

\[ \int_0^1 (u_\varepsilon - h)^+ dx + \int_0^t \int_0^1 |u_{tx}|^{p_1-1}(u_\varepsilon - h)^+_x \, dx \, dt \]

\[ \leq c \int_0^t \|\{(u_\varepsilon - h)^+(\cdot, t)\}|_{W^{1, \frac{p_1}{\tau}}((0, 1))} dt \]
Young’s inequality yields
\[
\int_0^1 (u_e - h)^{+2} \, dx + \int_0^t \int_0^1 |u_{xx}|^{p_1} (u_e - h)^{+2} \, dx \, dt
\leq c \|(u_e - h)^{+}\|_{L^{p_1+1}(Q_t)} + c \mu^{z_1},
\]
where
\[
z_1 = \frac{1}{p_1} + 1 \left( 1 - \frac{p_1}{p_1 + 1} \right) \times \frac{1}{1 - \frac{1}{p_1 + 1}}.
\]
By using Lemma 3.3, Holder’s inequality and Young’s inequality we have
\[
\|(u_e - h)^{+}\|_{L^{3(p_1+1)}(Q_t)} \leq c \|(u_e - h)^{+}\|_{L^{2(p_1+1)}(Q_t)}
\leq c \left( \|(u_e - h)^{+}\|_{L^{p_1+1}(Q_t)} + \mu^{z_1}(h) \right) + c \left( \|(u_e - h)^{+}\|_{L^{p_1+1}(Q_t)} + \mu^{z_1} \right) \frac{1}{p_1+1}
\leq c \left( \|(u_e - h)^{+}\|_{L^{p_1+1}(Q_t)} + \mu^{z_1}(h) \right) + c \mu^{z_1}(h) + c \left( \|(u_e - h)^{+}\|_{L^{p_1+1}(Q_t)} + \mu^{z_1} \right) \frac{1}{p_1+1}
\left( \frac{1}{p_1 + 1} \right)
\leq c \left( \|(u_e - h)^{+}\|_{L^{3(p_1+1)}(Q_t)} + \mu^{z_1}(h) + \mu^{z_1}(h) \right)
\leq \frac{1}{2} \|(u_e - h)^{+}\|_{L^{3(p_1+1)}(Q_t)} + c \mu^{z_1}(h) + c \mu^{z_1}(h) + c \mu^{z_1}(h) + c \mu^{z_1}(h),
\]
where
\[
z_2 = \frac{1}{p_1} + 1 \left( 1 - \frac{p_1}{3(p_1 + 1)} \right),
\]
\[
z_3 = \frac{1}{(p_1 + 1)(p_1 + 1)} \left( 1 - \frac{p_1}{3(p_1 + 1)} \right) \times \frac{1}{1 - \frac{1}{p_1 + 1}}.
\]
Hence,
\[ \|(u_e - h)\|_{L^{3(p_1+1)}(Q_T)} \leq c\mu^{2l}(h) + c\mu^{22}(h) + c\mu^{23}(h) + c\mu^{p_1+1}(h). \]

By using Holder’s inequality we have
\[
\int_0^t \int_0^1 (u_e - h)^+ \, dx \, dt \leq \|(u_e - h)\|_{L^{3(p_1+1)}(Q_T)} \cdot \mu^{1 - \frac{1}{3(p_1+1)}}(h)
\]
\[
\leq c[\mu^{\frac{1}{3(p_1+1)}} + \mu^{\frac{1}{3(p_1+1)} + z_2}(h) + \mu^{\frac{1}{3(p_1+1)} + z_3}(h) + \mu^{\frac{1}{3(p_1+1)} + (p_1+1)(h)}].
\]

Choose \(\tau > 1\) to be sufficiently large such that
\[
\frac{z_1}{2} - \frac{1}{3(p_1+1)} > 0, \quad \max\{z_2, z_3\} - \frac{1}{3(p_1+1)} > 0, \quad \frac{z_1}{p_1+1} - \frac{1}{3(p_1+1)} > 0.
\]

Then there exists \(\beta > 0\) such that
\[
\int_0^t \int_0^1 (u_e - h)^+ \, dx \, dt \leq c(\mu(h))^{1+\beta}, \quad \forall h \geq k_0 > 0.
\]

By Lemma 3.2 we have
\[
u_e(x, t) \leq k_0 + \left(1 + \frac{1}{\beta}\right)ct^\beta.
\]

Therefore, there exists \(T > 0\) independent of \(\varepsilon\) such that
\[
u_e(x, t) \leq M + 1 \quad \text{on } \tilde{Q}_T.
\]

Similarly, we have
\[
u_e(x, t) \leq M + 1 \quad \text{on } \tilde{Q}_T.
\]

Thus, we have the following:

**Lemma 3.4.** There exist \(T\) and a positive constant \(C\) independent of \(\varepsilon\) such that
\[
\delta \leq u_e(x, t), v_e(x, t) \leq C, \quad \|u_x\|_\infty, \|v_x\|_\infty \leq C \quad \text{on } \tilde{Q}_T. \quad (3.3)
\]

Next, we will get an \(L^2\)-estimates.
Lemma 3.5. There exists a positive constant $C$ independent of $\varepsilon$ such that

$$
\int_0^T \int_0^1 (u_{\varepsilon t}^2 + v_{\varepsilon t}^2) \, dx \, dt \leq C < + \infty. \tag{3.4}
$$

Proof of Proposition 3.1. Differentiate the first equation of (3.1) with respect to $t$ and multiply it by $\Phi_1(u_\varepsilon) u_{\varepsilon t}$. Then integrate it over $[0, 1] \times [0, T]$ to get

$$
\frac{1}{2} \int_0^1 (\Phi_1(u_\varepsilon) u_{\varepsilon t})^2(x, t) \, dx + \int_0^T \int_0^1 (u_{\varepsilon x}^2 + \varepsilon) \frac{p_{\varepsilon - 1}}{2} (p_1 u_{\varepsilon x}^2 + \varepsilon) u_{\varepsilon x}^2 \, dx \, dt
$$

$$
= \frac{1}{2} \int_0^1 (\Phi_1(u_\varepsilon) u_{\varepsilon t})^2(x, 0) \, dx - \int_0^T \int_0^1 (u_{\varepsilon x}^2 + \varepsilon) \frac{p_{\varepsilon - 1}}{2} (p_1 u_{\varepsilon x}^2 + \varepsilon) u_{\varepsilon x}^2 \, dx \, dt
$$

$$
\times \Phi'_1(u_\varepsilon) u_{\varepsilon x} u_{\varepsilon t} \, dx \, dt + \int_0^T ((u_{\varepsilon x}^2 + \varepsilon) \frac{p_{\varepsilon - 1}}{2} u_{\varepsilon x})_t \Phi_1(u_\varepsilon) u_{\varepsilon t}(1, t) \, dt.
$$

Using Holder’s inequality, we have

$$
\int_0^1 (\Phi_1(u_\varepsilon) u_{\varepsilon t})^2(x, t) \, dx + \int_0^T \int_0^1 (u_{\varepsilon x}^2 + \varepsilon) \frac{p_{\varepsilon - 1}}{2} (p_1 u_{\varepsilon x}^2 + \varepsilon) u_{\varepsilon x}^2 \, dx \, dt
$$

$$
\leq \int_0^1 (\Phi_1(u_\varepsilon) u_{\varepsilon t})^2(x, 0) \, dx
$$

$$
+ \int_0^T \int_0^1 (u_{\varepsilon x}^2 + \varepsilon) \frac{p_{\varepsilon - 1}}{2} (p_1 u_{\varepsilon x}^2 + \varepsilon) (\Phi'_1(u_\varepsilon) u_{\varepsilon x} u_{\varepsilon t})^2 \, dx \, dt
$$

$$
+ 2 \int_0^T ((u_{\varepsilon x}^2 + \varepsilon) \frac{p_{\varepsilon - 1}}{2} u_{\varepsilon x})_t \Phi_1(u_\varepsilon) u_{\varepsilon t}(1, t) \, dt. \tag{3.5}
$$

Using $u_\varepsilon(x, t), v_\varepsilon(x, t) \geq \delta$ and the boundary conditions in (3.1), we know that there exists $x_0 \in [0, 1]$ such that

$$
u_{\varepsilon x}(x, t) \geq \frac{\lambda^l_1 u_{\varepsilon x}}{2}, \quad v_{\varepsilon x}(x, t) \geq \frac{\lambda^l_1 u_{\varepsilon x}}{2}, \quad (x, t) \in [x_0, 1] \times [0, T].$$

Hence, we have

$$
\int_0^T \int_0^1 (u_{\varepsilon x}^2 + \varepsilon) \frac{p_{\varepsilon - 1}}{2} (p_1 u_{\varepsilon x}^2 + \varepsilon) u_{\varepsilon x}^2 \, dx \, dt \geq c \int_0^T \int_{x_0}^1 u_{\varepsilon x}^2 \, dx \tag{3.6}
$$

for some positive constant $c$ independent of $\varepsilon$.\]
Then, (3.5), together with (3.6), gives

\[
\int_0^1 (\Phi_1(u_\varepsilon)u_{t\varepsilon})^2(x, t) \, dx + c \int_0^T \int_{x_0}^1 u_{t\varepsilon}^2 \, dx \, dt \\
\leq \int_0^1 (\Phi_1(u_\varepsilon)u_{t\varepsilon})^2(x, 0) \, dx + c \int_0^T \int_0^1 u_{t\varepsilon}^2 \, dx \, dt \\
+ c \int_0^T (u_{t\varepsilon}^2 + v_{t\varepsilon}^2)(1, t) \, dt.
\]  

(3.7)

with the help of the boundary conditions and Young’s inequality.

Similarly, we have

\[
\int_0^1 (\Phi_2(v_\varepsilon)v_{t\varepsilon})^2(x, t) \, dx + c \int_0^T \int_{x_0}^1 v_{t\varepsilon}^2 \, dx \, dt \\
\leq \int_0^1 (\Phi_2(v_\varepsilon)v_{t\varepsilon})^2(x, 0) \, dx + c \int_0^T \int_0^1 v_{t\varepsilon}^2 \, dx \, dt + c \int_0^T (u_{t\varepsilon}^2 + v_{t\varepsilon}^2)(1, t) \, dt.
\]  

(3.8)

Using Sobolev’s inequalities, we have

\[
C \int_0^T (u_{t\varepsilon}^2 + v_{t\varepsilon}^2)(1, t) \, dt \\
\leq \tau_1 \int_0^T \int_{x_0}^1 (u_{t\varepsilon}^2 + v_{t\varepsilon}^2) \, dx \, dt + c(\tau_1) \int_0^T \int_{x_0}^1 (u_{t\varepsilon}^2 + v_{t\varepsilon}^2) \, dx \, dt
\]  

(3.9)

for any positive constant \(\tau_1\) independent of \(\varepsilon\).

