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On the Radical of a Group Algebra

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1. INTRODUCTION

Let G be a finite group, p a prime, F a splitting field for G with characteristic p, $S \in \mathrm{Syl}_p(G)$. $P(M)$ will denote the projective cover of the simple FG-module M. Let JFG be the Jacobson-radical of FG and $u := \dim_F P(1_G)$, where 1_G denotes the trivial FG-module. Then there is the following

LEMMA. (a) (Brauer and Nesbitt [2]). $\dim_F JFG \leqslant |G| - |G|/u$.

(b) (Wallace [21]). $S \trianglelefteq G \Rightarrow \dim_F JFG = |G| - |G|/u$.

The converse of (b) is true if G has a p-complement (Wallace [21]) or if S is cyclic (Motose [13]) or if $p = 2$ (Okuyama [16]). The main purpose of this paper is to obtain

THEOREM 1.1. $S \trianglelefteq G \Leftrightarrow \dim_F JFG = |G| - |G|/u$.

The proof uses Okuyama's result, the classification theorem of finite simple groups and the following

THEOREM 1.2 (Brockhaus and Michler [4]). *If G is a simple group of Lie-type and $p \neq 2$, then G has at least two p-blocks.*

Furthermore, the next lemma is needed.

LEMMA 1.3 (Wallace [21]). *Let G be p-solvable. Then:* $S \trianglelefteq G \Leftrightarrow \dim_F JFG = |G| - |G|/u$.

(For a short proof see Brockhaus [3].)

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2. Further Notation

Let $M_1, ..., M_k$ be representatives of simple FG-modules ($k \in \mathbb{N}$). If M, W are FG-modules, M simple, then $\chi_W\colon G \to F$ will denote the Frobenius character afforded by W, $\#(M, W)$ the multiplicity of M as a composition factor of W and $l(W)$ the composition length of W. $|\mathrm{Bl}_p(G)|$ will be the number of p-blocks of G and $\mathrm{Irr}(G)$ the set of irreducible complex characters of G. For $i, j \in \{1, ..., k\}$ let $c_{ij} := \#(M_j, P(M_i))$. Then $C := (c_{ij})$ is the Cartan matrix. If $H \subseteq G$, let $\hat{H} := \sum_{h \in H} h \quad (\in FG)$.

$$\alpha := \sum_{\substack{g \in G \\ g\ p\text{-element}}} g \quad \in ZFG,$$

the centre of FG. Define an F-linear map $T\colon FG \to F$ by

$$T(g) := \begin{cases} 1 & \text{if } g \text{ is a } p\text{-element} \\ 0 & \text{otherwise} \end{cases} \qquad (g \in G).$$

T is called "p-trace of FG."

If $\chi\colon FG \to F$ is a character or the p-trace of G, then $\ker \chi := \{x \in FG / \chi(gx) = 0\ \forall g \in G\}$.

$\ker \chi$ is an ideal of FG, because $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for quadratic matrices A, B and $|gh| = |(gh)^g| = |hg|$ for $g, h \in G$. (For these definitions see Formanek and Snider [7].)

Let $G = \{g_1, ..., g_n\}$.

$$t_{ij} := \begin{cases} 1 & \text{if } g_i^{-1} g_j \text{ is a } p\text{-element} \\ 0 & \text{otherwise} \end{cases} \qquad (\in F);$$

$i, j \in \{1, ..., n\}$. $T_G := (t_{ij})$, a symmetric matrix. Finally, $\omega(S)$ will denote the augmentation ideal of FS and $Z(G)$ the centre of G.

3. Results

Lemma 3.1. (Formanek and Snider [7]). (a) $JFG \subseteq \ker T$.

(b) *If W is an FG-module, then*: $JFG = \ker \chi_W \Leftrightarrow \forall i\colon p \nmid \#(M_i, W)$.

(c) $\chi_{(1_S)^G} = |N_G(S) \colon S|\, T$.

(d) $JFG = \ker T \Leftrightarrow \forall i\colon p \nmid \#(M_i, (1_S)^G)$.

