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On the distance from a rational power to the nearest integer

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Abstract

We prove that for any non-zero real number ξ the sequence of fractional parts $\{\xi(3/2)^n\}$, $n = 1, 2, 3, \ldots,$ contains at least one limit point in the interval $[0.238117\ldots, 0.761882\ldots]$ of length $0.523764\ldots$. More generally, it is shown that every sequence of distances to the nearest integer $||\xi(p/q)^n||$, $n=1, 2, 3, \ldots,$ where $p/q > 1$ is a rational number, has both 'large' and 'small' limit points. All obtained constants are explicitly expressed in terms of p and q. They are also expressible in terms of the Thue–Morse sequence and, for irrational ξ, are best possible for every pair $p > 1$, $q = 1$. Furthermore, we strengthen a classical result of Pisot and Vijayaraghavan by giving similar effective results for any sequence $||\xi\alpha^n||$, $n = 1, 2, 3, \ldots,$ where $\alpha > 1$ is an algebraic number and where $\xi \neq 0$ is an arbitrary real number satisfying $\xi \notin \mathbb{Q}(\alpha)$ in case α is a Pisot or a Salem number.

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1. Introduction

Let throughout $\alpha > 1$ be an algebraic number, and let $p > q \geqslant 1$ be two coprime positive integers. Write $[x]$ and $\{x\}$ for the integer and the fractional parts of a real number x, respectively. Let $||x||$ be the distance between x and the nearest integer to x, so that $||x|| = \min(\{x\}, 1 - \{x\})$. Let also $\xi \neq 0$ and η be fixed real numbers.

The distribution of the sequences $\{\xi\alpha^n + \eta\}$, $n = 1, 2, 3, \dots$, in general, and $\{\xi(p/q)^n\}$, $n = 1, 2, 3, \dots$, in particular, is a subject of intensive studies. The behavior of the sequences $\{\xi\alpha^n\}$, $n = 1, 2, 3, \dots$, and $||\xi\alpha^n||$, $n = 1, 2, 3, \dots$, is different depending on arithmetical nature of α. To be precise, it depends on whether α is (or is not) an algebraic integer which has no other conjugates outside the unit circle. This was noticed already by Pisot [25], Vijayaraghavan and Salem [27] (see also [8,11]). Later, such algebraic numbers were named after them. More precisely, an algebraic integer $\alpha > 1$ is called a *PV-number* (or a *Pisot and Vijayaraghavan* number, or simply a *Pisot* number) if its other conjugates (if any) lie in the open unit disc $|z| < 1$. An algebraic integer $\alpha > 1$ is called a *Salem number* if its other conjugates lie in the unit disc $|z| \leqslant 1$ with at least two conjugates lying on $|z| = 1$.

In terms of the distance to the nearest integer one can express their results as follows. Suppose that $\varepsilon > 0$ is an arbitrary positive number. Then, for each α which is a Pisot or a Salem number, there is a non-zero $\xi \in \mathbb{Q}(\alpha)$, such that $||\xi\alpha^n|| < \varepsilon$ for every $n \in \mathbb{N}$. See, e.g., [8,11] for a classical version of these results and also [16,34] for the 'fractional part' versions of this theorem for Pisot and Salem numbers, respectively. In all other cases, the sequence $||\xi\alpha^n||$, $n = 1, 2, 3, \dots$, has a limit point which is greater than a constant depending on α only. However, so far no such constant was given explicitly, so we begin with the following effective version of this statement.

Theorem 1. *Let $\alpha > 1$ be a real algebraic number and let ξ be a non-zero real number lying outside the field $\mathbb{Q}(\alpha)$ in case α is a Pisot or a Salem number. Then the sequence $||\xi\alpha^n||$, $n = 1, 2, 3, \dots$, has a limit point $\geqslant 1/\min(L(\alpha), 2\ell(\alpha))$.*

Here, $L(\alpha) = L(P) = \sum_{k=0}^{d} |a_k|$ is the length of the minimal polynomial

$$P(z) = a_d z^d + \cdots + a_1 z + a_0 \in \mathbb{Z}[z]$$

of α over $\mathbb{Q}$. The quantity $\ell(\alpha)$ is called the *reduced length* of α. It is defined by $\ell(\alpha) = \ell(P) = \inf L(PG)$, where the infimum is taken over every polynomial $G(z) \in \mathbb{R}[z]$ whose either leading or constant coefficient is equal to 1. This quantity was introduced by the author in [14] and then studied in detail by Schinzel [28].

The bound $1/2\ell(\alpha)$ of Theorem 1 follows from the next result.

Theorem 2. *Let $\alpha > 1$ be a real algebraic number, $\eta \in \mathbb{R}$, and let ξ be a non-zero real number lying outside the field $\mathbb{Q}(\alpha)$ in case α is a Pisot or a Salem number. Then the difference between the largest and the smallest limit points of the sequence $\{\xi\alpha^n + \eta\}$, $n = 1, 2, 3, \dots$, is at least $1/\ell(\alpha)$.*

The problems related to Theorem 2 were raised by Vijayaraghavan [32] and Mahler [24] who asked whether there exist $\xi > 0$, such that $\{\xi(3/2)^n\} < 1/2$ for each $n \in \mathbb{N}$. Such ξ, if exist, are called *Mahler's Z-numbers*. Despite some efforts, no serious progress towards showing that Mahler's Z-numbers do not exist (which is widely believed) was achieved until the work of Flatto et al. [19] (see also [18]). They were the first to prove an effective inequality for the difference between the largest and the smallest limit points of the sequence $\{\xi(p/q)^n\}$, $n = 1, 2, 3, \ldots$. To be precise, they proved that this difference is at least $1/p$ and, in general, no better bound is known, although there are several variations of their inequality that in some sense explain the phenomenon of $1/p$ [1,9,15,29]. Recently, the author proved Theorem 2 for $\eta = 0$. Since $\ell(p/q) = p$ (see [14] or [28]), the inequality of Flatto et al. [19] is a particular case Theorem 2 for $\eta = 0$ and $\alpha = p/q$. The proof of Theorem 2 is essentially the same as that of its particular case with $\eta = 0$ [14].

As we already said above, $\ell(p/q) = p$. Hence, for every rational number $\alpha = p/q > 1$, we have $2p = 2\ell(p/q) > p + q = L(p/q)$. (Although, for some α, the reverse inequality $2\ell(\alpha) < L(\alpha)$ holds.) Consequently, Theorem 1 implies that, for any non-zero ξ which is, in addition, irrational if $q = 1$, the sequence $||\xi(p/q)^n||$, $n = 1, 2, 3, \ldots$, has a limit point greater than or equal to $1/(p + q)$.

The aim of this paper is to improve this bound. Set

$$T(z) := \prod_{m=0}^{\infty} (1 - z^{2^m}) \tag{1}$$

and

$$E(z) := \frac{1 - (1 - z)T(z)}{2z}. \tag{2}$$

The main result of this paper is the following statement.

Theorem 3. *Let ξ be a non-zero real number and let $p/q > 1$, $\gcd(p, q) = 1$, be a rational number. Suppose that ξ is, in addition, irrational if $q = 1$. Then the sequence $||\xi(p/q)^n||$, $n = 1, 2, 3, \ldots$, has a limit point greater than or equal to $E(q/p)/p$, and a limit point smaller than or equal to $1/2 - (1 - e(q/p)T(q/p))/2q$, where $e(q/p) = 1 - q/p$ if $p + q$ is even and $e(q/p) = 1$ if $p + q$ is odd.*

Note that, by (1) and (2), $1/(p + q) < E(q/p)/p = (1 - (1 - q/p)T(q/p))/2q$, because $(1 + q/p)T(q/p) < 1$, so Theorem 3 improves the bound $1/(p + q)$ for every rational number $p/q > 1$.