Combining (3.7)–(3.9), we have

\[
\int_0^1 ((\Phi_1(u_\varepsilon)u_{t\varepsilon})^2 + (\Phi_2(v_\varepsilon)v_{t\varepsilon})^2)(x, t) \, dx \\
\leq \int_0^1 ((\Phi_1(u_\varepsilon)u_{t\varepsilon})^2 + (\Phi_2(v_\varepsilon)v_{t\varepsilon})^2)(x, 0) \, dx + \int_0^T \int_0^1 (u_{t\varepsilon}^2 + v_{t\varepsilon}^2) \, dx \, dt.
\]  

(3.10)

Gronwall’s inequality leads to the desired results.

Thus, we need only let \(\varepsilon \to 0\). From (3.3) (3.4) and (3.10) we know that \(u_\varepsilon, v_\varepsilon, u_{t\varepsilon}, v_{t\varepsilon}\) are weakly * compact in \(L^\infty(\bar{Q}_T)\) and \(u_{t\varepsilon}, v_{t\varepsilon}, \Phi_1(u_\varepsilon)u_{t\varepsilon}, \Phi_2(v_\varepsilon)v_{t\varepsilon}\) are weakly compact in \(L^2(\bar{Q}_T)\).

By Aubin’s Lemma (see [10, 11, 35]), it turns out that (for some subsequence, also denoted by \((u_\varepsilon, v_\varepsilon)\) in the sequel for simplicity)

\[(u_\varepsilon, v_\varepsilon) \rightarrow (u, v) \text{ in } (C(\bar{Q}_T))^2 \text{ uniformly on compact subsets of } \bar{Q}_T.\]
Therefore, \((u, v) \in C(\tilde{Q}_T)\), \(u_x, v_x \in L^\infty(\tilde{Q}_T), u_t, v_t \in L^2(\tilde{Q}_T)\) and
\[
\Phi_1(u_x)u_{xt} \to \Phi_1(u)u_t \quad \text{weakly in } L^2(\tilde{Q}_T),
\]
\[
\Phi_2(v_x)v_{xt} \to \Phi_1(v)v_t \quad \text{weakly in } L^2(\tilde{Q}_T).
\]

By the standard monotonicity argument it follows that
\[
a_1(u_x) \to |u_x|^{p_1-1}u_x \quad \text{weak} * - L^\infty(\tilde{Q}_T),
\]
\[
a_2(v_x) \to |v_x|^{p_2-1}v_x \quad \text{weak} * - L^\infty(\tilde{Q}_T).
\]

Therefore, it is easy to verify that function \((u(x, t), v(x, t))\) is a solution of (1.1).

The proof of Proposition 3.1 is completed. □

4. PROOF OF THE SUFFICIENCY

Assume that (1.4) holds, we will prove that all positive solutions of (1.1) exist globally.

We will divide proof of the sufficiency into some lemmas.
Throughout this section we denote
\[
\tau_j = \frac{p_j}{p_j + 1}, \quad \phi_j(x) = \frac{1}{2}(1 - \frac{1}{x^{\tau_j}}), \quad j = 1, 2
\]
and use \(\tilde{\lambda}\) to denote a positive constant satisfying \(\tilde{\lambda} > \lambda\).

**Lemma 4.1.** Assume that \(m_i < p_i, \ i = 1, 2\). If \(l_{11} < \frac{m_1}{p_1}, \ l_{22} < \frac{m_2}{p_2}\) and \(l_{12}l_{21} \leq \frac{(m_1/p_1 - l_{11})(m_2/p_2 - l_{22})}{(m_1/p_1 - l_{11}) + (m_2/p_2 - l_{22})}\), then the solution \((u, v)\) of (1.1) exists globally.

**Proof.** First, it is easy to prove that there exist \(l_1, l_2 > 1\) such that
\[
\frac{m_1}{p_1} - l_{11} - \frac{l_{12}}{1 - \frac{m_2}{p_2}} l_2 \geq 0,
\]
\[
- \frac{l_{21}}{1 - \frac{m_1}{p_1}} l_1 + \frac{m_2}{p_2} - \frac{l_{22}}{1 - \frac{m_2}{p_2}} l_2 \geq 0. \tag{4.1}
\]

By (1.4) and \(l_{12}, l_{21} > 0\) we have that \(\frac{m_1}{p_1} - l_{11}, \frac{m_2}{p_2} - l_{22}, 1 - \frac{m_1}{p_1}, 1 - \frac{m_2}{p_2}\) are positive. Set \(y_1(x, t) = a(1 + x^{\tau_1}) + e^{l_{11}(t+1)}, y_2(x, t) = b(1 + x^{\tau_2}) + e^{l_{12}(t+1)}\) and take
\[
w = y_1^{\frac{p_1}{m_1}}, \quad z = y_2^{\frac{p_2}{m_2}},
\]
where $l_1, l_2 > 1$ satisfy (4.1) and
\[
a = \frac{(p_1 - m_1)\tau_1}{p_1} 2^{p_2/m_2}, \quad b = \frac{(p_2 - m_2)\tau_2}{p_2} 2^{p_1/m_1},
\]
\[
l = \max\{\log(2a)/l_1, \log(2b)/l_2, \}
\]
\[
\left(\frac{p_1}{p_1 - m_1}\right)^{p_1 - 1} \frac{2a^{p_1}}{m_1 l_1 \tau_1^{p_1}} \left(1 + \frac{a p_1 m_1}{p_1 - m_1}\right),
\]
\[
\left(\frac{p_2}{p_2 - m_2}\right)^{p_2 - 1} \frac{2a^{p_2}}{m_2 l_2 \tau_2^{p_2}} \left(1 + \frac{a p_2 m_2}{p_2 - m_2}\right),
\]
\[
(p_1 - m_1) \log(\max u_0(x))/(l_1 p_1),
\]
\[
(p_2 - m_2) \log(\max v_0(x))/(l_2 p_2)\}.
\]
By the choices of $a, b$ and $l$ we have
\[
y_1 \leq 2e^{l_1(t+1)}, \quad y_2 \leq 2e^{l_2(t+1)}.
\] (4.2)

By direct computations and (4.2) we have, for $(x, t) \in (0, 1) \times [0, +\infty)$,
\[
(w^{m_1})_t = \frac{p_1 m_1 l_1}{p_1 - m_1} y_1^{p_1 - m_1 - 1} e^{l_1(t+1)},
\]
\[
w_x = \frac{a p_1}{(p_1 - m_1)\tau_1} y_1^{p_1 - m_1 - 1} \frac{1}{x^{\tau_1} - 1},
\]
\[
((w_x)^{p_1})_x = \frac{a p_1}{(p_1 - m_1)\tau_1} \left(\frac{y_1^{p_1 - m_1} x}{y_1^{p_1 - m_1}}\right)_x
\]
\[
= \frac{a p_1}{(p_1 - m_1)\tau_1} \left\{ y_1^{p_1 - m_1} + \frac{a p_1 m_1}{p_1 - m_1} y_1^{p_1 - m_1 - 1} \frac{1}{x^{\tau_1}}\right\}
\]
\[
= \frac{a p_1}{(p_1 - m_1)\tau_1} \left\{ y_1^{p_1 - m_1 - 1} + \frac{a p_1 m_1}{p_1 - m_1} \frac{1}{x^{\tau_1}}\right\}
\]
\[
\leq \frac{a p_1}{(p_1 - m_1)\tau_1} \left\{ y_1^{p_1 - m_1 - 1} + \frac{a p_1 m_1}{p_1 - m_1} \frac{1}{x^{\tau_1}}\right\}
\]
\[
\leq \frac{p_1 m_1 l_1}{p_1 - m_1} y_1^{p_1 - m_1 - 1} e^{l_1(t+1)} \left(1 + \frac{a p_1 m_1}{p_1 - m_1}\right)
\]
\[
= (w^{m_1})_t,
\]
Similarly, we have
\[(z^\beta)^x \leq (z^\alpha)^x.\] (4.4)

And
\[w_x|_{x=0} = 0, \quad z_x|_{x=0} = 0,\]

\[w_x = \frac{a p_1}{(p_1 - m_1)\tau_1} \frac{m_1}{p_1-m_1} y_1^{m_1-p_1 l_1} - \frac{p_1 l_2}{p_2-m_2} y_2^{p_1 l_2} - \frac{p_2 l_1}{p_1-m_1} y_1^{p_2 l_1},\]

\[w_x \geq w^{l_1} z^{l_2} \frac{a p_1}{(p_1 - m_1)\tau_1} 2^{p_1 l_2} \frac{p_2 l_1}{p_2-m_2} \exp\left\{ \left[ \frac{m_1}{p_1} - \frac{l_1}{1 - \frac{m_1}{p_1}} \right] l_1 - \frac{l_2}{1 - \frac{m_2}{p_2}} \right\} l(t+1) \}

\[\geq w^{l_1} z^{l_2} \frac{a p_1}{(p_1 - m_1)\tau_1} 2^{p_1 l_2} \frac{p_2 l_1}{p_2-m_2} = \tilde{\lambda} w^{l_1} z^{l_2}, \quad x = 1, \quad t > 0.\] (4.5)

Similarly, we have
\[z_x \geq \tilde{\lambda} w^{l_1} z^{l_2}, \quad x = 1, \quad t > 0.\] (4.6)

For \(x \in [0, 1],\) we have
\[w(x, 0) = (a(1 + x^{l_1}) + e^{\frac{1}{p_1} - \frac{p_1}{m_1}}) \geq \frac{a^{l_1} - p_1}{p_1 - m_1} u_0(x),\] (4.7)

\[z(x, 0) = (a(1 + x^{l_2}) + e^{\frac{1}{p_2} - \frac{p_2}{m_2}}) \geq \frac{a^{l_2} - p_2}{p_2 - m_2} v_0(x).\] (4.8)

From (4.3)–(4.8) we see that \((w,z)\) is an upper solution of (1.1) with \(\tilde{\lambda}.\) Therefore, \((u,v) \leq (w,z),\) and hence \((u,v)\) exists globally. 

**Lemma 4.2.** Assume \(m_1 = p_1, \quad m_2 < p_2.\) If \(l_{11} < 1, \quad l_{22} < \frac{m_2}{p_2} \quad \text{and} \quad l_{12} l_{21} \leq (1 - l_{11})\left(\frac{m_2}{p_2} - l_{22}\right),\) then the solution \((u,v)\) of (1.1) exists globally.
Proof. First, it is easy to prove that there exist $l_1, l_2 > 1$ such that

\[
(1 - l_{11})l_1 - \frac{l_{12}}{1 - \frac{m_2}{p_2}} l_2 \geq 0,
\]

\[
-l_{21} l_1 + \frac{m_2}{p_2} - \frac{l_{22}}{1 - \frac{m_2}{p_2}} l_2 \geq 0.
\]

(4.9)

Set

\[
w = a(1 + x^{\frac{1}{p_1}})e^{ll_1(t+1)}, \quad z = [b(1 + x^{\frac{1}{p_2}}) + e^{ll_2(t+1)}]^{\frac{p_2}{p_2 - m_2}},
\]

where $l_1, l_2$ satisfy (4.9).