Proof (Formanek and Snider). (a) T annihilates every nilpotent element (see Passman [17, p. 47]).

(b) $R := FG/JFG$, $\chi_i := \chi_{M_i} \forall i$. M_i becomes a simple R-module via $m(x + JFG) := mx$ ($m \in M_i$, $x \in FG$).

(1) $\mathrm{Ann}_R(M_i) = \mathrm{Ann}_{FG}(M_i)/JFG$ (where "Ann" denotes the annihilator).

By Wedderburn's theorem:

(2) There are orthogonal central primitive idempotents $e_1, \dots, e_k \in R$ having the following properties:

$R = e_1 R \oplus \cdots \oplus e_k R$; $\mathrm{Ann}_R(M_i) = (1 - e_i) R$, a maximal ideal of R.
If B is an F-basis for M_i and $d_i := \dim_F M_i$, then for each i:

$$\phi_i \colon e_i R \to F_{d_i}$$
$$x \mapsto \text{matrix of } \begin{Bmatrix} M_i \to M_i \\ m \mapsto mx \end{Bmatrix}$$

with respect to B is a ring isomorphism.
Obviously

(3) $\mathrm{Ann}_{FG}(M_i) \subseteq \ker \chi_i \ \forall i$.

(4) $\ker \chi_i \neq FG$.

Proof. Choose $x \in FG$ with $x + JFG \in e_i R$ and

$$\phi_i(x + JFG) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{pmatrix} \qquad (\in F_{d_i})$$

$$\Rightarrow \chi_i(x) = \mathrm{tr} \begin{Bmatrix} M_i \to M_i \\ m \mapsto mx = m(x + JFG) \end{Bmatrix}$$
$$= \mathrm{tr}\, \phi_i(x + JFG) = 1 \neq 0$$
$$\Rightarrow x \notin \ker \chi_i.$$

Therefore (4) holds. $\mathrm{Ann}_{FG}(M_i)$ is a maximal ideal of FG by (1), (2). It follows from (3), (4) that

(5) $\ker \chi_i = \mathrm{Ann}_{FG}(M_i) \ \forall i$.

Let $m_i := \#(M_i, W)$. Then

(6) $\chi_W = \sum_{i=1}^{k} m_i \chi_i$.

(7) $\ker \chi_W = \bigcap_{p \nmid m_i} \ker \chi_i$.

Proof. "$\supseteq$" by (6).

"$\subseteq$": Suppose (7) is not true.

$\Rightarrow \exists x \in FG\ \exists j \in \{1,\dots,k\}\ \forall g \in G$: $\chi_W(gx)=0$, $\chi_j(x)\neq 0$, $p \nmid m_j$.

Choose $e_j' \in FG$: $e_j = e_j' + JFG$; let $i \in \{1,\dots,k\}$.

$$\chi_i(e_j'x) = \operatorname{tr}\left\{\begin{matrix} M_i \to M_i \\ m \mapsto me_j'x = [m(e_j'+JFG)]\,x = (me_j)\,x \end{matrix}\right\},$$

$$\left.\begin{matrix} me_j = m & \text{if}\quad i=j \\ = 0 & \text{otherwise} \end{matrix}\right\} \quad (\text{by } (2)).$$

Therefore

$$\begin{aligned} \chi_i(e_j'x) &= \chi_i(x) && \text{if}\quad i=j \\ &= 0 && \text{otherwise,} \end{aligned}$$

$$\begin{aligned} 0 &= \chi_W(e_j'x) \qquad (\text{by assumption}) \\ &\underset{(6)}{=} \sum_{i=1}^{k} m_i\chi_i(e_j'x) = m_j\chi_j(x) \neq 0 \qquad (\text{because } p \nmid m_j), \end{aligned}$$

a contradiction!

(8) $\bigcap_{i=1}^{k} \operatorname{Ann}_{FG}(M_i) \neq \bigcap_{i\in I} \operatorname{Ann}_{FG}(M_i)$, if $I \subsetneqq \{1,\dots,k\}$.