Usually, the powers of $3/2$ are of additional interest, because of their connection with Mahler's and Waring's problems (see, e.g., [31] for more references concerning the latter). So we begin explaining the implications of Theorem 3 with its simple numerical restatement for $p/q = 3/2$. (Note that $e(3/2) = 1$, since $3 + 2 = 5$ is odd.)

Corollary 1. *If* $\xi \neq 0$ *then the sequence* $||\xi(3/2)^n||$, $n = 1, 2, 3, \ldots,$ *has a limit point greater than or equal to* $(3 - T(2/3))/12 = 0.238117\ldots$ *and a limit point smaller than or equal to* $(1 + T(2/3))/4 = 0.285647\ldots.$

In other words, the first part of Corollary 1 says that, for any $\xi \neq 0$, the interval $[0.238117\ldots, 0.761882\ldots]$ of length $0.523764\ldots$ contains a limit point of the sequence $\{\xi(3/2)^n\}$, $n = 1, 2, 3, \ldots$. This shows the progress towards Mahler's conjecture which can be also stated in the following stronger form: prove that the interval $(1/2, 1]$ (or even any subinterval of $[0, 1]$ of length $1/2$) always contains a limit point of the sequence $\{\xi(3/2)^n\}$, $n = 1, 2, 3, \ldots,$ where $\xi \neq 0$. Although the interval is 'wrong', the progress from earlier results proving the same for all intervals of length $2/3$ to the interval of length $0.523764\ldots$ which is only just greater than $1/2$ is obvious. (Formally, this result for all intervals of length $2/3$ only follows from Theorem 2 and not from earlier results.) In the opposite direction, Akiyama et al. proved recently [2] that there exists a non-zero ξ, such that $||\xi(3/2)^n|| < 1/3$ for every $n \in \mathbb{N}$. So the constant $(3-T(2/3))/12 = 0.238117\ldots$ of Corollary 2 cannot be replaced by a constant greater than $1/3$. On the other hand, Pollington [26] showed that there is a non-zero ξ, such that $||\xi(3/2)^n|| > 4/65$ for every $n \in \mathbb{N}$, so the constant $(1+T(2/3))/4 = 0.285647\ldots$ cannot be replaced by a constant smaller than $4/65$.

Apparently, Theorem 3 is the best result which one can obtain with the tools developed in [14–17]. This is shown by the following corollary stating that the bounds of Theorem 3 for $q = 1$ and $p > 1$ are sharp.

Corollary 2. *Let* ξ *be an irrational number and let* $p > 1$ *be an integer. Then the sequence* $||\xi p^n||$, $n = 1, 2, 3, \ldots,$ *has a limit point greater than or equal to* $\xi_p := E(1/p)/p$, *and a limit point smaller than or equal to* $\hat{\xi}_p := e(1/p)T(1/p)/2$, *where* $e(1/p) = 1 - 1/p$ *if* p *is odd, and* $e(1/p) = 1$ *if* p *is even. Furthermore, both bounds are best possible: in particular,* $\xi_p, \hat{\xi}_p \notin \mathbb{Q}$ *and* $||\xi_p p^n|| < \xi_p$, $||\hat{\xi}_p p^n|| > \hat{\xi}_p$ *for every* $n \in \mathbb{N}$.

As an example, we give a numerical version of the lower bound of Corollary 2 corresponding to $p = 10$.

Corollary 3. *If* ξ *is an irrational number then the sequence* $||\xi 10^n||$, $n = 1, 2, 3, \ldots,$ *has a limit point greater than or equal to*

$$\xi_{10} = E(1/10)/10 = 0.099090099009099090090990099909009\ldots.$$

Furthermore, $\xi_{10} \notin \mathbb{Q}$ *and* $||\xi_{10}10^n|| < \xi_{10}$ *for every* $n \in \mathbb{N}$.

A reader having experience with automatic sequences will recognize the sequence corresponding to the digits 0 and 9 of ξ_{10} immediately. It is the Thue–Morse sequence (see Section 3 for definitions), because, by (1) and (2), $E(z) = (1 - z)(z + z^2 + z^4 + z^6 + \ldots)$, where the coefficients $0, 1$ of the series correspond to the $0, 1$ elements in the Thue–Morse sequence. We shall use Thue–Morse and some other automatic sequences in the proof of Theorem 3. A version of Corollary 3 (although not equivalent to Corollary 3) concerning the upper bound $\hat{\xi}_2$ corresponding to $p = 2$ was known before. See, e.g., [4,5]. The same result for any other pair $p > 1$, $q = 1$ can be derived using certain extremal properties of the Thue–Morse sequence [23]. The main difficulties in the proof of Theorem 3 arise from the case when q/p is large, say, greater than $(\sqrt{5} - 1)/2$, because the proof for small q/p, say for $q/p \leqslant 1/2$, can be obtained by combining the ideas of [14] with the results of combinatorics on words [4–6,23]. (Then, in the sense of Section 3, a greater value is attached to a greater word; this is not true for 'large' $r = q/p$.)

The problems concerning fractional parts of rational powers are closely related to corresponding problems for integer parts. For instance, Mahler's Z-numbers do not exist if for each $\xi > 0$ the sequence $[\xi(3/2)^n]$, $n = 1, 2, 3, \dots,$ contains infinitely many odd numbers. Surprisingly, it is not known whether, for each fixed $\xi > 0$, the sequence $[\xi(p/q)^n]$, $n = 1, 2, 3, \dots,$ contains infinitely many composite numbers or not (see, e.g., [21, Problem E19]). This was only proved for $p/q = 3/2$, $p/q = 4/3$ [20], and for $p/q = 5/4$ [17]. See also [3,7,10,13] for other results about prime and composite numbers of the form $[\xi\alpha^n]$. One should mention that the problems concerning $[\xi\alpha^n]$ and $\{\xi\alpha^n\}$ with real $\alpha > 1$ and $\xi \neq 0$ are extremely difficult and the progress is slow only when one considers specific values of ξ and α. Metrical results are well-known from the work of Weyl [33] and Koksma [22]; see, e.g., [7] for an example of such result and also [30] for a result concerning any (not necessarily algebraic) $\alpha > 1$.

Recall that the sequence $s_1, s_2, s_3, \dots$ is called *ultimately periodic* if there is $t \in \mathbb{N}$, such that $s_{n+t} = s_t$ for all sufficiently large n. We shall derive Theorem 3 from the next result which is of independent interest.

Theorem 4. *Let $s_1, s_2, s_3, \dots$ be a sequence of integers which is not ultimately periodic, and let r be a fixed real number satisfying $0 < r < 1$. Then, for each $\varepsilon > 0$, there are infinitely many $l \in \mathbb{N}$, such that*

$$|s_l + s_{l+1}r + s_{l+2}r^2 + \dots| > E(r) - \varepsilon. \tag{3}$$

Similarly, if $s_1, s_2, s_3, \dots$ is a sequence of odd integers which is not ultimately periodic, then, for each $\varepsilon > 0$, there are infinitely many $l \in \mathbb{N}$ for which

$$|s_l + s_{l+1}r + s_{l+2}r^2 + \dots| > (1 - T(r))/r - \varepsilon. \tag{4}$$

Furthermore, both inequalities (3) and (4) are best possible. In Section 4, we will construct sequences of integers $s_1, s_2, s_3, \dots$ and of odd integers $\hat{s}_1, \hat{s}_2, \hat{s}_3, \dots$ (in terms of the Thue–Morse sequence) which are not ultimately periodic and satisfy $|s_l + s_{l+1}r + s_{l+2}r^2 + \dots| < E(r)$ and $|\hat{s}_l + \hat{s}_{l+1}r + \hat{s}_{l+2}r^2 + \dots| < (1 - T(r))/r$ for every $l \in \mathbb{N}$.