Similar to the proof of Lemma 4.1 we can prove that there exist $a, b, l > 0$ such that $(w, z)$, defined by this manner, is an upper solution of (1.1) with $\tilde{\lambda}$, therefore $(u, v)$ exists globally. \]

**Lemma 4.3.** Assume $m_1 < p_1, m_2 = p_2$. If $l_{11} < \frac{m_1}{p_1}, l_{22} < 1$ and $l_{12} l_{21} \leq (\frac{m_1}{p_1} - l_{11})(1 - l_{22})$, then the solution $(u, v)$ of (1.1) exists globally.

**Proof.** The proof is similar to that of Lemma 4.2. \]

**Lemma 4.4.** Assume $m_1 = p_1, m_2 = p_2$. If $l_{11} < 1, l_{22} < 1$ and $l_{12} l_{21} \leq (1 - l_{11})(1 - l_{22})$, then the solution $(u, v)$ of (1.1) exists globally.

**Proof.** First, it is easy to prove that there exist $l_1, l_2 > 1$ such that

\[
(1 - l_{11})l_1 - l_{12} l_2 \geq 0,
\]

\[-l_{21} l_1 + (1 - l_{22}) l_2 \geq 0.
\]

(4.10)

Set

\[
w = a(1 + x^{\frac{1}{p_1}})e^{ll_1(t+1)}, \quad z = b(1 + x^{\frac{1}{p_2}})e^{ll_2(t+1)},
\]

where $l_1, l_2$ satisfy (4.10).

Similar to the above arguments it is easy to prove that there exist $a, b, l > 0$ such that $(w, z)$, defined by this manner, is an upper solution of (1.1) with $\tilde{\lambda}$, therefore $(u, v)$ exists globally. \]

**Lemma 4.5.** Assume that $m_i > p_i, i = 1, 2$. If $l_{11} < \frac{m_i + 1}{p_i + 1}, l_{22} < \frac{m_i + 1}{p_i + 1}$ and $l_{12} l_{21} \leq (\frac{m_i + 1}{p_i + 1} - l_{11})(\frac{m_i + 1}{p_i + 1} - l_{22})$, then the solution $(u, v)$ of (1.1) exists globally.

**Proof.** It suffices to prove that for any $T > 0$ there exists $C(T) > 0$ such that

\[
u(x, t), v(x, t) \leq C(T) < + \infty, \quad (x, t) \in \tilde{Q}_T.
\]

(4.11)
To this aim, denote \( \theta = \frac{p_1}{m_1-p_1}, \) \( l = 1, \) \( \sigma = \frac{p_2}{m_2-p_2} \) and

\[
M = \max \left\{ \frac{k^{n_1} \theta^{n_1-1} (2\tau_1 + p_1(\theta + 2))}{8p_1 \tau_1^{n_1+1} m_1}, \frac{\sigma^{n_1-1} (2\tau_2 + p_2(\sigma + 2))}{8p_2 \tau_2^{n_1+1} m_2} \right\},
\]

\[
k = 2^{n_1+2(l_1,\theta+\sigma_l_2) + 3 + 2(l_1,\theta+\sigma_l_2)} \frac{\lambda \tau_1^{n_1+1}}{\theta} \left( \frac{\tau_2 \lambda^{n_1}}{(1-l_2)\sigma+\tau_2} \right)^{\sigma_l_1_2}.
\]

It is obvious that for any given \( T: 0 < T < +\infty \) there exists natural number \( N = N(T) \) such that \( \frac{N \log^2 2}{M} > T > \frac{(N-1) \log^2 2}{M} \). Take

\[
w_i(x,t) = \left\{ \varepsilon_i e^{-M(t-T_i-1)} - \frac{1}{4} (1 - \phi_1(x)) \right\} \left\{ \frac{k}{l} \right\} = \varepsilon_i^{-\theta} y_i^{-\theta}
\]

and

\[
z_i(x,t) = \left\{ \eta_i e^{-M(t-T_i-1)} - \frac{1}{4} (1 - \phi_2(x)) \right\} \left\{ \frac{l}{k} \right\} = \eta_i^{-\sigma} y_i^{-\sigma},
\]

where \( T_0 = 0, T_i = \frac{i \log 2}{M}, i = 1, \ldots, N \) and \( \varepsilon_i, \eta_i > 0 \) are all constants to be determined later.

It is obvious that for any given \( i = 1, \ldots, N, y_j(x,t), j = 1, 2 \) are well defined on \( Q_i := [0,1] \times [T_{i-1}, T_i] \cap [0,1] \times [0, T] \) and \( \frac{1}{4} \leq y_j \leq 1, j = 1, 2. \)

By the direct calculations we have for \( (x,t) \in Q_i \)

\[
(w_{ix}^{p_1})_x = M\theta m_1 \varepsilon_i^{-m_1 \theta} y_i^{-m_1 \theta-1} \geq \frac{M\theta m_1}{2} \varepsilon_i^{-m_1 \theta},
\]

\[
w_i = \frac{k \theta}{8 \tau_1} \varepsilon_i^{-(\theta + \tau_1)} y_i^{-\theta-1} (1 - \phi_1(x)) \frac{k}{l} \frac{1}{x^{-\tau_1}},
\]

\[
(w_{ix}^{p_1})_x = \left( \frac{k \theta}{8 \tau_1} \right)^{p_1} \varepsilon_i^{-p_1(\theta + \tau_1)} (y_i^{-p_1(\theta+1)} (1 - \phi_1(x)) \frac{k}{l} \frac{1}{x^{-\tau_1}} )_x
\]

\[
= \left( \frac{k \theta}{8 \tau_1} \right)^{p_1} \varepsilon_i^{-p_1(\theta + \tau_1)} \left\{ \frac{y_i^{-p_1(\theta+1)} (1 - \phi_1(x)) \frac{k}{l} \frac{1}{x^{-\tau_1}}}{y_i^{-p_1(\theta+1)} (1 - \phi_1(x)) \frac{k}{l} \frac{1}{x^{-\tau_1}}} \right\}
\]

\[
+ \frac{p_1}{2 \tau_1} \left( \frac{k}{l} \right) y_i^{-p_1(\theta+1)} (1 - \phi_1(x)) \frac{k}{l} \frac{1}{x^{-\tau_1}}
\]

\[
+ \frac{p_1 k (\theta + 1)}{8 \tau_1 \varepsilon_i^{\tau_1}} y_i^{-p_1(\theta+1)-1} (1 - \phi_1(x)) \frac{k}{l} \frac{1}{x^{-\tau_1}}
\]
\[
\begin{align*}
\epsilon_i^{p_i} & \frac{k\theta}{8\tau_1} e_i^{p_i} - p_i(\theta + \tau_1)y_i^{p_i} - p_i(\theta + 1) \left\{ 1 + \frac{p_1 k}{2\tau_1 e_i^{p_i}} + \frac{p_1 k(\theta + 1)}{2\tau_1 e_i^{p_i}} \right\} \\
\epsilon_i^{p_i} & \frac{k\theta}{8\tau_1} e_i^{-\theta} 4p_i(\theta + 1) \left\{ e_i^{\epsilon_i^{p_i}} + \frac{p_1 k}{2\tau_1} + \frac{p_1 k(\theta + 1)}{2\tau_1} \right\} \\
\epsilon_i^{p_i} & \frac{k\theta}{8\tau_1} e_i^{-\theta} 4p_i(\theta + 1) \left\{ k + \frac{p_1 k}{2\tau_1} + \frac{p_1 k(\theta + 1)}{2\tau_1} \right\} \\
\frac{M\theta m_1}{2} & e_i^{-\theta} \leq (w_i^m)_t.
\end{align*}
\]

Similarly, we have
\[
(z_i^{m_2})_t \geq (z_i^{m_2})_x, \quad (x, t) \in Q_i.
\]

Obviously,
\[
w_{ix} |_{x=0} = 0, \quad z_{ix} |_{x=0} = 0,
\]

\[
w_{ix} = e_i^{-(\theta + \tau_1)} \frac{k\theta}{8\tau_1} e_i^{-\theta - 1}
\]

\[
= w_i^{l_1} z_i^{l_2} e_i^{-(\theta + \tau_1) + l_1 \eta_i^{l_2} \sigma \frac{k\theta}{8\tau_1}} \left( e^{-M(t-T_{i-1})} - \frac{1}{4} \right)^{(l_1-1)\theta + \sigma l_1 - 1}
\]

\[
\geq \tilde{\lambda} w_i^{l_1} z_i^{l_2}, \quad x = 1, \quad T_{i-1} \leq t \leq T_i
\]

holds if the following inequality holds:
\[
e_i^{-(\theta + \tau_1) + l_1 \eta_i^{l_2} \sigma \frac{k\theta}{8\tau_1}} 4^{-(l_1-1)\theta + \sigma l_1 - 1} \geq \tilde{\lambda}. \quad (4.12)
\]

Similarly, we have
\[
z_{ix} \geq \tilde{\lambda} w_i^{l_1} z_i^{l_2}
\]

holds if the following inequality holds:
\[
e_i^{l_1 \theta} \eta_i^{l_2 \theta - (\sigma + \tau_2) + l_2 \sigma \frac{l_1 \theta}{8\tau_2}} 4^{-(l_1-1)\theta + \sigma l_2} \geq \tilde{\lambda}. \quad (4.13)
\]

By the choices of \(k, l\) one see that (4.12) and (4.13) hold if
\[
\epsilon_i^{l_1 \theta} \eta_i^{l_2 \theta - (\sigma + \tau_2) + l_2 \sigma \frac{l_1 \theta}{8\tau_2}} 4^{-(l_1-1)\theta + \sigma l_2} \geq 1,
\]

\[
\eta_i = \left( \frac{\sigma 4^{-(l_1 \theta + \sigma l_2)} (1(l_1-1)\theta + \sigma l_2) e_i^{l_1 \theta}}{8\tau_2 \tilde{\lambda}} \right) \epsilon_i^{l_2 \theta - (\sigma + \tau_2) + l_2 \sigma \frac{l_1 \theta}{8\tau_2}}. \quad (4.14)
\]
Set \( \varepsilon_i = a \left( \frac{1}{4} \right)^i \), \( \eta_i = b \left( \frac{1}{4} \right)^i \), \( i = 1, \ldots, N \), then (4.14) holds if

\[
\left( a \left( \frac{1}{4} \right)^i \right)^{-\left[(1-l_{11})\theta + \tau_1\right] + \frac{l_{12}l_{21}\theta\sigma}{(1-l_{22})\sigma + \tau_2}} \geq 1
\]

\[
b = 4^i \left( \frac{\sigma 4^{-\left(1-l_1\theta + \sigma l_2\right)}}{8\tau_2^2} \right)^{\frac{l_{12}l_{21}\theta \sigma}{(1-l_{22})\sigma + \tau_2}} \left( a \left( \frac{1}{4} \right)^i \right)^{\frac{l_{12}l_{21}\theta}{(1-l_{22})\sigma + \tau_2}}.
\]

(4.15)