Proof. Let $j \in \{1,\dots,k\}\setminus I$. Choose $e_j' \in FG$ with $e_j = e_j' + JFG$. $\forall i \in I$: $i \neq j$, so

$$M_ie_j' = M_i(e_j'+JFG) = M_ie_j \underset{(2)}{=} 0$$

$$\begin{aligned} &\Rightarrow e_j' \in \operatorname{Ann}_{FG}(M_i) \qquad \forall i \in I \\ &\Rightarrow e_j' \in \bigcap_{i\in I} \operatorname{Ann}_{FG}(M_i). \end{aligned}$$

But $M_je_j' = M_j(e_j'+JFG) = M_je_j =_{(2)} M_j \neq 0 \Rightarrow e_j' \notin \operatorname{Ann}_{FG}(M_j) \Rightarrow e_j' \notin \bigcap_{i=1}^{k} \operatorname{Ann}_{FG}(M_i)$. Therefore (8) holds.

$$JFG = \bigcap_{i=1}^{k} \operatorname{Ann}_{FG}(M_i) \subseteq \bigcap_{p\nmid m_i} \operatorname{Ann}_{FG}(M_i) \underset{(5)}{=} \bigcap_{p\nmid m_i} \ker\chi_i \underset{(7)}{=} \ker\chi_W.$$

It follows from (8): $JFG = \ker\chi_W \Leftrightarrow \forall i$: $p \nmid m_i$.

(c) Let $g \in G$. An easy calculation shows

$$\begin{aligned} \chi_{(1_S)^G}(g) &= |\{P \in \operatorname{Syl}_p(G) \mid Pg = P\}| \cdot 1 \\ &= |N_G(S) : S| \cdot |\{P \in \operatorname{Syl}_p(G) \mid g \in P\}| \cdot 1. \end{aligned}$$

The statement is true, if g is not a p-element. Let g be a p-element. Then $g \in P \Leftrightarrow g \in N_G(P)$. Therefore

(9) $\chi_{(1_S)^G}(g) = |N_G(S): S|\ |\{P \in \mathrm{Syl}_p(G) \mid P^g = P\}| \cdot 1.$

(10) $|\{P \in \mathrm{Syl}_p(G) \mid P^g = P\}| \equiv 1 \pmod p.$

Proof. $\langle g \rangle$ acts on $\mathrm{Syl}_p(G)$ by conjugation. Let $Y_1, \ldots, Y_t$ be the orbits

$$\Rightarrow \mathrm{Syl}_p(G) = \bigcup_{|Y_i|=1} Y_i \cup \bigcup_{p \mid |Y_i|} Y_i = \{P \in \mathrm{Syl}_p(G) \mid P^g = P\} \cup \bigcup_{p \mid |Y_i|} Y_i$$

$$\Rightarrow 1 \equiv |\mathrm{Syl}_p(G)| \equiv |\{P \in \mathrm{Syl}_p(G) \mid P^g = P\}| \pmod p.$$

Part (d) follows from (b), (c). Q.E.D.

LEMMA 3.2.

$$\ker T = \mathrm{Ann}_{FG}(\alpha) = \left\{ \sum_{i=1}^{n} c_i g_i \,\middle|\, c_1, \ldots, c_n \in F,\ T_G \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = 0 \right\}.$$

In particular $\dim_F \mathrm{Ann}_{FG}(\alpha) = |G| - \mathrm{rk}(T_G)$.

Remark. Lemma 3.2 is essentially contained in Tsushima [20] (where the matrix T_G is defined in a slightly different manner).