In the next section, we will recall our earlier results and prove Theorems 1 and 2. All results related to automatic sequences are given in Section 3. Section 4 contains the proofs of Theorems 3, 4 and Corollary 2. (Recall that Corollary 1 is just a numerical version of Theorem 3 for $p/q = 3/2$, whereas Corollary 3 is a numerical version of Corollary 2 for $p = 10$.)

2. Earlier results

Let throughout $x_n = [\xi\alpha^n + \eta]$ and $y_n = \{\xi\alpha^n + \eta\}$. Since $\alpha^n P(\alpha) = a_d\alpha^{n+d} + \cdots + a_1\alpha^{n+1} + a_0\alpha^n = 0$ and $\xi\alpha^n = x_n + y_n - \eta$, we have

$$s_n := a_d x_{n+d} + \cdots + a_1 x_{n+1} + a_0 x_n = -a_d y_{n+d} - \cdots - a_1 y_{n+1} - a_0 y_n + P(1)\eta. \quad (5)$$

In particular, setting $\eta = 1/2$, we see that $x_n = [\xi\alpha^n + 1/2]$ is the nearest integer to $\xi\alpha^n$ and $|y_n - 1/2| = |\{\xi\alpha^n + 1/2\} - 1/2| = ||\xi\alpha^n||$.

The key lemma in [14] was the following:

Lemma 1. *The sequence $s_1, s_2, s_3, \ldots$ is not ultimately periodic, unless α is a Pisot number or a Salem number and $\xi \in \mathbb{Q}(\alpha)$.*

Its proof is given in [14] for $\xi > 0$ and $\eta = 0$. It is completely independent of η (since, assuming that it is periodic with period t, we work with the difference $s_{n+t} - s_n$ cancelling the term depending on η in (5)) and carries over without change to arbitrary real η and to arbitrary real $\xi \neq 0$.

We will combine this lemma with a simple combinatorial result of [14]:

Lemma 2. *Assume that an infinite sequence of letters which belong to a finite alphabet $\{\kappa_1, \ldots, \kappa_n\}$ is not ultimately periodic. Then, for every $N \in \mathbb{N}$, there is a pattern U of length N and two different letters κ_i and κ_j, such that the sequence contains infinitely many patterns of the form $\kappa_i U$ and $\kappa_j U$. Similarly, there is a pattern U' of length N and two different letters $\kappa_{i'}$ and $\kappa_{j'}$, such that the sequence contains infinitely many patterns of the form $U'\kappa_{i'}$ and $U'\kappa_{j'}$.*

An alternative proof of Lemma 2 can be given using Theorem 10.2.6 of Allouche and Shallit [6]. In [14], Lemma 2 is stated with 'of length N' replaced by a weaker statement 'of length at least N'. Evidently, the weaker statement implies the stronger statement immediately, because we can disregard the end of U and the beginning of U'.

Lemma 3. *Let d be a fixed positive integer, and let $q_d, \ldots, q_1, q_0$, where $q_d q_0 \neq 0$, be real numbers. Suppose that $\vartheta_n \in \mathbb{R}$, $n = 1, 2, \ldots,$ satisfy the linear recurrent relation*

$$q_d\vartheta_{n+d} + \cdots + q_1\vartheta_{n+1} + q_0\vartheta_n = s_n,$$

where each element of the sequence $s_1, s_2, s_3, \ldots$ belongs to a set of real numbers $\mathcal{S}$. Let $Q(z) = q_d z^d + \cdots + q_1 z + q_0$. If $|s_n| \geqslant \hat{s}$ for infinitely many n then

$$|\vartheta_n| \geqslant \hat{s}/L(Q) \quad (6)$$

for infinitely many $n \in \mathbb{N}$. *Moreover, if* $\mathcal{S}$ *is finite and the sequence* $s_1, s_2, s_3, \dots$ *is not ultimately periodic then*

$$\limsup_{n\to\infty} \vartheta_n - \liminf_{n\to\infty} \vartheta_n \geqslant s^*/\ell(Q), \tag{7}$$

where s^* *is the smallest non-zero distance between two elements of* $\mathcal{S}$.

Proof. Observing that there are infinitely many n for which

$$\hat{s} \leqslant |s_n| \leqslant L(Q) \max_{0\leqslant k\leqslant d} |\vartheta_{n+k}|$$

we obtain (6) immediately.

Set $L^+(F)$ and $-L^-(F)$ for the sum of positive and negative coefficients of a polynomial $F(z) \in \mathbb{R}[z]$, respectively, so that $L^+(F) + L^-(F) = L(F)$. Let also $\mu = \limsup_{n\to\infty} \vartheta_n$ and $\lambda = \liminf_{n\to\infty} \vartheta_n$. For the proof of (7), we fix $\epsilon > 0$ and assume that there is a polynomial $G(z) = 1+b_1z+\dots+b_mz^m \in \mathbb{R}[z]$, such that $\ell(Q) > L(QG) - \epsilon$. By Lemma 2, there are infinitely many n (say, of the first kind), such that $\overline{s_ns_{n+1}\dots s_{n+m}} = s''U$, and infinitely many n (say, of the second kind) for which $\overline{s_ns_{n+1}\dots s_{n+m}} = s'U$, where $s''-s' \geqslant s^*$. Fix $\varepsilon > 0$. By choosing a sufficiently large n of the first kind and multiplying the equalities $q_d\vartheta_{n+j+d} + \dots + q_1\vartheta_{n+j+1} + q_0\vartheta_{n+j} = s_{n+j}$, where $j = 0, 1, \dots, m$, by $1, b_1, \dots, b_m$, respectively, and adding them we obtain $L^+(QG)(\mu+\varepsilon) - L^-(QG)(\lambda-\varepsilon) \geqslant s'' + c(U)$, where $c(U)$ is a constant depending on U and on $b_1, \dots, b_m$, but not on n. Similarly, by taking a large n of the second kind, we get $L^-(QG)(\mu+\varepsilon) - L^+(QG)(\lambda-\varepsilon) \geqslant -s' - c(U)$. Adding both inequalities we get $L(QG)(\mu-\lambda+2\varepsilon) \geqslant s''-s' \geqslant s^*$. Since $L(QG) < \ell(Q)+\epsilon$, and both ϵ and ε can be taken arbitrarily small, this yields $\ell(Q)(\mu-\lambda) \geqslant s^*$, that is (7). □

The alternative case, when there is a polynomial $G(z) = b_0+b_1z+\dots+b_{m-1}z^{m-1}+z^m \in \mathbb{R}[z]$, such that $\ell(Q) > L(QG)-\epsilon$, can be treated in the same manner using the second part of Lemma 2.