A direct calculation shows

\[
- \left[(1-l_{11})\theta + \tau_1\right] + \frac{l_{12}l_{21}\theta\sigma}{(1-l_{22})\sigma + \tau_2} - \frac{\sigma\theta}{(1-l_{22})\sigma + \tau_2} \left[l_{12}l_{21} - \left(\frac{m_1 + 1}{p_1 + 1} - l_{11}\right)\left(\frac{m_2 + 1}{p_2 + 1} - l_{22}\right)\right] \leq 0.
\]

Hence, for fixed \( i \) we can choose \( a \) and \( b \) sufficiently small such that (4.15) holds and

\[
w_i(x, 0) = \left\{ \varepsilon_1 \left[ 1 - \frac{1}{4}(1 - \phi_1)^{k/4} \right] \right\}^{-\theta} \geq \varepsilon_{1}^{-\theta} = \left( a \left( \frac{1}{4} \right)^i \right)^{-\theta} \geq \max_{0 \leq x \leq 1} u_0(x) \geq u_0(x),
\]

\[
z_i(x, 0) = \left\{ \eta_1 \left[ 1 - \frac{1}{4}(1 - \phi_2)^{l/4} \right] \right\}^{-\sigma} \geq \eta_{1}^{-\sigma} = \left( b \left( \frac{1}{4} \right)^i \right)^{-\sigma} \geq \max_{0 \leq x \leq 1} v_0(x) \geq v_0(x).
\]

For the above fixed \( a \) and \( b \), when \( i = 2, \ldots, N \), we have

\[
w_{i-1}(x, T_{i-1}) = \left\{ \varepsilon_{i-1} \left[ e^{-M(T_{i-1}-T_{i-2})} - \frac{1}{4}(1 - \phi_1)^{k/4} \right] \right\}^{-\theta} \leq \left( \varepsilon_{i-1} \left( \frac{1}{4} \right)^i \right)^{-\theta} = \varepsilon_i^{-\theta} \leq w_i(x, T_{i-1})
\]

and

\[
z_{i-1}(x, T_{i-1}) = \left\{ \eta_{i-1} \left[ e^{-M(T_{i-1}-T_{i-2})} - \frac{1}{4}(1 - \phi_1)^{l/4} \right] \right\}^{-\sigma} \leq \left( \eta_{i-1} \left( \frac{1}{4} \right)^i \right)^{-\sigma} = \eta_i^{-\sigma} \leq z_i(x, T_{i-1}).
\]
This shows that \((w, z)\) is an upper solution of (1.1) with \(\tilde{\lambda}\) on \(Q_t\). By comparison principle we have \((u, v) \leq (w_i, z_i)\) on \(Q_t, i = 1, \ldots, N\). It is obvious that there exist \(C(T) > 0\) such that (4.11) holds. This completes the proof of Lemma 4.5.

**Lemma 4.6** Assume \(m_1 < p_1, m_2 > p_2\). If \(l_{11} = \frac{m_1}{p_1}, l_{22} = \frac{m_2 + 1}{p_2 + 1}\) and \(l_{12}l_{21} \leq \left(\frac{m_1 - 1}{p_1 - 1}\right)^2\left(\frac{m_2 + 1}{p_2 + 1} - l_{22}\right)\), then the solution \((u, v)\) of (1.1) exists globally.

**Proof.** First, we consider the following cases:

1. \(p_1 \leq 1\); 2. \(p_1 > 1\) and \(m_1 \leq 1\); 3. \(p_1 > 1, m_1 > 1\) and \(l_{11} > \frac{m_1 - 1}{p_1 - 1}\).

As in Lemma 4.5, it suffices to prove that for any \(T > 0\) there exists \(C(T) > 0\) such that (4.11) holds. To this aim, denote \(\theta = \frac{1}{(1 - l_{11})} > 1, \sigma = \frac{p_1}{m_1 - 1} - p_2, \alpha = p_1 - m_1, k = \frac{1}{p_1 - (p_1 - m_1)}, c = \max\{1, \left(\frac{p_1}{\tau_1}\right)^{p_1} (1 + 2 p_1 (\theta - 1))\}/(km_1)\), \(M = \max\{\frac{L_{12}^{\sigma p_2 - 1} \log 2 + 1}{M_2^{2 + \sigma 2}}, 1\}\) and

\[
l = 2^{3 + \frac{2\sigma l_{12} + (\sigma + 1)(1 - l_{12}) + \sigma + 2\sigma 2}{\sigma l_{12}}} \left[\frac{\tilde{\lambda} \tau_2 [1 + c(\log 2 + 1)]}{\tau_2}\right]^{\sigma 2} \times \left(\frac{1 - \frac{\lambda}{\tau_1}}{\frac{\lambda}{\tau_1}}\right)^{\frac{1 - l_{12} \sigma + \sigma 2}{\tau_2}}.
\]

It is obvious that for any given \(T, 0 < T < + \infty\) there exists natural number \(N = N(T)\) such that \(N \log M > T > (N - 1) \log \frac{2}{M}\). Take

\[w_i(x, t) = \varepsilon_i \left\{ [1 + c(1 + \log 2)]^{t} - \phi_1(x) \right\}^{\theta} = \varepsilon_i y_1^{\theta}
\]

and

\[z_i(x, t) = \left\{ \eta_i \left[ e^{-M(t - T_j, \varepsilon)} - \frac{1}{4} (1 - \phi_2(x)) \right]^{\frac{1}{2}} \right\}^{-\sigma} = \eta_i^{-\sigma} y_2^{-\sigma}
\]

where \(T_0 = 0, T_i = \frac{i \log 2}{M}, i = 1, \ldots, N\) and \(\varepsilon_i, \eta_i > 0\) are all constants to be determined later.

It is also obvious that for any given \(i = 1, \ldots, N, y_j(x, t), j = 1, 2\) are well defined on \(Q_j := [0, 1] \times [T_{i-1}, T_i] \cap [0, 1] \times [0, T]\) and

\[\frac{1}{4} \leq y_1 \leq \varepsilon_i^{k_2} [1 + c(1 + \log 2)], \quad \frac{1}{4} \leq y_2 \leq 1 \quad (4.16)
\]

if \(\varepsilon_i\) is sufficiently large.
Thus, by direct calculations we have for \((x,t) \in Q_t, \ i = 1, \ldots, N\)

\[
(w_i^{m_1})_t = e_i^{m_1+\varphi} c k m_1 \theta y_1^{m_1 \theta-1} [1 + c e_i^\varphi (t + 1 - T_{i-1})]^{k-1},
\]

\[
w_{ix} = e_i \frac{\theta}{2 \tau_1} x_i^{\frac{1}{\nu}} y_1^{\theta-1},
\]

\[
(w_{ix}^{m_1})_x = \left( \frac{\theta}{2 \tau_1} \right)^{p_1} e_i^{p_1} (y_1^{p_1 (\theta-1)} x)_x
\]

\[
= \left( \frac{\theta}{2 \tau_1} \right)^{p_1} e_i^{p_1} \left\{ y_1^{p_1 (\theta-1)} + \frac{p_1 (\theta - 1)}{2 \tau_1} y_1^{p_1 (\theta-1)-1} \frac{1}{x_i^{\varphi}} \right\}
\]

\[
\leq \left( \frac{\theta}{2 \tau_1} \right)^{p_1} e_i^{m_1+\varphi} y_1^{m_1 \theta-1} y_1^{(p_1 - m_1) \theta - p_1 + 1} \left( 1 + \frac{2 p_1 (\theta - 1)}{\tau_1} \right).
\]

For the present case, a direct calculation shows that \((p_1 - m_1) \theta - p_1 + 1 \geq 0\) and \(k > 1\) if \((p_1 - m_1) \theta - p_1 + 1 > 0\) while \(k = 1\) if \((p_1 - m_1) \theta - p_1 + 1 = 0\). Therefore,

\[
(w_{ix}^{m_1})_x \leq \left( \frac{\theta}{2 \tau_1} \right)^{p_1} e_i^{m_1+\varphi} y_1^{m_1 \theta-1}
\]

\[
\times [1 + c e_i^\varphi (t + 1 - T_{i-1})]^{k[(p_1 - m_1) \theta - p_1 + 1]} \left( 1 + \frac{2 p_1 (\theta - 1)}{\tau_1} \right)
\]

\[
= \left( \frac{\theta}{2 \tau_1} \right)^{p_1} e_i^{m_1+\varphi} y_1^{m_1 \theta-1} [1 + c e_i^\varphi (t + 1 - T_{i-1})]^{k-1} \left( 1 + \frac{2 (\theta - 1)}{\tau_1} \right)
\]

\[
\leq e_i^{m_1+\varphi} y_1^{m_1 \theta-1} [1 + c e_i^\varphi (t + 1 - T_{i-1})]^{k-1} c k m_1 \theta
\]

\[
= (w_i^{m_1})_t.
\]

As in Lemma 4.5, we have

\[
(z_i^{m_2})_t \geq (z_{ix}^{m_2})_x, \quad (x,t) \in Q_t.
\]

Obviously,

\[
w_{ix}|_{x=0} = 0, \quad z_{ix}|_{x=0} = 0.
\]
Using \( \theta(1 - l_{11}) - 1 = 0 \) we see that

\[
\begin{align*}
\omega_{ix} &= \frac{\theta}{2\tau_1} y_1^{\theta - 1} = w_i^{l_{11}} z_i^{l_{12}} e_i^{l_{11} - l_{12}} \eta_i^{l_{12}} \frac{\theta}{2\tau_1} y_1^{\theta(1 - l_{11}) - 1} y_2^{\sigma l_{12}} \\
\geq w_i^{l_{11}} z_i^{l_{12}} e_i^{l_{11} - l_{12}} \eta_i^{l_{12}} \frac{\theta}{2\tau_1} \left( \frac{1}{4} \right)^{\sigma l_{12}} \geq \tilde{\lambda} w_i^{l_{11}} z_i^{l_{12}}, & x = 1, \quad T_{i-1} \leq t \leq T_i
\end{align*}
\]

holds if the following inequality holds:

\[
\varepsilon_i^{l_{11} - l_{12}} \eta_i^{l_{12}} \frac{\theta}{\tau_1 2^{\sigma l_{12} + 1}} \geq \tilde{\lambda}. \tag{4.17}
\]

Using (4.16) we know that

\[
\begin{align*}
z_{ix} &= \eta_i^{-(\sigma + \tau_2)} \frac{\sigma l}{8\tau_2} y_2^{-\sigma - 1} \\
&= w_i^{l_{21}} z_i^{l_{22}} e_i^{l_{21}} \eta_i^{-(\sigma + \tau_2) + l_{22} \sigma} \frac{l_{22}}{8\tau_2} y_2^{-\theta l_{21}} y_2^{l_{22} - 1} y_2^{-\sigma - 1} \\
&\geq w_i^{l_{21}} z_i^{l_{22}} e_i^{l_{21} + (1 + 2k\theta)l_{21}} \eta_i^{l_{22} \sigma} \frac{(\sigma + \tau_2) + l_{22} \sigma + l_{21} + (1 + c(\log 2 + 1))^{-\theta l_{21}}}{\tau_2 2^{l_{22} \sigma + 2}} \\
&\geq \tilde{\lambda} w_i^{l_{21}} z_i^{l_{22}}, & x = 1, \quad T_{i-1} \leq t \leq T_i
\end{align*}
\]

holds if the following inequality holds:

\[
\varepsilon_i^{l_{21} + (1 + 2k\theta)l_{21}} \eta_i^{-(\sigma + \tau_2) + l_{22} \sigma} \frac{l_{21} + (1 + c(\log 2 + 1))^{-\theta l_{21}}}{\tau_2 2^{l_{22} \sigma + 2}} \geq \tilde{\lambda}. \tag{4.18}
\]