Proof. Let $x \in FG$. $\exists f_j \in F: x = \sum_j f_j g_j$

$$x \in \ker T \Leftrightarrow \forall i: \sum_j f_j T(g_i^{-1} g_j) = 0$$

$$\Leftrightarrow (t_{ij}) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = 0 \Leftrightarrow x \in \left\{ \sum_i c_i g_i \,\middle|\, \cdots \right\}.$$

T_G defines an F-linear map

$$\varphi: FG \to FG$$

$$g_j \mapsto \sum_i t_{ij} g_i$$

$$\varphi(g_j) = \sum_{\substack{i \\ g_i^{-1} g_j\ p\text{-el.}}} g_i = \sum_{\substack{g \in G \\ g^{-1}\ p\text{-el.}}} g_j g \qquad (g_i = g_j g)$$

$$= g_j \sum_{g^{-1}\ p\text{-el.}} g = g_j \alpha.$$

Therefore $\varphi(x) = x\alpha\ \forall x \in FG$. It follows: $x \in \{\sum_i c_i g_i \mid \cdots\} \Leftrightarrow \varphi(x) = 0 \Leftrightarrow x \in \mathrm{Ann}_{FG}(\alpha)$. Q.E.D.

LEMMA 3.3. (a) $\#(M_i, FG) = \dim_F P(M_i)\ \forall i.$

(b) $\dim_F P(M_i) = |S|\ \#(M_i, (1_S)^G)\ \forall i.$

(c) $u = |S| \Leftrightarrow \#(1_G, (1_S)^G) = 1.$

Proof.

$$\text{(a)}\quad \sum_j c_{ij} \dim_F M_j = \sum_j \#(M_j, P(M_i)) \dim_F M_j = \dim_F P(M_i),$$

$$\sum_j c_{ji} \dim_F M_j = \sum_j \#(M_i, P(M_j)) \dim_F M_j = \#(M_i, FG),$$

since $\dim_F M_j$ is the multiplicity of $P(M_j)$ as a direct summand of FG. $c_{ij} = c_{ji}$ by symmetry of the Cartan-matrix.

(b) Let $O = W_0 < W_1 < \cdots < W_{|S|} = FS$ be an FS-composition series of FS. Then $O = W_0^G < W_1^G < \cdots < W_{|S|}^G = (FS)^G \cong FG$, where $W_i^G/W_{i-1}^G \cong (W_i/W_{i-1})^G \cong (1_S)^G\ \ \forall i \in \{1,..., |S|\}$. It follows that $|S|\ \#(M_i, (1_S)^G) = \#(M_i, FG) =_{(a)} \dim_F P(M_i)$.

Part (c) follows from (b). Q.E.D.

LEMMA 3.4. (a) $JFG \subseteq \mathrm{Ann}_{FG}(\alpha)$, $\alpha FG \subseteq \mathrm{soc}\ FG$.

(b) *The following statements are equivalent*:

(i) $JFG = \mathrm{Ann}_{FG}(\alpha)$.

(ii) $\mathrm{soc}\ FG = \alpha FG$.

(iii) $p \nmid (\dim_F P(M_i))/|S|\ \forall i.$ (Note: $(\dim_F P(M_i))/|S| \in \mathbb{N}$ by Lemma 2.4(b).)

(iv) $P(M_i)\, \alpha \neq 0\ \forall i.$

(v) $P(M_i)\, \alpha \cong M_i\ \ \forall i.$

(vi) $1 \in \mathrm{Supp}(e\alpha)$ for every primitive idempotent e.

Proof. (a) $JFG \subseteq \mathrm{Ann}_{FG}(\alpha)$ by Lemmas 3.1(a), 3.2 (see also Tsushima [19]). $\alpha FG \cong FG/\mathrm{Ann}_{FG}(\alpha) \cong (FG/JFG)/(\mathrm{Ann}_{FG}(\alpha)/JFG)$ is semisimple (since $FG/JFG \cong \mathrm{soc}\ FG$). $\Rightarrow \alpha FG \subseteq \mathrm{soc}\ FG$.

(b) (i) $\Leftrightarrow$ (ii) is clear (see proof of (a)).