Proof of Theorem 2. Let us write (5) in the form $a_dy_{n+d} + \dots + a_1y_{n+1} + a_0y_n = -s_n+P(1)\eta$. Here, the right-hand sides, $-s_n+P(1)\eta$, take values of the form $\mathbb{Z}+P(1)\eta$. Their moduli are bounded from above by $L(P)$, so there are only finitely many of them. By Lemma 1, the sequence, $-s_n + P(1)\eta$, $n = 1, 2, 3, \dots$, is not ultimately periodic. The difference between two distinct values of this sequence is at least 1. Now, using (7) we deduce that $\limsup_{n\to\infty} y_n - \liminf_{n\to\infty} y_n \geqslant 1/\ell(P)$, as claimed. □

Proof of Theorem 1. If the largest limit point of the sequence $||\xi\alpha^n||$, $n = 1, 2, 3, \dots$, is strictly smaller than $1/2\ell(\alpha)$, then the limit points of the sequence $\{\xi\alpha^n + 1/2\}$, $n = 1, 2, 3, \dots$, all belong to the open interval $((1 - 1/\ell(\alpha))/2, (1 + 1/\ell(\alpha))/2)$ of length $1/\ell(\alpha)$, a contradiction with Theorem 2. (In fact, by the results of Section 6 in [14] and more general results of Schinzel [28], $\ell(\alpha) \geqslant 2$ for each non-zero algebraic α.)

The bound $1/L(P)$ follows from (6). Indeed, by Lemma 1, there are infinitely many n for which $|s_n| \geqslant 1$. Using $P(1) = a_d + \dots + a_1 + a_0$ we can write (5) in the form

$$s_n = -a_d(y_{n+d} - 1/2) - \dots - a_1(y_{n+1} - 1/2) - a_0(y_n - 1/2),$$

where $|y_n - 1/2| = ||\xi\alpha^n||$. Now, (6) implies that $||\xi\alpha^n|| \geqslant 1/L(P)$ for infinitely many $n \in \mathbb{N}$ which is more than required. $\square$

3. Automatic sequences

In this section, several infinite sequences will be used. (N.J.A. Sloane in his on-line encyclopedia of integer sequences http://www.research.att.com/~njas/sequences/ assigned to them the numbers $A001285$, $A026465$, $A003159$, respectively.) The best known is the *Thue–Morse sequence* usually given by

$$0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, \dots .$$

It begins with 0 and is obtained by making infinitely many steps, where at each step 0 is replaced by the pattern $0, 1$ and 1 is replaced by the pattern $1, 0$. There are many equivalent definitions of this sequence: see, e.g., [5]. Throughout, we will denote the elements of the Thue–Morse sequence by $t_0, t_1, t_2, t_3, \dots$.

Less well known, but most important to us, is the sequence of the number of consecutive identical symbols in the Thue–Morse sequence ($A026465$)

$$1, 2, 1, 1, 2, 2, 2, 1, 1, 2, 1, 1, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 2, 1, 1, 2, 1, 1, 2,$$
$$1, 1, 2, 2, \dots$$

which we will call the *pairs in Thue–Morse sequence*. Its elements will be denoted by $p_0, p_1, p_2, p_3, \dots$.

Let us order the words (finite and infinite) of the alphabet $\{1, 2\}$ as follows. If $\mathbf{v} \neq \mathbf{v}'$ and neither word is a prefix (beginning) of the other, then there is a smallest positive integer k, such that the first $k-1$ symbols in both $\mathbf{v}$ and $\mathbf{v}'$ coincide, but their kth symbols are different, say 2 and 1, respectively. Then, we define their order by $\mathbf{v} > \mathbf{v}'$ if k is odd, and $\mathbf{v}' > \mathbf{v}$ if k is even. For example, $21111 > 21122222$ and $122211\ldots > 121221$. In the sequel, we will write $\mathbf{v}_m$ for the word obtained from an infinite word $\mathbf{v}$ by deleting its first m letters. In particular, $\mathbf{v}_0 = \mathbf{v}$.

Lemma 4. *The word* $\mathbf{w} = 211222112112112221122211222112112112 2\ldots$ *corresponding to the pairs in Thue–Morse sequence* (*but without the first symbol* $p_0 = 1$) *is the smallest non-periodic infinite word satisfying* $\mathbf{w} > \mathbf{w}_m$ *for each* $m \in \mathbb{N}$.

Proof. Let W be the set of all infinite non-periodic words $\mathbf{v}$ of the alphabet $\{1, 2\}$ satisfying $\mathbf{v} > \mathbf{v}_m$ for each $m \in \mathbb{N}$. We need to show that, firstly, for any $\mathbf{v} \in W \setminus \{\mathbf{w}\}$, we have $\mathbf{v} > \mathbf{w}$, and, secondly, $\mathbf{w} \in W$.

The fact that the word $\mathbf{v}$ is non-periodic implies that there are infinitely many symbols 2 in $\mathbf{v}$. So $\mathbf{v}$ begins with 2. Similarly, as $\mathbf{v}$ contains infinitely many patterns 21, $\mathbf{v}$ begins with 21. If $\mathbf{v}$ begins with 212 then $\mathbf{v} > \mathbf{w}$. So in the search of the least word of W we can only consider the subset of W (denoted by W_1) of words which begin with 211. Similarly, since $2111 > 2112$, we can only consider the words of W_1 which begin with 2112 (denoted by W_1 again). Hence, $\mathbf{v} \in W_1$ contains no patterns of the form 2111 and 212, so only words composed of the blocks $A_1 = 211$ and $A_0 = 2$ which start from A_1 belong to W_1. Now, since each word of W_1 is non-periodic it contains infinitely many patterns A_1A_0. Note that $A_1A_0A_0 > A_1A_1$, and $A_1A_0A_1 > A_1A_1$, so each $\mathbf{v} \in W_1$ begins with A_1A_0. But $A_1A_0A_1 > A_1A_0A_0A_1$ and $A_1A_0A_1 > A_1A_0A_0A_0$, so all words of W_1 that begin with the pattern $A_1A_0A_1$ are greater than $\mathbf{w}$. Assume therefore that each $\mathbf{v} \in W_2 \subset W_1$ begins with $A_1A_0A_0$, where W_2 contains the words composed of the patterns $A_1A_0A_0$ and A_1 only. Similarly, as $A_1A_0A_0A_0A_1 > A_1A_0A_0A_1$ and $A_1A_0A_0A_0A_0 > A_1A_0A_0A_1$, we define W_2 as containing only the words composed from the blocks $A_2 = A_1A_0A_0$ and A_1 only that begin with A_2. Since the lengths of A_2 and A_1 are odd, we can repeat the same argument with A_2 and A_1 (as we did with A_1 and A_0) and so on. We thus obtain a sequence of sets $\ldots \subset W_3 \subset W_2 \subset W_1 \subset W$, where $\mathbf{v} > \mathbf{w}$ for every $\mathbf{v} \in W_k$ and $\mathbf{v} \in W$. But $\bigcap_{j=1}^{\infty} W_j$ contains at most one element, so it must be $\mathbf{w}$ in case $\mathbf{w} \in W$. Indeed, $\mathbf{w}$ is non-periodic (which is well-known and follows from one of the definitions of the sequence corresponding to $\mathbf{w}$ as starting with 2, and at each step replacing 2 by 211 and 1 by 2), so that $\mathbf{w} \neq \mathbf{w}_m$ for any $m \in \mathbb{N}$. Furthermore, $\mathbf{w}$ contains no patterns of the form 212, 2111, $A_kA_{k-1}A_k$, $A_kA_{k-1}A_{k-1}A_{k-1}A_{k-1}$, where $k \in \mathbb{N}$, so $\mathbf{w}$ cannot be smaller than $\mathbf{w}_m$, where $m \geqslant 1$. This completes the proof of the lemma. □

The word $\mathbf{w}$ can be also constructed as follows. (This is exactly what we did in our proof.) We start with $A_1 = 211$, $A_0 = 2$, and then, for each $m \geqslant 2$, define $A_m = A_{m-1}A_{m-2}A_{m-2}$. Then each word A_{m-1} is a prefix of A_m and the word $\mathbf{w}$ begins with the pattern A_m for every $m \in \mathbb{N}$. (We can also write this as $\mathbf{w} = A_\infty$.) This argument shows that each pattern A_m (and, moreover, each subpattern of $\mathbf{w}$) appears in $\mathbf{w}$ infinitely often. By the construction, A_m is of length f_m, where $f_0 = 1$, $f_1 = 3$, and $f_{k+1} = f_k + 2f_{k-1}$ for $k = 1, 2, 3, \ldots$. Hence $\mathbf{w} = A_m\mathbf{w}_{f_m}$ for every $m \geqslant 0$. The construction of A_m in the lemma also implies the following corollary.