By the choice of \( l \) one see that (4.17) and (4.18) hold if

\[
\varepsilon_i^{l_{11} - l_{12} l_{21} (1 + 2k\theta) \sigma} (1 - l_{22}) \sigma + \tau_2 \geq 1,
\]

\[
\eta_i = \left( \frac{l \sigma}{\tau_2 2^{l_{22} \sigma + 2} [1 + c(\log 2 + 1)]^{-\theta l_{21}}} \right)^{\frac{1}{1 - l_{11} - l_{12} l_{21} (1 + 2k\theta) \sigma}} \times \varepsilon_i^{l_{21} (1 + 2k\theta) \sigma} (1 - l_{22}) \sigma + \tau_2. \tag{4.19}
\]

Set \( \varepsilon_i = a(2 + \log 2)^{\frac{k\theta l_{21}}{1 + 2k\theta}}, \quad \eta_i = b(\frac{l}{\tau_2})^i, \quad i = 1, \ldots, N \), then (4.19) holds if

\[
(a(2 + \log 2)^{\frac{k\theta l_{21}}{1 + 2k\theta}})^{1 - l_{11} - l_{12} l_{21} (1 + 2k\theta) \sigma} (1 - l_{22}) \sigma + \tau_2 \geq 1.
\]
there exist a comparison principle we have

\[
(1 - l_{11} - l_{12} l_{21} (1 + a k \theta) \sigma)/(1 - l_{22}) \sigma + \tau_2
\]

A direct calculation shows that

\[
1 - l_{11} - l_{12} l_{21} (1 + a k \theta) \sigma/(1 - l_{22}) \sigma + \tau_2
\]

Hence, for fixed \(i\) we can choose \(a\) sufficiently large and then choose \(b\) sufficiently small to satisfy (4.20) and

\[
w_1(x, 0) = \varepsilon_i [1 + c \varepsilon_i^2] - \phi_1(x)_{x}^{\theta} \geq \varepsilon_i (\frac{1}{4})^{\theta} = a(2 + \log 2)^{k\theta/(1 + a k \theta)} (\frac{1}{4})^{\theta}
\]

\[
\geq \max_{0 \leq x \leq 1} u_0(x) \geq u_0(x),
\]

\[
z_1(x, 0) = \left\{ \eta_1 [1 - \frac{1}{4} (1 - \phi_2)^{y(t)} \right\}^{-\sigma} \geq (\eta_1)^{-\sigma} = (\frac{3}{4})^{\sigma} \geq \max_{0 \leq x \leq 1} v_0(x) \geq v_0(x).
\]

For the above fixed \(a, b\) and \(i = 2, \ldots, N\), we have

\[
w_{i-1}(x, T_{i-1}) \leq \varepsilon_i^{1 + a k \theta} [1 + c (\log 2 + 1)]^{k \theta}
\]

\[
\leq e^{k \theta} \varepsilon_i^{1 + a k \theta} [2 + \log 2]^{k \theta} = e^{k \theta} \varepsilon_i^{1 + a k \theta} \leq w_i(x, T_{i-1}),
\]

\[
z_{i-1}(x, T_{i-1}) = \left\{ \eta_i [1 - \frac{1}{4} (1 - \phi_2)^{y(t)} \right\}^{-\sigma}
\]

\[
\leq (\eta_i)^{-\sigma} = (\frac{1}{2})^{-\sigma} \leq z_i(x, T_{i-1}).
\]

This shows that \((w_i, z_i)\) is an upper solution of (1.1) with \(\bar{x}\) on \(Q_i\). By comparison principle we have \((u, v) \leq (w_i, z_i)\) on \(Q_i\), \(i = 1, \ldots, N\). Therefore, there exist \(C(T)\): \(0 < C(T) < +\infty\) such that (4.11) holds. And hence the solution \((u, v)\) of (1.1) exists globally.

Secondly, we consider the following case:

\[(4)\text{ } p_1 > 1, \text{ } m_1 > 1 \text{ and } l_{11} < \frac{m_1 - 1}{p_1 - 1}.
\]
As above, it suffices to prove that there exist \( C(T) : 0 < C(T) < + \infty \) such that (4.11) holds. To this aim, denote \( \theta = \frac{p_{i}-1}{m_{1}} > 1, \quad x = \frac{(1-m_{1}/p_{i})}{m_{1}-l_{11}} > 0, \)
\( \sigma = \frac{p_{i}}{m_{2}-p_{i}}, \quad c = \max \{ 2, (\frac{\theta}{\tau_{1}})^{p_{1}}(1 + p_{1}(\theta-1)/(m_{1} \theta)), \quad \tau = \frac{\tau_{1}-(m_{1}-m_{2})}{\tau_{1}} \} \) and \( M, I \) as above.

For the present case, a direct calculation shows that \( 0 < x < p_{1} - m_{1}, \tau < 0, \tau + \tau_{1}(1 - l_{11}) \theta - 1 = 0 \) and \( \frac{1}{p_{1} - m_{2}} = \frac{m_{1}}{p_{1}} - l_{11} \). Take

\[
,w_{i}(x, t) = e_{i} \{ 1 + c \varepsilon_{i}^{2}(t + 1 - T_{i-1}) + (1 - \phi_{1}(x))^{\varepsilon_{i}} \}^{\theta} = e_{i} y_{i}^{\theta},
\]

and

\[
z_{i}(x, t) = \{ \eta_{i} e^{-(l_{i} - T_{i-1})} - \frac{1}{4} (1 - \phi_{2}(x))^{\eta_{i}} \}^{-\sigma} = \eta_{i}^{\sigma} y_{i}^{-\sigma}.
\]

where \( T_{0} = 0, T_{i} = \frac{i \log 2}{M}, \quad i = 1, \ldots, N \) and \( \varepsilon_{i}, \eta_{i} > 0 \) are all constants to be determined later.

It is obvious that for any given \( i = 1, \ldots, N, \quad y_{i}(x, t), \quad j = 1, 2, \) are well defined on \( Q_{i} : = [0, 1] \times [T_{i-1}, T_{i}] \cap [0, 1] \times [0, T] \) and

\[
1 \leq c\varepsilon_{i}^{2} \leq y_{i} \leq \varepsilon_{i}^{2}[1 + c(1 + \log 2)], \quad \frac{1}{4} \leq y_{2} \leq 1
\]

(4.21)

if \( \varepsilon_{i} \) is sufficiently large.

By direct calculations we have for \( (x, t) \in Q_{i} \)

\[
(w_{m_{i}}^{l_{1}})_{x} = c^{m_{i}+x} m_{1} \theta \cdot y_{m_{i}}^{m_{1} \theta-1},
\]

\[
w_{ix} = c_{i}^{1+\tau} \frac{\theta}{2 \tau_{1}} (1 - \phi_{1}(x))^{\varepsilon_{i}^{1-1} x^{1/\tau_{1}-1} y_{i}^{\theta-1}},
\]

\[
(w_{p_{i}}^{l_{1}})_{x} = \left( \frac{\theta}{2 \tau_{1}} \right)^{p_{i}} c_{i}^{p_{i}(1+\tau)} y_{i}^{p_{i}(\theta-1)} (1 - \phi_{1}(x))^{p_{i}(\varepsilon_{i}^{1-1} x)}
\]

\[
= \left( \frac{\theta}{2 \tau_{1}} \right)^{p_{i}} c_{i}^{p_{i}(1+\tau)} \left\{ y_{i}^{p_{i}(\theta-1)} (1 - \phi_{1}(x))^{p_{i}(\varepsilon_{i}^{1-1})} + \frac{p_{i}(\varepsilon_{i}^{1-1})}{2 \tau_{1}} y_{i}^{p_{i}(\theta-1)-1} (1 - \phi_{1}(x))^{p_{i}(\varepsilon_{i}^{1-1})-1} x^{1/\tau_{1}} + \frac{p_{i}(\theta-1)\varepsilon_{i}^{1-1}}{2 \tau_{1}} y_{i}^{p_{i}(\theta-1)-1} (1 - \phi_{1}(x))^{p_{i}(\varepsilon_{i}^{1-1})-1} x^{1/\tau_{1}} \right\}
\]
\[
\left( \frac{\theta}{2\tau_1} \right)^{p_i} e_i^{p_i(1+\tau)} y_1^{p_i(\theta-1)} \left( 2^{p_i} + \frac{p_i(\theta - 1)2^{p_i+1}}{\tau_1} \right) \\
= \left( \frac{\theta}{\tau_1} \right)^{p_i} e_i^{m_1} y_1^{m_1(\theta-1)} y_1^{(p_i-m_1)\theta-p_i+1} \left( 1 + \frac{p_i(\theta - 1)}{\tau_1} \right) \\
\times e_i^{m_1+\gamma} c m_1 \theta y_1^{m_1(\theta-1)} \leq (w_i^{m_1})_x.
\]

(Here, we have used that \(e_i^2 \leq 1, y_1 \geq 1, (p_1 - m_1)\theta - p_1 + 1 = 0\).

As in Lemma 4.5, we have

\[
(z_i^{m_2})_t \geq (z_i^{m_2})_{XX}, \quad (x,t) \in Q_i.
\]

Obviously,

\[
w_{ix}|_{x=0} = 0, \quad z_{ix}|_{x=0} = 0.
\]

Using \(\tau + \alpha(\theta(1 - l_{11}) - 1] = 0, \frac{1 - l_{11}}{1 + 2\theta} = \frac{m_1}{p_1} - l_{11}\) and (4.21) we have

\[
w_{ix} = c_i^{1+\tau} \theta \frac{\theta}{\tau_1} y_1^{\theta-1} \leq w_i^{l_{11}} e_i^{l_{12}} \theta^i \eta_i^{l_{12}} \theta \frac{\theta}{\tau_1} y_1^{(1-l_{11})\theta-1} y_2^{\sigma l_{12}} \\
\geq w_i^{l_{11}} e_i^{l_{12}} \eta_i^{l_{12}} \theta^{1-l_{11}+\tau+\alpha(1-l_{11})\theta-1} \theta \frac{\theta}{\tau_1} y_1^{(1-l_{11})\theta-1} y_2^{\sigma l_{12}} \\
= w_i^{l_{11}} e_i^{l_{12}} \eta_i^{l_{12}} \theta \frac{1}{\tau_1} y_1^{\sigma l_{12}}, \quad x = 1, \quad T_{i-1} \leq t \leq T_i.
\]

Therefore, if (4.17) and (4.18) with \(k = 1\) hold, then for \(x = 1, T_{i-1} \leq t \leq T_i\)

\[
w_{ix} \geq \tilde{\lambda} w_i^{l_{11}} z_i^{l_{12}}, \quad z_{ix} \geq \tilde{\lambda} w_i^{l_{21}} z_i^{l_{22}}
\]

hold.