(i) $\Leftrightarrow$ (iii):

$$p \nmid \frac{\dim_F P(M_i)}{|S|}\ \forall i \underset{\text{Lemma 3.3(b)}}{\Leftrightarrow} p \nmid \#(M_i, (1_S)^G)\ \forall i$$

$$\underset{\text{Lemmas 3.1(d), 3.2}}{\Leftrightarrow} JFG = \mathrm{Ann}_{FG}(\alpha).$$

(ii) $\Leftrightarrow$ (iv), (ii) $\Leftrightarrow$ (v): Let $FG = \sum_{i=1}^{t} \oplus P_i$, P_i indecomposable right ideals. $P_i\alpha \leqslant P_i \cap \alpha FG \leqslant_{\text{Lemma 3.2}} P_i \cap \operatorname{soc} FG = \operatorname{soc} P_i$; therefore $\alpha FG = \sum_{i=1}^{t} \oplus P_i\alpha \leqslant \sum_{i=1}^{t} \oplus \operatorname{soc} P_i = \operatorname{soc} FG$. It follows that $\alpha FG = \operatorname{soc} FG \Leftrightarrow \forall i\colon P_i\alpha = \operatorname{soc} P_i \Leftrightarrow \forall i\colon P_i\alpha \neq 0$.

(iv) $\Leftrightarrow$ (vi): $e\alpha \neq 0 \Leftrightarrow 1 \in \operatorname{Supp}(e\alpha)$ by Okuyama [14, (1.G)].

Q.E.D.

COROLLARY 3.5. (Formanek and Snider [7]; Tsushima [19]). *Let G be p-solvable. Then $JFG = \operatorname{Ann}_{FG}(\alpha)$.*

Proof. $\forall i\colon p \nmid (\dim_F P(M_i))/|S|$ by Hamernik, Michler [8, Corollary 2.3b)]. The statement follows from Lemma 3.4. Q.E.D.

LEMMA 3.6. (a) (Wallace [21]). *$\dim_F JFG = |G| - |G|/u \Leftrightarrow \forall i\colon P(M_i) \cong P(1_G) \otimes_F M_i$.*

(b) *$\dim_F JFG = |G| - |G|/u \Leftrightarrow \forall i, j\colon \quad \#(1_G, M_i^* \otimes_F M_j) = \delta_{ij}$. ($\delta_{ij}$ denotes the Kronecker symbol.)*

(c) *$\dim_F JFG = |G| - |G|/u \Rightarrow p \nmid \dim_F M_i \,\forall i$. In particular every p-Block has maximal defect.*

Remark. Part (b) is due to M. Lorenz. Part (c) may be found in Okuyama [16].

Proof. (a) $P(1_G) \otimes_F M_i$ is projective and $M_i \cong 1_G \otimes_F M_i \leqslant P(M_i) \otimes_F M_i$. Therefore

$$\forall i\colon P(M_i) \mid P(1_G) \otimes_F M_i. \tag{$*$}$$

$$\dim_F FG/JFG = \sum_i (\dim_F M_i)^2 = \frac{1}{u} \sum_i (\dim_F M_i)(u \dim_F M_i)$$
$$\underset{(*)}{\geqslant} \frac{1}{u} \sum_i (\dim_F M_i)(\dim_F P(M_i)) = \frac{|G|}{u}$$

(because $\dim_F M_i$ is the multiplicity of $P(M_i)$ as a direct summand of FG). Therefore $P(M_i) \cong P(1_G) \otimes_F M_i \;\forall i \Leftrightarrow \dim_F FG/JFG = |G|/u$.

(b) $$\#(1_G, M_i^* \otimes_F M_j) = \dim_F \operatorname{Hom}_{FG}(P(1_G), M_i^* \otimes_F M_j)$$
$$= \dim_F \operatorname{Hom}_{FG}(P(1_G) \otimes_F M_i, M_j)$$

"$\Rightarrow$": $\#(1_G, M_i^* \otimes_F M_j) \underset{(a)}{=} \dim_F \operatorname{Hom}_{FG}(P(M_i), M_j) = \delta_{ij}$.