Corollary 4. *Let $m \geqslant 2$ be a fixed integer, and let $\mathbf{v}$ be a non-periodic word. Then $\mathbf{v}$ contains either A_m infinitely many times or it contains a finite word $\mathbf{u}$ satisfying $\mathbf{u} > \mathbf{w}$ infinitely many times.*

Proof. Indeed, if A_m appears in $\mathbf{v}$ only finitely many times then $\mathbf{v}$ contains infinitely many patterns either 212, or 2111, or $A_kA_{k-1}A_k$, where $0 < k < m$, or $A_kA_{k-1}A_{k-1}A_{k-1}A_{k-1}$, where $0 < k < m - 1$. Since $A_k > A_{k-1}A_k$ and $A_k > A_{k-1}A_{k-1}$, any of the above patterns is greater than $\mathbf{w}$ which implies the corollary.

We remark that Lemma 4 can be also derived from a result of Allouche and Cosnard [4] on some extremal property of the Thue–Morse sequence or from [23], where a similar result is given for the words of the alphabet $\{-1, 1\}$ instead of $\{1, 2\}$.

Now, to each finite or infinite word $\mathbf{v} = v_1 v_2 v_3 \ldots$ of the alphabet $\{1, 2\}$ and to each number r, where $0 < r < 1$, we attach the real number

$$E(\mathbf{v}, r) = 1 - r^{v_1} + r^{v_1+v_2} - r^{v_1+v_2+v_3} + r^{v_1+v_2+v_3+v_4} - r^{v_1+v_2+v_3+v_4+v_5} + \ldots.$$

In particular,

$$E(\mathbf{w}, r) = 1 - r^2 + r^3 - r^4 + r^6 - r^8 + r^{10} - r^{11} + r^{12} - r^{14} + r^{15} - r^{16} + r^{18} - r^{19} + \ldots.$$

We remark that the powers in $rE(\mathbf{w}, r)$, that is, the sequence of partial sums of the word $1\mathbf{w}$ $p_0, p_0 + p_1, p_0 + p_1 + p_2, \ldots$ is another sequence from the above mentioned web page of N.J.A. Sloane ($A003159$)

$$1, 3, 4, 5, 7, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 23, 25, 27, 28, 29, 31, 33, 35, 36, 37, 39, 41, \ldots.$$

(This is known as the sequence with the property that for each $n \in A$ we have $2n \notin A$, where A and $2A$ form a partition of $\mathbb{N}$.) Subtracting 1, let us write $a_1 = 2$, $a_2 = 3$, $a_3 = 4$, $a_4 = 6$, etc., where the sum of the first k symbols of $\mathbf{w}$ is denoted by a_k, i.e. $a_k = p_1 + p_2 + \cdots + p_k$. The above series for $E(\mathbf{w}, r)$ will be expressed in the form

$$E(\mathbf{w}, r) = 1 - r^{a_1} + r^{a_2} - r^{a_3} + r^{a_4} - r^{a_5} + \ldots. \tag{8}$$

It is well-known that the quantity $T(r)$ given in (1) can be expressed by the Thue–Morse sequence as follows

$$(-1)^{t_0} + (-1)^{t_1} r + (-1)^{t_2} r^2 + (-1)^{t_3} r^3 + \ldots$$
$$= (1 - r)(1 - r^2)(1 - r^4)(1 - r^8) \ldots = T(r)$$

(see, e.g., [5]). By the definition of $\mathbf{w}$ and (8), the connection between $E(\mathbf{w}, r)$ and the generating function of the Thue–Morse sequence is given by

$$rE(\mathbf{w}, r) = (1 - r)(t_0 + t_1 r + t_2 r^2 + t_3 r^3 + \ldots).$$

Since $1 - (-1)^{t_m} = 2t_m$, we deduce that

$$\frac{1}{1 - r} - T(r) = \frac{2rE(\mathbf{w}, r)}{1 - r},$$

giving $E(\mathbf{w}, r) = (1-(1-r)T(r))/2r$. Hence

$$E(r) = E(\mathbf{w}, r) < 1/2r \tag{9}$$

(see (1) and (2)). Likewise, for each $m \geqslant 0$, the definition of $\mathbf{w}$ as the pairs in Thue–Morse sequence yields

$$E(\mathbf{w}_m, r) = (1-r)F_{a_m+1}(r) = (1-r)F_{p_0+\dots+p_m}(r), \tag{10}$$

where $\mathbf{w}_m$ denotes the word obtained from $\mathbf{w}$ by deleting its first m letters and where

$$F_k(z) := \begin{cases} t_k + t_{k+1}z + t_{k+2}z^2 + t_{k+3}z^3 + \dots & \text{if } t_k = 1, \\ \bar{t}_k + \bar{t}_{k+1}z + \bar{t}_{k+2}z^2 + \bar{t}_{k+3}z^3 + \dots & \text{if } t_k = 0. \end{cases} \tag{11}$$

Here, $\bar{t}_j = 1 - t_j$ for each $j \geqslant 0$. For instance, since $p_0 + p_1 + p_2 + p_3 = 1 + 2 + 1 + 1 = 5$, $F_5(r) = 1 + r + r^4 + r^5 + r^7 + \dots$ corresponds to $E(\mathbf{w}_3, r) = 1 - r^2 + r^4 - r^6 + r^7 - r^8 + \dots$. □

Lemma 5. *Let r be a fixed real number, $0 < r < 1$, and let $u, v \geqslant 0$. Then $F_u(r) - rF_v(r) \geqslant T(r)$. In particular, $rF_v(r) < F_u(r)$.*

Proof. It is sufficient to prove that each sum of the form either $t_u + t_{u+1}r + t_{u+2}r^2 + \dots$ or $\bar{t}_u + \bar{t}_{u+1}r + \bar{t}_{u+2}r^2 + \dots$ is at least $\bar{t}_0 + \bar{t}_1 r + \bar{t}_2 r^2 + \dots$ if it starts with the first coefficient 1 and that it is at most $t_1 r + t_2 r^2 + t_3 r^3 + \dots$ if it starts with the first coefficient 0. Then, since $\bar{t}_k - t_k = (-1)^{t_k}$, the difference between two such infinite sums will be at least $\sum_{k=0}^{\infty} (-1)^{t_k} r^k = T(r)$, as claimed. Clearly, since $1/(1-r) - t_u - t_{u+1}r - t_{u+2}r^2 - \dots = \bar{t}_u + \bar{t}_{u+1}r + \bar{t}_{u+2}r^2 + \dots$, it is sufficient to prove only 'half' of this, namely, that each sum $t_u + t_{u+1}r + t_{u+2}r^2 + \dots$ starting with $t_u = 0$ (and each sum $\bar{t}_u + \bar{t}_{u+1}r + \bar{t}_{u+2}r^2 + \dots$ starting with $\bar{t}_u = 0$) is at most $t_0 + t_1 r + t_2 r^2 + t_3 r^3 + \dots$.