Thus, as above we can get our results.

This completes the proof of Lemma 4.6.

**Lemma 4.7.** Assume \(m_1 > p_1, m_2 < p_2\). If \(l_{11} < \frac{m_1+1}{p_1+1}, l_{22} < \frac{m_2}{p_2}\) and \(l_{12} l_{21} \leq \left( \frac{m_1+1}{p_1+1} - l_{11} \right) \left( \frac{m_2}{p_2} - l_{22} \right)\), then the solution \((u, v)\) of (1.1) exists globally.

**Proof.** The proof is similar to that of Lemma 4.6.

**Lemma 4.8.** Assume \(m_1 = p_1, m_2 > p_2\). If \(l_{11} < 1, l_{22} < \frac{m_2+1}{p_2+1}\) and \(l_{12} l_{21} \leq \left( 1 - l_{11} \right) \left( \frac{m_2+1}{p_2+1} - l_{22} \right)\), then the solution \((u, v)\) of (1.1) exists globally.

**Proof.** As in Lemma 4.5, it suffices to prove that there exist \(C(T): 0 < C(T) < +\infty\) such that (4.11) holds. To this aim, take

\[
w_i(x, t) = e_i \left\{ \left[ 1 + c(t + 1 - T_{i-1}) \right]^k - \phi_1(x) \right\} \theta = e_i y_1^\theta
\]
and
\[ z_i(x, t) = \eta_i [e^{-M(i - T_{i-1})} - \frac{1}{4}(1 - \phi_2(x))^{\frac{1}{\eta_i^2}}]^{-\sigma} = \eta_i^\sigma y_2^{-\sigma}, \]
where \( T_0 = 0, T_i = \frac{i \log 2}{M}, i = 1, \ldots, N, \theta = \frac{1}{1 - l_1}, \sigma, l, M, c \) as above, and
\[ k = \begin{cases} 1, & m_1 = p_1 \geq 1, \\ \frac{1}{m_1}, & m_1 = p_1 < 1. \end{cases} \]

Similar to the above arguments we can show that there exist positive constants \( \varepsilon_i \) and \( \eta_i \) such that \((w_i, z_i)\) is an upper solution of (1.1) with \( \hat{\lambda} \) on \( Q_i \).
By comparison principle we have \((u, v) \leq (w_i, z_i)\) on \( Q_i \), \( i = 1, \ldots, N \). Therefore, there exist \( C(T): 0 < C(T) < +\infty \) such that (4.11) holds. And hence the solution \((u, v)\) of (1.1) exists globally.

This completes the proof of Lemma 4.8.  

**Lemma 4.9.** Assume \( m_1 > p_1, m_2 = p_2 \). If \( l_{11} < \frac{m_1 + 1}{p_1 + 1}, l_{22} < 1 \) and \( l_{12} l_{21} \leq (\frac{m_1 + 1}{p_1 + 1} - l_{11})(1 - l_{22}) \), then the solution \((u, v)\) of (1.1) exists globally.

**Proof.** The proof is similar to that of Lemma 4.8.

By Lemmas 4.1–4.9 we get the proof of the sufficiency.

**5. PROOF OF THE NECESSITY**

We will complete the proof of the necessity by a series of lemmas.
In this section we denote \( \tau_i = \frac{p_i}{p_i + 1}, i = 1, 2 \) and use \( \hat{\lambda} \) to denote a positive constant satisfying \( \hat{\lambda} < \lambda \).
First we will prove a result, which are useful as we proceed, from linear algebra.

**Lemma 5.1.** Assume that \( a_{11}, a_{22} \geq 0, a_{12}, a_{21} > 0 \) and \( a_{12} a_{21} > a_{11} a_{22} \). Then for any given \( a_1, a_2 \in R^1 \), there exist \( k, l > 1 \) such that
\[ a_{11} k - a_{12} l + a_1 \leq 0, \]
\[ -a_{21} k + a_{22} l + a_2 \leq 0. \]

**Proof.** When \( a_{11} > 0, a_{22} > 0 \), by \( a_{12} a_{21} > a_{11} a_{22} \) we have \( 0 < a_{11} / a_{12} < a_{21} / a_{22} \). Therefore, there exists \( \mu > 0 \) such that \( 0 < a_{11} / a_{12} < \mu < a_{21} / a_{22} \).
Set
\[ k = \max\left\{ 2, \frac{a_1}{a_{12}/\mu} , \frac{a_2}{a_{22} / \left( \mu - \frac{a_{11}}{a_{12}} \right)} \right\}, \quad l = k\mu, \]
then \( k, l > 1 \) satisfy (5.1).

When \( a_{11} = 0, \quad a_{22} > 0 \), set
\[ l = \max\left\{ 2, \frac{a_1}{a_{12}} \right\}, \quad k = \max\{ 2, (a_{22}l + a_2)/a_{21} \}. \]

When \( a_{11} = 0, \quad a_{22} = 0 \), set
\[ l = \max\left\{ 2, \frac{a_1}{a_{12}} \right\}, \quad k = \max\left\{ 2, \frac{a_2}{a_{21}} \right\}. \]

When \( a_{11} > 0, \quad a_{22} = 0 \), set
\[ k = \max\left\{ 2, \frac{a_2}{a_{21}} \right\}, \quad l = \max\{ 2, (a_{11}k + a_1)/a_{12} \}. \]

Then \( k, l > 1 \) satisfy (5.1). \( \blacksquare \)

**Lemma 5.2.** If \( l_{11} > \min\{\frac{m_i}{p_i}, \frac{m_i + 1}{p_i + 1}\} \) or \( l_{22} > \min\{\frac{m_3}{p_3}, \frac{m_2 + 1}{p_2 + 1}\} \). Then the solution \((u, v)\) blows up in finite time.

**Proof.** When \( l_{11} > \min\{\frac{m_i}{p_i}, \frac{m_i + 1}{p_i + 1}\} \), the local solution \((u, v)\) of (1.1) satisfies \((u, v) \geq (\delta, \delta)\), it follows that
\[ u_x = \lambda u^{l_{11}} v^{l_{12}} \geq \lambda \delta^{l_{12}} u^{l_{11}}, \quad x = 1, \quad t > 0. \]

Consider the following single parabolic problem:
\[ \left( |w|^{m_i-1} w \right)_t = \left( |w_x|^{p_i-1} w_x \right)_x, \quad 0 < x < 1, \quad t > 0, \]
\[ w_x|_{x=0} = 0, \quad w_x|_{x=1} = \lambda \delta^{l_{12}} w^{l_{11}}|_{x=1}, \quad t > 0, \]
\[ w(x, 0) = u_0(x) \geq \delta > 0, \quad 0 \leq x \leq 1. \]

Then \( w(x, t) \) blows up in finite time (see [47]). Choose \((u, v) = (w, \delta)\), by using \((u, v) \geq (\delta, \delta)\), it is easy to verify that \((u, v)\) is a lower solution of (1.1) with \( \lambda < \lambda \). By the comparison principle we have that \((u, v) \geq (u, v)\), and hence \((u, v)\) blows up in finite time.

When \( l_{22} > \min\{\frac{m_3}{p_3}, \frac{m_2 + 1}{p_2 + 1}\} \), the proof is similar. \( \blacksquare \)

**Lemma 5.3.** Assume that \( m_i < p_i, \quad i = 1, 2 \). If \( l_{11} \leq \frac{m_i}{p_i}, \quad l_{22} \leq \frac{m_2}{p_2} \) and \( l_{12} l_{21} > \left( \frac{m_3}{p_3} - l_{11} \right) \left( \frac{m_2}{p_2} - l_{22} \right) \), then the solution \((u, v)\) blows up in finite time.
Proof. By Lemma 5.1 we see that there exist \( k, l > 1 \) satisfying

\[
\begin{align*}
& s = \frac{m_1 - l_{11}}{1 - \frac{m_1}{p_1}} k - \frac{l_{12}}{1 - \frac{m_2}{p_2}} l - \frac{l_{12}}{p_2 - m_2} + \frac{1 - l_{11}}{p_1 - m_1} \leq 0, \\
& r = -\frac{l_{21}}{1 - \frac{m_1}{p_1}} k + \frac{m_2 - l_{22}}{1 - \frac{m_2}{p_2}} l - \frac{l_{21}}{p_1 - m_1} + \frac{1 - l_{22}}{p_2 - m_2} \leq 0. 
\end{align*}
\]

Let \( k, l \) satisfy (5.2), define \( y(x, t) = a(1 + \frac{1}{x^2}) + (b - ct)^{-k} \), \( z(x, t) = \frac{1}{x} \) \( (1 + x^2) + (b - ct)^{-l} \), \( \theta = (p_1 + \frac{1}{k})/(p_1 - m_1) \), \( \sigma = (p_2 + \frac{1}{k})/(p_2 - m_2) \) and take

\[
\begin{align*}
& u(x, t) = y^0, \\
& v(x, t) = z^\sigma,
\end{align*}
\]

where

\[
\begin{align*}
& b = \max\{(\delta^{-1} 2^{\theta})^{1/k\theta}, (\delta^{-1} 2^{\sigma})^{1/\ell\sigma}\}, \\
& a = \min\{2^{-1} b^{-k}, 2^{-1} b^{-l}, \frac{1}{x^2} \beta^{-1} b^\alpha 2^{1+l_1(\theta-1)}, \frac{1}{x^2} \beta^{-1} b^\alpha 2^{1+l_2(\sigma-1)}\}, \\
& c = \min\left\{\frac{a^{p_1} \beta^{p_1-1}}{km_1^{\tau_1^{p_1}}}, \frac{a^{p_2} \sigma^{p_2-1}}{lm_2^{\tau_2^{p_2}}}\right\}.
\end{align*}
\]

By direct computations we have for \( x \in (0, 1), 0 < t < b/c \)

\[
\begin{align*}
& (u^{m_1})_t = m_1 \theta \kappa y^{m_1, \theta-1}(b - ct)^{-k}, \\
& \quad \frac{u_x}{x} = \frac{a^\theta}{\tau_1} y^{\theta-1} \frac{1}{x^{\tau_1-1}}, \\
& ((u_x)^{p_1})_x = \left( \frac{a^\theta}{\tau_1} \right)^{p_1} (y^{p_1(\theta-1)} x)_x \\
& \quad = \left( \frac{a^\theta}{\tau_1} \right)^{p_1} \left( y^{p_1(\theta-1)} + \frac{a p_1 (\theta - 1)}{\tau_1} y^{p_1(\theta-1)-1} \frac{1}{x^{\tau_1-1}} \right).
\end{align*}
\]