"$\Leftarrow$": Assume $\dim_F JFG \neq |G| - |G|/u$

$$\underset{(a)}{\Rightarrow} \exists j\colon P(M_i) \oplus P(M_j) \mid P(1_G) \otimes_F M_i$$

$$\#(1_G, M_i^* \otimes_F M_j) \geq \dim_F \operatorname{Hom}_{FG}(P(M_i), M_j) + \dim_F \operatorname{Hom}_{FG}(P(M_j), M_j) = \dim_F \operatorname{Hom}_{FG}(P(M_i), M_j) + 1.$$

Case 1. $i = j \Rightarrow \#(1_G, M_i^* \otimes_F M_j) \geq 2$, a contradiction.

Case 2. $i \neq j \Rightarrow \#(1_G, M_i^* \otimes_F M_j) \geq 1$, a contradiction.

(c) Suppose $\exists i: p \mid \dim_F M_i \Rightarrow \#(1_G, M_i^* \otimes_F M_i) \geq 2$ (see Puttaswamaiah, Dixon [18, p. 133]). This contradicts (b). Q.E.D.

G. Michler observed the following fact.

Remark 3.7. Let G be p-solvable. Then: $p \nmid \dim_F M_i \ \forall i \Leftrightarrow S \trianglelefteq G$.

Proof.

$$p \nmid \dim_F M_i \Leftrightarrow \dim_F P(M_i) = |S| \dim_F M_i$$

(Hamernik, Michler [8, Corollary 2.3(a, b)])

$$\Leftrightarrow \dim_F P(M_i) = u \dim_F M_i$$

(since $u = |S|$)

$$\Leftrightarrow P(M_i) \cong P(1_G) \otimes_F M_i$$

(see (*) in the proof of Lemma 3.6(a)).

The result follows from Lemmas 1.3 and 3.6(a). Q.E.D.

LEMMA 3.8. *Let* $\dim_F JFG = |G| - |G|/u$.

(a) $u = |S|, \quad C \begin{pmatrix} \dim_F M_1 \\ \vdots \\ \dim_F M_k \end{pmatrix} = |S| \begin{pmatrix} \dim_F M_1 \\ \vdots \\ \dim_F M_k \end{pmatrix}.$

(b) Let $S \leq U \leq G$. Then $P(1_U) \cong P(1_G)|_U$. In particular $P(1_G)|_S \cong FS$.

Remark. After I finished the proof Professor Wallace informed me that Lemma 3.8(a) was already known to him. His (unpublished) proof is different from the following.

Proof. (a)

$$C \begin{pmatrix} \dim_F M_1 \\ \vdots \\ \dim_F M_k \end{pmatrix} = \begin{pmatrix} \dim_F P(M_1) \\ \vdots \\ \dim_F P(M_k) \end{pmatrix} \underset{\text{Lemma 3.6(a)}}{=} u \begin{pmatrix} \dim_F P(M_1) \\ \vdots \\ \dim_F P(M_k) \end{pmatrix}. \qquad (*)$$

Let $f(X)$ be the characteristic polynomial of C.

$$f(X) = \det \quad C - \begin{pmatrix} X & & 0 \\ & \ddots & \\ 0 & & X \end{pmatrix}$$
$$= a_k X^k + a_{k-1} X^{k-1} + \cdots + a_1 X + a_0; \qquad a_0, \ldots, a_k \in \mathbb{Z}.$$

By the Euclidean algorithm $f(X) = (X-u)(b_{k-1}X^{k-1} + \cdots + b_1 X + b_0) + c$ for certain $b_0, \ldots, b_{k-1}, c \in \mathbb{Z}$. $c = f(u) = 0$ by (*). Comparing coefficients: $-ub_0 = a_0 = \pm \det C$.

$\det C$ is a power of p (see Puttaswamaiah and Dixon [18, p. 106])

$\Rightarrow \exists b \in \mathbb{N}: u = p^b$

$\Rightarrow u \mid |S|$ (otherwise $\dim_F JFG = |G| - |G|/u \notin \mathbb{N}$).

By Lemma 3.3(b), $|S| \mid u$. Therefore $u = |S|$.

(b) $P(1_G)|_U$ is projective and $1_U \leqslant P(1_G)|_U$

$$\Rightarrow P(1_U) \mid P(1_G)|_U$$
$$\Rightarrow |S| \leqslant \dim_F P(1_U) \qquad \text{(Lemma 3.3(b))}$$
$$\leqslant \dim_F P(1_G)|_U = u = |S|.$$

Therefore $P(1_U) \cong P(1_G)|_U$. Q.E.D.