Assume for the contradiction that $t_u + t_{u+1}r + t_{u+2}r^2 + \dots > t_0 + t_1 r + t_2 r^2 + \dots$, where $t_u = 0$. Then there is a $k \in \mathbb{N}$ so large that

$$\begin{aligned} T_{u,k}(r) &:= t_u + t_{u+1}r + t_{u+2}r^2 + \dots + t_{u+2^k-1}r^{2^k-1} \\ &> t_0 + t_1 r + t_2 r^2 + \dots + t_{2^k-1}r^{2^k-1}. \end{aligned}$$

We will prove, however, that $T_{u,k}(r) \leqslant T_{0,k}(r)$ and $\overline{T}_{u,k}(r) \leqslant T_{0,k}(r)$ for each $k \in \mathbb{N}$ and for each u satisfying $t_u = 0$ and $\bar{t}_u = 0$, respectively. Here, $\overline{T}_{u,k}(r) := \bar{t}_u + \bar{t}_{u+1}r + \bar{t}_{u+2}r^2 + \dots + \bar{t}_{u+2^k-1}r^{2^k-1}$.

This certainly holds for $k = 1$. Suppose that this holds for each $j < k$. Using the fact that $t_{2i} = t_i$ and $t_{2i+1} = \bar{t}_i$ (this is one of the definitions of the Thue–Morse sequence), we can write, for even u, $T_{u,k}(r) = T_{u/2,k-1}(r^2) + r\overline{T}_{u/2,k-1}(r^2)$. We need to show that

$$T_{u/2,k-1}(r^2) + r\overline{T}_{u/2,k-1}(r^2) \leqslant T_{0,k}(r) = T_{0,k-1}(r^2) + r\overline{T}_{0,k-1}(r^2).$$

But the sum $T_{u,k}(r)+\overline{T}_{u,k}(r) = 1+r+\cdots+r^{2^k-1}$ is independent of u, so $T_{u/2,k-1}(r^2)+\overline{T}_{u/2,k-1}(r^2) = T_{0,k-1}(r^2) + \overline{T}_{0,k-1}(r^2)$ and the above inequality is equivalent to $(1-r)(T_{0,k-1}(r^2) - T_{u/2,k-1}(r^2)) \geqslant 0$, which holds by induction on k. If u is odd, say $u = 2l+1$, then $T_{u,k}(r) = \overline{T}_{l,k-1}(r^2) + rT_{l+1,k-1}(r^2) \leqslant r^2\overline{T}_{l+1,k-1}(r^2) + rT_{l+1,k-1}(r^2)$, because $\bar{t}_l = t_{2l+1} = 0$. Now, either $t_{l+1} = 0$ or $\bar{t}_{l+1} = 0$. In the first case, $r^2\overline{T}_{l+1,k-1}(r^2) + rT_{l+1,k-1}(r^2) \leqslant r\overline{T}_{l+1,k-1}(r^2) + T_{l+1,k-1}(r^2)$, and the proof follows from $T_{l+1,k-1}(r^2) \leqslant T_{0,k-1}(r^2)$, as above. In the second case, $\bar{t}_{l+1} = 0$, using $r^2\overline{T}_{l+1,k-1}(r^2) + rT_{l+1,k-1}(r^2) \leqslant \overline{T}_{l+1,k-1}(r^2) + rT_{l+1,k-1}(r^2)$, we obtain the required inequality from $\overline{T}_{l+1,k-1}(r^2) \leqslant T_{0,k-1}(r^2)$. The proof of $\overline{T}_{u,k}(r) \leqslant T_{0,k}(r)$, where $\bar{t}_u = 0$, is similar. □

Lemma 6. *Let r be a fixed real number, $0 < r < 1$, and let $j, i \geqslant 0$. Then $rE(\mathbf{w}_j, r) < E(\mathbf{w}_i, r) \leqslant E(\mathbf{w}, r)$.*

Proof. The first inequality follows by (10), (11) and Lemma 5. Suppose that there is $i \in \mathbb{N}$ such that $E(\mathbf{w}_i, r) \geqslant E(\mathbf{w}, r)$. By Lemma 4, $\mathbf{w} > \mathbf{w}_i$, so there is a smallest index, say k, such that the first difference between the words $\mathbf{w}$ and $\mathbf{w}_i$ occurs at the kth place. These kth symbols of $\mathbf{w}$ and $\mathbf{w}_i$ should be 2 and 1, respectively, if k is odd and 1 and 2, respectively, if k is even. In the first case, by (8),

$$E(\mathbf{w}, r) - E(\mathbf{w}_i, r) = -r^{a_k}E(\mathbf{w}_{k-1}, r) + r^{a_k-1}E(\mathbf{w}_{i+k-1}, r)$$

which is positive by the first inequality of this lemma. In the second case, $E(\mathbf{w}, r) - E(\mathbf{w}_i, r) = r^{a_k}E(\mathbf{w}_{k-1}, r) - r^{a_k+1}E(\mathbf{w}_{i+k-1}, r)$, which is positive, by the first inequality of this lemma again. This proves more than required, namely, $E(\mathbf{w}_i, r) < E(\mathbf{w}, r)$ for every $i > 0$. □

Lemma 7. *Let r be a fixed real number, $0 < r < 1$, and let $\mathbf{v}$ be any non-periodic word. Then, for any $\varepsilon > 0$, there are infinitely many $l \in \mathbb{N}$, such that $E(\mathbf{v}_l, r) > E(r) - \varepsilon$.*

Proof. Fix m so large that $r^{a_k+1} < \varepsilon$, where $k = f_m$. (Recall that f_m is the length of the word A_m; f_m is an odd number.) Then, for any word $\mathbf{v}$ which begins with A_m (in particular, for $\mathbf{w}$), we have $E(A_m, r) < E(\mathbf{v}, r) < E(A_m, r) + \varepsilon$, so $E(\mathbf{v}, r) > E(r) - \varepsilon$. Hence, if $\mathbf{v}$ contains infinitely many subwords A_m, the lemma is proved.

By Corollary 4, the only alternative is that there is a finite word $\mathbf{u}$, $\mathbf{u} > \mathbf{w}$, which occurs in $\mathbf{v}$ infinitely many times. Without loss of generality we can assume that the length of $\mathbf{u}$ is k, and that first $k-1$ symbols of $\mathbf{u}$ coincide with the first $k-1$ symbols of $\mathbf{w}$. Then the kth symbols in $\mathbf{u}$ and $\mathbf{w}$ are 2 and 1 if k is odd, and 1 and 2 if k is even. Suppose also that $\mathbf{u}$ starts at the lth place of $\mathbf{v}$. In the first case, as in Lemma 6, we can write the value attached to the finite word consisting of $k-1$ first symbols of $\mathbf{w}$ as

$$E(\mathbf{w}, r) + r^{a_k} E(\mathbf{w}_{k-1}, r) = E(\mathbf{v}_{l-1}, r) + r^{a_k+1} E(\mathbf{v}_{l+k-2}, r).$$

For $E(\mathbf{v}_{l-1}, r) \leqslant E(\mathbf{w}, r)$ and $E(\mathbf{v}_{l+k-2}, r) \leqslant E(\mathbf{w}, r)$, this implies that $E(\mathbf{w}_{k-1}, r) \leqslant rE(\mathbf{w}, r)$, a contradiction with Lemma 6. Therefore, at least one of the numbers $E(\mathbf{v}_{l-1}, r)$, $E(\mathbf{v}_{l+k-2}, r)$ is greater than $E(\mathbf{w}, r) = E(r)$.