Since \( \theta = (p_1 + \frac{1}{k})/(p_1 - m_1) > \frac{p_1}{p_1 - m_1} > 1 \), by \( y \geq (b - ct)^{-k} \) we have

\[
\begin{align*}
((u_x)^{p_1})_x \geq \left( \frac{a^\theta}{\tau_1} \right)^{p_1} y^{m_1, \theta-1} = \left( \frac{a^\theta}{\tau_1} \right)^{p_1} y^{m_1(\theta-1)} y^{(p_1 - m_1)\theta - p_1 + 1} \\
\quad = \left( \frac{a^\theta}{\tau_1} \right)^{p_1} y^{m_1(\theta-1)} y^{k+1} y^k \geq \left( \frac{a^\theta}{\tau_1} \right)^{p_1} y^{m_1(\theta-1)} (b - ct)^{-k-1} \\
\quad \geq m_1 \kappa \theta y^{m_1, \theta-1} (b - ct)^{-(k+1)} = (u^{m_1})_t, \quad x \in (0, 1), \quad 0 < t < b/c,
\end{align*}
\]
i.e.,

\[(u_x)^m_n \geq (u^m_n)_r, \quad x \in (0, 1), \quad 0 < t < b/c. \] (5.3)

Similarly, we have

\[(v_x)^m_n \geq (v^m_n)_r, \quad x \in (0, 1), \quad 0 < t < b/c. \] (5.4)

Obviously,

\[u_x|_{x=0} = v_x|_{x=0} = 0, \quad 0 < t < b/c. \] (5.5)

By the choices of \(a, b\) and \(c\) we have

\[(b - ct)^{-k} \leq y \leq 2(b - ct)^{-k}, \quad (b - ct)^{-l} \leq z \leq 2(b - ct)^{-l}. \] (5.6)

By the expressions of \(\theta, \sigma\), the assumptions of this lemma and (5.2) we have

\[\theta(1 - l_{11}) > 1, \quad -k[(1 - l_{11})\theta - 1] + l_{12}\theta = -s \geq 0. \] (5.7)

Using (5.6) and (5.7) we get

\[u_x = \frac{a\theta}{\tau_1} y^{\theta - 1} = u_i^{l_{12}} v_i^{l_{12}} \frac{a\theta}{\tau_1} y^{\theta(1 - l_{11}) - 1} z^{-l_{12}l_{12}}\]

\[\leq u_i^{l_{11}} v_i^{l_{12}} \frac{a\theta}{\tau_1} 2^{\theta(1 - l_{11}) - 1} (b - ct)^{-k[\theta(1 - l_{11}) - 1] + l_{12}l_{12}}\]

\[= u_i^{l_{11}} v_i^{l_{12}} \frac{a\theta}{\tau_1} 2^{\theta(1 - l_{11}) - 1} (b - ct)^{-s}\]

\[\leq u_i^{l_{11}} v_i^{l_{12}} \frac{a\theta}{\tau_1} 2^{\theta(1 - l_{11}) - 1} b^{-s} \leq u_i^{l_{11}} v_i^{l_{12}}, \quad x = 1, \quad 0 < t < b/c,\]

i.e.,

\[u_x \leq u_i^{l_{11}} v_i^{l_{12}}, \quad x = 1, \quad 0 < t < b/c. \] (5.8)

Similarly, we have

\[v_x \leq u_i^{l_{21}} v_i^{l_{22}}, \quad x = 1, \quad 0 < t < b/c. \] (5.9)
By the choices of $a$ and $b$ we have that for $x \in [0, 1]$,
\[
y(x, 0) = (a(1 + x^{rac{1}{m_1}}) + b^{-k})^\theta \leq (2b^{-k})^\theta \leq \delta \leq u_0(x),
\]
\[
v(x, 0) = (a(1 + x^{rac{1}{m_1}}) + b^{-l})^\sigma \leq (2b^{-l})^\sigma \leq \delta \leq v_0(x).
\]
(5.10)

From (5.3)–(5.5), (5.8)–(5.10) we see that $(y, v)$ is a lower solution of (1.1) with $\hat{\gamma}$. Therefore, $(u, v) \geq (y, v)$. Obviously, $(y, v)$ blows up in finite time. And hence $(u, v)$ blows up in finite time.

**Lemma 5.4.** Assume $m_1 < p_1$, $m_2 = p_2$. If $l_1 \leq \frac{m_1}{p_1}$, $l_2 \leq 1$ and $l_1 l_2 > \frac{(m_1 - l_1)(1 - l_2)}{(p_1 - l_1)/p_1}$, then the solution $(u, v)$ of (1.1) blows up in finite time.

**Proof.** It follows from Lemma 5.1 that there exist $k, l > 1$ such that
\[
s = \frac{m_1 - l_1}{1 - m_1/p_1} k - l_1 k + \frac{1 - l_1}{p_1 - m_1} \leq 0,
\]
\[
r = -\frac{l_2}{1 - m_1/p_1} k + (1 - l_2)l - \frac{l_2}{p_1 - m_1} + 1 \leq 0.
\]
(5.11)

Set $y(x, t) = a(1 + x^{\frac{1}{m_1}}) + (b - ct)^{-k}$, $z(x, t) = (b^{-1} + d(x^2 + x - 2))^{-1} - ct$, $\theta = (p_1 + \frac{1}{k})/(p_1 - m_1)$ and let
\[
y(x, t) = y^\theta(x, t), \quad v(x, t) = z^{-l}(x, t),
\]
where $k, l$ satisfy (5.11) and
\[
b = \max\{\delta^{-1/l}, (\delta^{-1/2})^{1/(k\theta)}\}, \quad a = \min\left\{\frac{1}{2} b^{-k}, \frac{\hat{\gamma} \tau}{\theta} b^{2l - (1 - l_1)\theta}\right\}
\]
\[
d = \min\left\{\frac{1}{4b^2} \frac{\hat{\gamma}^2 \tau^2}{3l}\right\}, \quad c = \min\left\{(dl)^{p_1} b^{p_2+2} d^2 p_2 (l - 1) a^{p_1} \tau^{p_1-1}\right\}/\left(2p_1 l m_2, \frac{km_1 \tau_1^{p_1}}{km_1 \tau_1^{p_1}}\right).
\]

By the expression of $d$ we have for $0 \leq x \leq 1, \ 0 < t < b/c$
\[
\frac{1}{2b} \leq \frac{1}{b} + d(x^2 + x - 2) \leq \frac{1}{b^2}, \quad 0 \leq z \leq 2b.
\]
(5.12)

Using (5.12), by direct computations we have
\[
(y^{m_2})_t = clm_2 z^{-m_2 l - 1}, \quad v_x = dlz^{-l-1}(b^{-1} + d(x^2 + x - 2))^{-2}(2x + 1),
\]
\[(v_p^{p^2})_x = (dl)^{p_1}(z^{-p_2(l+1)}(b^{-1} + d(x^2 + x - 2))^{-2p_2}(2x + 1)^{p_2})_x
\]
\[= (dl)^{p_1}(z^{-p_2(l+1)}(b^{-1} + d(x^2 + x - 2))^{-2p_2}p_2(2x + 1)^{p_2-1} + z^{-p_2(l+1)-1}
\]
\[\times (b^{-1} + d(x^2 + x - 2))^{-2p_2-2}(2x + 1)^{p_2+1}dp_2
\]
\[\times [l + 1 - 2z(b^{-1} + d(x^2 + x - 2))]\}
\[\geq (dl)^{p_1}z^{-p_2l-1}(2b)^{-p_2b^2p_2+2}dp_2(l - 1)
\]
\[= (dl)^{p_1}z^{-m_2l-1}(2b)^{-p_2b^2p_2+2}dp_2(l - 1)
\]
\[\geq clm_2z^{-m_2l-1} = (u^m_2)_t, \quad x \in (0, 1), \quad 0 < t < b/c,
\]
i.e.,
\[(v_p^{p^2})_x \geq (u^m_2)_t, \quad x \in (0, 1), \quad 0 < t < b/c. \quad (5.13)\]

As in Lemma 5.3, we have
\[(u^m_2)_t \geq (u^m_2)_t, \quad x \in (0, 1), \quad 0 < t < b/c. \quad (5.14)\]

Obviously,
\[u_x|_{x=0} \geq 0, \quad u|_{x=0} \geq 0, \quad 0 < t < b/c. \quad (5.15)\]

Using (5.11) we get for \(x = 1, \quad 0 < t < b/c\)
\[u_x = \frac{a\theta}{\tau_1}y^{\theta-1} \leq u^{l_{11}}_{l_{12}} \frac{a\theta}{\tau_1} y^{\theta(l(l_{11})-1)}_{l_{12}}
\]
\[\leq \frac{a\theta}{\tau_1} 2^{\theta(l(l_{11})-1)}(b - ct)^{-k[\theta(l(l_{11})-1)]+l_{12}}u^{l_{11}}_{l_{12}}
\]
\[= \frac{a\theta}{\tau_1} 2^{\theta(l(l_{11})-1)}(b - ct)^{-s}u^{l_{11}}_{l_{12}}
\]
\[\leq \frac{a\theta}{\tau_1} 2^{\theta(l(l_{11})-1)}b^{-s}u^{l_{11}}_{l_{12}}
\]
\[\leq \dot{u}^{l_{11}}_{l_{12}}, \quad x = 1, \quad 0 < t < b/c,
\]
i.e.,
\[u_x \leq \dot{u}^{l_{11}}_{l_{12}}, \quad x = 1, \quad 0 < t < b/c. \quad (5.16)\]
And
\[ v_x = 3lb^2(b - ct)^{l-1} \]
\[ = u^l v^{l_22} 3lb^2 y^{-0l_22} (b - ct)^{l_22(l-1)-1} \]
\[ = u^l v^{l_22} 3lb^2 (b - ct)^{-r} \]
\[ = u^l v^{l_22} 3lb^2 b^{-r} \leq \hat{u} u^{l_21} v^{l_22}, \]
i.e.,
\[ v_x \leq \hat{u} u^{l_21} v^{l_22}, \quad x = 1, \quad 0 < t < b/c. \quad (5.17) \]

For \( x \in [0, 1] \), we have
\[ y(x, 0) = (a(1 + x^{m_1}) + b^{-k})^0 \leq (2b^{-k})^0 \leq \delta \leq u_0(x), \]
\[ y(x, 0) = (b^{-1} + d(x^2 + x - 2))^l \leq b^{-l} \leq \delta \leq v_0(x). \quad (5.18) \]

From (5.13)–(5.18) we see that \((u, v)\) is a lower solution of (1.1) with \( \hat{u} \). Therefore, \((u, v) \geq (y, v)\). Obviously, \((u, v)\) blows up in finite time. And hence \((u, v)\) blows up in finite time.

**Lemma 5.5.** Assume \( m_1 = p_1, \ m_2 < p_2 \). If \( l_{11} \leq l_{12} \leq \frac{m_2}{p_2} \) and \( l_{12} l_{21} > (1 - l_{11})(\frac{m_3}{p_3} - l_{22}) \), then the solution \((u, v)\) of (1.1) blows up in finite time.

**Proof.** The proof is similar to that of Lemma 5.4.

**Lemma 5.6.** Assume \( m_1 = p_1, \ m_2 = p_2 \). If \( l_{11} \leq l_{12} < 1 \) and \( l_{12} l_{21} > (1 - l_{11})(1 - l_{22}) \), then the solution \((u, v)\) of (1.1) blows up in finite time.