Alternatively, if k is even,

$$E(\mathbf{w}, r) - r^{a_k} E(\mathbf{w}_{k-1}, r) = E(\mathbf{v}_{l-1}, r) - r^{a_k-1} + r^{a_k+\delta} E(\mathbf{v}_{l+k-1}, r),$$

where $\delta \in \{0, 1\}$. So

$$E(\mathbf{v}_{l-1}, r) - E(\mathbf{w}, r) = r^{a_k-1}(1 - r^{1+\delta} E(\mathbf{v}_{l+k-1}, r) - rE(\mathbf{w}_{k-1}, r)).$$

If $E(\mathbf{v}_{l+k-1}, r) \leqslant E(\mathbf{w}, r)$, then, by (9) and Lemma 6, we have $r^{1+\delta} E(\mathbf{v}_{l+k-2}, r) + rE(\mathbf{w}_{k-1}, r) \leqslant 2rE(r) < 1$, so $E(\mathbf{v}_{l-1}, r) > E(\mathbf{w}, r)$. Consequently, at least one of the numbers $E(\mathbf{v}_{l+k-1})$, $E(\mathbf{v}_{l-1}, r)$ is greater than $E(\mathbf{w}, r) = E(r)$.

Summarizing, we see that if $\mathbf{v}$ contains a finite word $\mathbf{u} > \mathbf{w}$ infinitely many times then the stronger inequality $E(\mathbf{v}_l, r) > E(\mathbf{w}, r) = E(r)$ holds for infinitely many l. □

Lemma 8. *Let r be a fixed real number, $0 < r < 1$. Then $E(r) < 1/(1 + r^3)$.*

Proof. By (9) we have $E(r) < 1/2r$. This is less than or equal to $1/(1 + r^3)$ for $r \geqslant (\sqrt{5} - 1)/2$. It remains to prove the lemma for $r < (\sqrt{5} - 1)/2$. Then, as $E(r) < 1 - r^2 + r^3 - r^4 + r^6$,

$$\begin{aligned}(1 + r^3)E(r) &< 1 - r^2 + 2r^3 - r^4 - r^5 + 2r^6 - r^7 + r^9 < 1 - r^2 + 2r^3 - r^4 + r^6 \\ &= 1 - r^2((1 - r)^2 - r^4) = 1 - r^2(1 - r + r^2)(1 - r - r^2)\end{aligned}$$

is less than 1 if $r < (\sqrt{5} - 1)/2$. This proves the lemma. □

4. Proofs

Proof of Theorem 4. Let $s_1, s_2, s_3, \ldots$ be a sequence of integers. We define

$$U_l(z) := s_l + s_{l+1}z + s_{l+2}z^2 + s_{l+3}z^3 + \ldots . \tag{12}$$

If there are infinitely many $l \in \mathbb{N}$ such that $|s_l| \geqslant 2$, we have

$$2 \leqslant |s_l| = |U_l(r) - rU_{l+1}(r)| \leqslant |U_l(r)| + r|U_{l+1}(r)|.$$

Hence at least one of the numbers $|U_l(r)|$, $|U_{l+1}(r)|$ is greater than or equal to

$$2/(r+1) > 1 - r^2 + r^3 > E(\mathbf{w}, r) = E(r),$$

so (12) implies (3).

If, say there are infinitely many $l \in \mathbb{N}$ for which $s_l = \pm 1$, $s_{l+1} = 0$, $s_{l+2} = 0$, then

$$1 = |s_l| = |U_l(r) - r^3 U_{l+3}(r)| \leqslant |U_l(r)| + r^3 |U_{l+3}(r)|.$$

Now, at least one of the numbers $|U_l(r)|$, $|U_{l+3}(r)|$ is greater than or equal to $1/(1+r^3)$ which is strictly greater than $E(r)$, by Lemma 8. This proves (3) for such sequences too. If there are infinitely many $l \in \mathbb{N}$ such that $s_l = 1$, $s_{l+1} = 0$, $s_{l+2} = 1$ then $U_l(r) - r^3 U_{l+3}(r) = 1 + r^2$, so at least one of the numbers $|U_l(r)|$, $|U_{l+3}(r)|$ is greater than or equal to $(1 + r^2)/(1 + r^3) > 1 > E(r)$. (Evidently, Lemma 8 implies that $E(r) < 1$.) Similarly, we obtain the required inequality (3) in case there are infinitely many $l \in \mathbb{N}$ such that $s_l = -1$, $s_{l+1} = 0$, $s_{l+2} = -1$. Finally, if there are infinitely many $l \in \mathbb{N}$ for which $s_l = 1$, $s_{l+1} = 1$, then $U_l(r) - r^2 U_{l+2}(r) = 1 + r$, and so at least one of the numbers $|U_l(r)|$, $|U_{l+2}(r)|$ is greater than or equal to $(1+r)/(1+r^2) > 1 > E(r)$. The case with infinitely many $l \in \mathbb{N}$ for which $s_l = -1$, $s_{l+1} = -1$ is similar.

Therefore, we can assume without loss of generality that, starting with certain place, say n_0, the non-periodic sequence $s_{n_0}, s_{n_0+1}, s_{n_0+2}, \ldots$ begins with $s_{n_0} = 1$ and is of the form $1, -1, 1, -1, \ldots$, with some units (either 1 and -1 or -1 and 1) separated by one 0. Omitting the first symbol $s_{n_0} = 1$, we will write 2 if two units are separated by 0 and 1 if they are not. This translates such a sequence into a non-periodic word of the alphabet $\{1, 2\}$. For instance, the sequence $1, 0, -1, 1, -1, 0, 1, 0, -1, 0, 1, -1, 1, \ldots$ translates into the word $21122211\ldots$. For such sequences, each $|U_l(r)|$, where $s_l = \pm 1$, $l \geqslant n_0$, is equal to $E(\mathbf{v}_s, r)$, where $\mathbf{v}$ is the word corresponding to $s_{n_0} = 1, s_{n_0+1}, s_{n_0+2}, \ldots$. (Here, $n - n_0 - s$ is the number of zeros among $s_{n_0+1}, \ldots, s_{n_0+l-1}$.) Similarly, since two zeros in a row do not occur, $|U_l(r)| = rE(\mathbf{v}_s, r)$ if $s_l = 0$. Inequality (3) now follows from Lemma 7.

The proof of (4) is similar. We will show that s_n (which now are odd integers), starting from a certain place, take only values ± 1 and no more than two values in a row have the same sign. Set $V_l(z) := s_l + s_{l+1} z + s_{l+2} z^2 + \ldots$. Evidently, if $|s_l| \geqslant 3$ for infinitely many l, then $3 = |s_l| = |V_l(r) - rV_{l+1}(r)|$. So at least one of the numbers $|V_l(r)|$, $|V_{l+1}(r)|$ is greater than $3/(1 + r)$. We will show that this is greater than $(1 - T(r))/r$. Indeed, by (2), $T(r) = (1 - 2rE(r))/(1 - r)$, so inequality $3/(1 + r) > (1 - T(r))/r$ is equivalent to the inequality $E(r) < (2 - r)/(1 + r)$ which follows from Lemma 8 combined with $1 < (2 - r)(1 - r + r^2)$. Similarly, if three values 1 in row (or three values -1 in a row) occur infinitely often, then writing $1 + r + r^2 = |V_l(r) + r^3 V_{l+3}(r)|$ we deduce that at least one of the numbers $|V_l(r)|$, $|V_{l+3}(r)|$ is greater than $(1 + r + r^2)/(1 + r^3)$. This is greater than $(1 - T(r))/r$, because $(1 + r + r^2)/(1 + r^3) > (1 - T(r))/r$ transforms into $E(r) < 1/(1 + r^3)$ which holds by Lemma 8.

So, starting from a certain n_0, s_n takes only two values $1, -1$ with at most two equal values in a row. Let h be a map taking any $H(r)$ into $(1+(1-r)H(r))/2$. Assuming, without loss of generality that $n_0 = 0$, $s_0 = 1$, we can transform the sequence of $1, -1$ with the above properties into the sequence of $1, -1, 0$ considered in the previous part on applying h to $V_0(r)$ which will take $V_0(r) \to U_0(r)$. Since $h : H(r) \to (1+(1-r)H(r))/2$, this map will transform the right-hand side of (4), $(1-T(r))/r$, into $(1+(1-r)(1-T(r))/r)/2 = E(r)$, which is the right-hand side of (3). This completes the proof of (4) and of Theorem 4. □

Both inequalities (3) and (4) are sharp. We can define, for instance, the sequence $s_1, s_2, \ldots,$ as the coefficients of $s_0 + s_1 z + s_2 z^2 + s_3 z^3 + \ldots := E(\mathbf{w}, z) = 1 - z^2 + z^3 - z^4 + z^6 - z^8 + \ldots$ Then, for each $l \in \mathbb{N}$, $|s_l + s_{l+1} r + s_{l+2} r^2 + \ldots| < E(r)$, by the inequality $E(\mathbf{w}_l, r) < E(\mathbf{w}, r) = E(r)$ (see Lemma 6), so (4) is sharp. Similarly, we can define odd integers by the formula

$$\begin{aligned} \hat{s}_0 + \hat{s}_1 z + \hat{s}_2 z^2 + \hat{s}_3 z^3 + \ldots &:= (1 - T(z))/z = (2E(z) - 1)/(1 - z) \\ &= \sum_{k=0}^{\infty} (-1)^{t_{k+1}+1} z^k = \sum_{k=0}^{\infty} (2t_{k+1} - 1) z^k \\ &= 1 + z - z^2 + z^3 - z^4 - z^5 + \ldots. \end{aligned}$$

This gives $|\hat{s}_l + \hat{s}_{l+1} r + \hat{s}_{l+2} r^2 + \ldots| < (1 - T(r))/r$ for every $l \in \mathbb{N}$. So two 'extreme' sequences of integers $s_1, s_2, \ldots$ and of odd integers $\hat{s}_1, \hat{s}_2, \ldots$ showing that (3) and (4) are best possible can be given in terms of the Thue–Morse sequence as $s_k = t_{k+1} - t_k$, $k = 1, 2, \ldots,$ and $\hat{s}_k = 2t_{k+1} - 1$, $k = 1, 2, \ldots,$ respectively.

Proof of Theorem 3. The proof is a combination of Lemma 1 and Theorem 4. We will first prove that the sequence $||\xi(p/q)^n||$, $n = 1, 2, 3, \ldots,$ has a 'large' limit point and then that it has a 'small' limit point. Throughout the proof of this theorem, $r := q/p$.

For $\alpha = p/q$, we have $P(z) = -p + qz$. Now, equality (5) with $\eta = 1/2$ implies that

$$s_n = -q g_{n+1} + p g_n,$$

where $g_n := y_n - 1/2 = \{\xi(p/q)^n + 1/2\} - 1/2$, so that $||\xi\alpha^n|| = |g_n|$. Hence $g_n = s_n/p + r g_{n+1}$ with $r = q/p < 1$. By expressing g_{n+1} by g_{n+2} and so on, this yields

$$g_n = (1/p)(s_n + s_{n+1} r + s_{n+2} r^2 + s_{n+3} r^3 + \ldots).$$

By Lemma 1, the sequence of integers s_n, $n = 1, 2, 3, \ldots,$ is not ultimately periodic. Using (3) we deduce that there are infinitely many integers n, such that $|g_n| > (E(r) - \varepsilon)/p$. This proves the first part of Theorem 3.

To show that the sequence $||\xi(p/q)^n||$, $n = 1, 2, 3, \ldots$, has a 'small' limit point, we write the fractional part $\{\xi(p/q)^n\}$ in the form $1/2 + g_n$, where $-1/2 \leqslant g_n < 1/2$. Set $g := \limsup_{n\to\infty} |g_n|$. We need to show that $g \geqslant (1-e(r)T(r))/2q$, where $e(r) = 1-r$ if $p+q$ is even and $e(r) = 1$ if $p+q$ is odd. This time,

$$\begin{aligned} p[\xi(p/q)^n] - q[\xi(p/q)^{n+1}] &= -p(1/2 + g_n) + q(1/2 + g_{n+1}) \\ &= (q-p)/2 - pg_n + qg_{n+1}, \end{aligned}$$

so $s_n := -qg_{n+1} + pg_n$ belongs to $\mathbb{Z}$ if $p+q$ is even and to $1/2+\mathbb{Z}$ if $p+q$ is odd. As above, $s_1, s_2, s_3, \ldots$ is not ultimately periodic, by Lemma 1. The case with $p+q$ being even corresponds to the case which we just considered and follows by (3). Suppose $p+q$ is odd and s_n take the values of the form $1/2+\mathbb{Z}$. We need to prove that then $g \geqslant (1-T(r))/2q$.

Indeed, as above, $g_n = (1/p)(s_n + s_{n+1}r + s_{n+2}r^2 + s_{n+3}r^3 + \ldots)$ with the difference being that s_n are of the form $1/2+\mathbb{Z}$. Multiplying both sides of this equality by 2 we see that $2s_n$ are odd integers and derive from (4) that $2g \geqslant (1-T(r))/rp$. Since $rp = q$, this yields $g \geqslant (1-T(r))/2q$ and completes the proof of Theorem 3. □

Proof of Corollary 2. Both $\xi_p = E(1/p)/p$ and $\hat{\xi}_p = e(1/p)T(1/p)/2$ are linear forms in $t_0 + t_1p^{-1} + t_2p^{-2} + t_3p^{-3} + \ldots$ and 1 with rational coefficients, so ξ_p and $\hat{\xi}_p$ are irrational numbers, because the Thue–Morse sequence $t_0, t_1, t_2, t_3, \ldots$ is not ultimately periodic. (Transcendence of such constants was proved in [12]. The number $t_0 + t_1/2 + t_2/2^2 + t_3/2^3 + \ldots = 0.412454\ldots$ corresponding to $p = 2$ is usually called the Thue–Morse constant.) Using (1), (2), (8), (10), (11) one can easily see that the constants ξ_p and $\hat{\xi}_p$ can be written in the form

$$\begin{aligned} \xi_p &= 1/p - 1/p^3 + 1/p^4 - 1/p^5 + 1/p^7 - 1/p^9 + 1/p^{11} - \ldots \\ &= (1-1/p)(t_0 + t_1/p + t_2/p^2 + t_3/p^3 + t_4/p^4 + t_5/p^5 + \ldots), \end{aligned}$$

where the powers form the sequence $A003159$ mentioned above, and

$$\hat{\xi}_p = \begin{cases} 1/2 - \xi_p & \text{if } p \text{ is odd}, \\ (1/2)(1 - 1/p - 1/p^2 + 1/p^3 - 1/p^4 + p^5 + 1/p^6 - 1/p^7 - \ldots) & \text{if } p \text{ is even}. \end{cases}$$

The inequalities $||\xi_p p^n|| < \xi_p$ and $||\hat{\xi}_p p^n|| > \hat{\xi}_p$ for $n \in \mathbb{N}$ follow from Lemmas 5 and 6. □

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