**Proof.** The proof is similar to that of Lemma 5.4.

**Lemma 5.7.** Assume that \( m_i > p_i, \ i = 1, 2 \). If \( l_{11} < \frac{m_{i+1}}{p_{i+1}}, l_{22} < \frac{m_{i+1}}{p_{i+1}} \) and \( l_{12} l_{21} > (\frac{m_{i+1}}{p_{i+1}} - l_{11})(\frac{m_{i+1}}{p_{i+1}} - l_{22}) \), then the solution \((u, v)\) blows up in finite time.
Proof. By Lemma 5.1 we see that there exist $k, l > 1$ satisfying

$$s = \frac{m_1 + 1}{p_1 + 1} - l_{11} - k - \frac{l_{12}}{m_2 - p_2} - \frac{l_{12}}{m_1 - p_1} \leq 0,$$

$$r = -\frac{l_{21}}{m_1 + 1} - 1 + \frac{m_2 + 1}{p_2 + 1} - l_{22} - \frac{l_{21}}{p_1 - m_1} + 1 - \frac{l_{22}}{p_2 - m_2} \leq 0. \quad (5.19)$$

Define $y(x, t) = a(1 - x) + (b - ct)^k$, $z(x, t) = a(1 - x) + (b - ct)^l$, $\theta = (p_1 + \frac{k}{k})/(m_1 - p_1)$, $\sigma = (p_2 + \frac{l}{l})/(m_2 - p_2)$, $y(x, t) = y^{-\theta}$, $\varphi(x, t) = z^{-\sigma}$, where $k, l$ satisfy (5.19) and

$$b = \max\{(\delta^2)^{-1/(k\theta)}, (\delta^2)^{-1/(l\sigma)}\},$$

$$a = \min\left\{b^k, b^l, \frac{b^k}{\theta}, \frac{b^l}{\sigma}\right\},$$

$$c = \min\left\{\frac{(a\theta)^{p_1}}{m_1\theta}, \frac{(a\sigma)^{p_2}}{m_2\sigma} (\sigma + 1)\right\}.$$ By direct computations we have

$$(u^{m_1})_t = m_1\theta cy^{-m_1\theta - 1}(b - ct)^{k - 1}, \quad u_s = a\theta y^{-\theta - 1},$$

$$(u_x)^{p_1} = (a\theta)^{p_1} (y^{p_1(-\theta - 1)})_x$$

$$= (a\theta)^{p_1} a p_1 (\theta + 1) y^{-m_1\theta - 1} y^{(m_1 - p_1)\theta - p_1}$$

$$\geq (a\theta)^{p_1} a p_1 (\theta + 1) y^{-m_1\theta - 1} (b - ct)^{p_1(m_1 - p_1)\theta - p_1}$$

$$= (a\theta)^{p_1} a p_1 (\theta + 1) y^{-m_1\theta - 1} (b - ct)^{k - 1}$$

$$\geq m_1\theta cy^{-m_1\theta - 1}(b - ct)^{k - 1} = (u^{m_1})_t, \quad x \in (0, 1), \quad 0 < t < b/c$$

(here we have used that $\theta > \frac{p_1}{m_1 - p_1}$)

i.e.,

$$(u_x)^{p_1} \geq (u^{m_1})_t, \quad x \in (0, 1), \quad 0 < t < b/c. \quad (5.20)$$

Similarly, we have

$$(u_x)^{p_2} \geq (u^{m_2})_t, \quad x \in (0, 1), \quad 0 < t < b/c. \quad (5.21)$$
Obviously,
\[ u_x|_{x=0} = u_x|_{x=0} = 0, \quad 0 < t < b/c. \] (5.22)

By the expressions of \( \theta, \sigma \), the assumptions of this lemma and (5.19) we have
\[ l_{12} l_\sigma + k(l_{11} \theta - 1 - \theta) = -s \geq 0. \] (5.23)

Using (5.23) we get
\[
\begin{align*}
    u_x &= \theta(b - ct)^{-k(\theta+1)} = u_x^{l_{11}} u_x^{l_{12}} \theta(b - ct)^{-k(\theta+1) + kl_{11} \theta + l_{12} \sigma l} \\
    &= u_x^{l_{11}} u_x^{l_{12}} \theta(b - ct)^{-s} \leq u_x^{l_{11}} u_x^{l_{12}} \theta b^{-s} \leq u_x^{l_{11}} u_x^{l_{12}}, \quad x = 1, \quad 0 < t < b/c,
\end{align*}
\]
i.e.,
\[ u_x \leq u_x^{l_{12}}, \quad x = 1, \quad 0 < t < b/c. \] (5.24)

Similarly, we have
\[ v_x \leq v_x^{l_{12}}, \quad x = 1, \quad 0 < t < b/c. \] (5.25)

By the choices of \( a \) and \( b \) we have that for \( x \in [0, 1] \),
\[
\begin{align*}
    u(x, 0) &= (a(1 - x) + b^k)^{-\sigma} \leq (2b)^{-\sigma} \leq \delta \leq u_0(x), \\
    v(x, 0) &= (a(1 - x) + b^{-l})^{-\sigma} \leq (2b)^{-\sigma} \leq \delta \leq v_0(x).
\end{align*}
\] (5.26)

From (5.20)–(5.22), (5.24)–(5.26) we see that \( (u, v) \) is a lower solution of (1.1) with \( \delta \). Therefore \( (u, v) \geq (u, v) \). Obviously, \( (u, v) \) blows up in finite time. And hence \( (u, v) \) blows up in finite time. \( \blacksquare \)

**Lemma 5.8.** Assume that \( m_1 = p_1 \) and \( m_2 > p_2 \). If \( l_{11} \leq 1, \ l_{22} \leq \frac{m_1 + 1}{p_2 + 1} \), and \( l_{12} l_{21} > (1 - l_{11})(\frac{m_2 + 1}{p_2 + 1} - l_{22}) \), then the solution \( (u, v) \) blows up in finite time.

**Proof.** By Lemma 5.1, we see that there exist \( k, l > 1 \) satisfying
\[
\begin{align*}
    s &= (1 - l_{11})k - \frac{l_{12}}{\frac{m_2 + 1}{p_2 + 1} - 1} l + \frac{l_{12}}{m_2 - p_2} + 1 \leq 0, \\
    r &= -l_{21} k + \frac{m_2 + 1}{p_2 + 1} - l_{22} \frac{m_2 + 1}{p_2 + 1} - 1 l + \frac{1 - l_{22}}{p_2 - m_2} \leq 0.
\end{align*}
\]

Let \( k, l \) be as above, define \( y(x, t) = (b^{-1} + d(x^2 + x - 2))^{-1} - ct \), \( z(x, t) = a(1 - x) + (b - ct)^l \), \( \sigma = (p_2 + \frac{l_{11} k}{l_{12}})/(m_2 - p_2) \), \( u(x, t) = y^{-k}, v(x, t) = z^{-\sigma} \).
As in Lemmas 5.4 and 5.7, it is easy to verify that there exist positive constants $a, b, c, d$ such that $(u, v)$ is a lower solution of (1.1) with $\hat{v}$. Therefore, $(u, v) \not\geq (u, v)$. Obviously, $(u, v)$ blows up in finite time. And hence $(u, v)$ blows up in finite time.  

**Lemma 5.9.** Assume that $m_1 > p_1$ and $m_2 = p_2$. If $l_{11} \leq \frac{m_1 + 1}{p_1 + 1}$, $l_{22} \leq 1$ and $l_{12}l_{21} > \left(\frac{m_1 + 1}{p_1 + 1} - l_{11}\right)(1 - l_{22})$, then the solution $(u, v)$ blows up in finite time.

**Proof.** The proof is similar to that of Lemma 5.8.

**Lemma 5.10.** Assume that $m_1 > p_1$ and $m_2 < p_2$. If $l_{11} \leq \frac{m_1 + 1}{p_1 + 1}$, $l_{22} \leq \frac{m_2}{p_2}$ and $l_{12}l_{21} > \left(\frac{m_1 + 1}{p_1 + 1} - l_{11}\right)\left(\frac{m_2}{p_2} - l_{22}\right)$, then the solution $(u, v)$ blows up in finite time.

**Proof.** By Lemma 5.1 we see that there exist $k, l > 1$ satisfying

$$s = \frac{m_1 + 1}{p_1 + 1} - \frac{l_{11}}{1 - \frac{m_1 + 1}{p_1 + 1}} - 1 - l_{12}, \quad r = -\frac{l_{21}}{1 - \frac{m_1 + 1}{p_1 + 1}} - 1 + \frac{m_2}{p_2} - l_{22}.$$

Let $k_2l$ be as above, define $y(x, t) = a(1 - x) + (b - ct)^k$, $z(x, t) = a(1 + x^2) + (b - ct)^{-l}$, $\sigma = (p_2 + \frac{1}{l})/(p_2 - m_2)$, $\theta = (p_1 + 1 - \frac{1}{k})/(m_1 - p_1)$, $\varphi(x, t) = y^\sigma$, $\psi(x, t) = z^\theta$.

As in Lemmas 5.3 and 5.7, it is easy to verify that there exist positive constants $a, b, c$ such that $(u, v)$ is a lower solution of (1.1) with $\hat{v}$. Therefore, $(u, v) \not\geq (u, v)$. Obviously, $(u, v)$ blows up in finite time. And hence $(u, v)$ blows up in finite time.

**Lemma 5.11.** Assume that $m_1 < p_1$ and $m_2 > p_2$. If $l_{11} \leq \frac{m_1}{p_1}$, $l_{22} \leq \frac{m_2 + 1}{p_2 + 1}$ and $l_{12}l_{21} > \left(\frac{m_1}{p_1} - l_{11}\right)\left(\frac{m_2 + 1}{p_2 + 1} - l_{22}\right)$, then the solution $(u, v)$ blows up in finite time.

**Proof.** The proof is similar to that of Lemma 5.10.

By Lemmas 5.2–5.11 we get that the necessity holds.

**Remark 5.1.** The above blow up results turn out to be valid for the following $N$-dimensional form of problem (1.1):

$$(u^{m_1})_t = \sum_{i=1}^{N}(|u_{x_i}|^{m_1 - 1}u_{x_i})_{x_i}, \quad x \in \Omega, \quad t > 0,$$

$$(v^{m_2})_t = \sum_{i=1}^{N}(|v_{x_i}|^{m_2 - 1}v_{x_i})_{x_i}, \quad x \in \Omega, \quad t > 0,$$
\[ \begin{align*}
\frac{\partial u}{\partial \eta} &= u^{11} v^{12}, \quad x \in \partial \Omega, \quad t > 0 \\
\frac{\partial v}{\partial \eta} &= u^{21} v^{22}, \quad x \in \partial \Omega, \quad t > 0 \\
 u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \tilde{\Omega},
\end{align*} \]

where \( \Omega \subset \mathbb{R}^N \) is a bounded open set with smooth boundary \( \partial \Omega \), \( \eta \) stands the unit outward normal of \( \partial \Omega \) to \( \Omega \).

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**REFERENCES**