LEMMA 3.9. (a) $u = |S| \Rightarrow \forall i$: $\#(M_i, (1_S)^G) \leqslant \#(M_i, \text{soc } FG)$ $(= \dim_F M_i)$.

(b) $\dim_F JFG = |G| - |G|/u \Leftrightarrow (1_S)^G$ *and* soc FG *have the same composition factors* (*multiplicities included*).

(c) $\dim_F JFG = |G| - |G|/u \Leftrightarrow \omega(S)\, FG$ *and* JFG *have the same composition factors* (*multiplicities included*).

(d) $\dim_F JFG = |G| - |G|/u \Rightarrow JFG = \text{Ann}_{FG}(\alpha)$, soc $FG = \alpha FG$.

Proof. (a)

$$|S| \; \#(M_i, (1_S)^G) = \dim_F P(M_i) \qquad \text{(Lemma 3.3(b))}$$
$$\leqslant u \dim_F M_i \qquad \text{(see (*) in the proof of Lemma 3.6)}$$
$$= u \; \#(M_i, \text{soc } FG)$$
$$= |S| \; \#(M_i, \text{soc } FG).$$

(b) *Case* 1. $\dim_F JFG = |G| - |G|/u \Rightarrow u = |S|$ (Lemma 3.8(a)).

Case 2. $(1_S)^G$ and soc FG have same composition factors

$$\Rightarrow 1 = \#(1_G, (1_S)^G) = u/|S| \qquad \text{(Lemma 3.3(b))}.$$
$$\Rightarrow u = |S|.$$

Therefore $u = |S|$ in each case. It follows that $\#(M_i, (1_S)^G) \leqslant \#(M_i, \text{soc } FG)\ \forall i$ and

$$\text{“}=\text{”} \Leftrightarrow \forall i\colon \dim_F P(M_i) = u \dim_F M_i$$
$$\Leftrightarrow \dim_F JFG = |G| - |G|/u \qquad \text{(Lemma 3.6(a))}.$$

Part (c) follows from (b), since $FG/\omega(S)\, FG \cong (1_S)^G$ (see Passman [17, Lemma 1.2ii, p. 68]) and $FG/JFG \cong \text{soc } FG$.

(d) By Lemma 3.6(c), $p \nmid \dim_F M_i = \#(M_i, \text{soc } FG) \underset{(b)}{=} \#(M_i, (1_S)^G)$ $\forall i$ (Lemmas 3.1(d), 3.2)

$$\Rightarrow JFG = \text{Ann}_{FG}(\alpha)$$
$$\Rightarrow \alpha FG \cong FG/\text{Ann}_{FG}(\alpha) = FG/JFG \cong \text{soc } FG$$
$$\Rightarrow \text{soc } FG = \alpha FG \qquad \text{Lemma 3.4(a))}.$$

Q.E.D.

LEMMA 3.10. (a) $p \nmid \dim_F M_i\ \ \forall i \Rightarrow \alpha\hat{K} = |K|\,\alpha$ *for each p-conjugacy class K.*

(b) $p \nmid \dim_F M_i\ \forall i \Leftrightarrow JFG = \ker \chi_{\text{soc } FG}$.

(c) $$\dim_F JFG = |G| - \frac{|G|}{u} \Rightarrow \chi_{\text{soc } FG}(g)$$
$$= \begin{cases} |G\colon S| \cdot 1_F & \textit{if g p-element} \\ 0 & \textit{otherwise} \end{cases} \quad (g \in G).$$

(d) $\sum_{i=1}^{k} [u - l(P(M_i))] \dim_F M_i \geqslant 0$. “=” $\Leftrightarrow \dim_F JFG = |G| - |G|/u$.

(e) $\dim_F JFG = |G| - |G|/u \Rightarrow \text{rk } T_G = |G\colon S|$.

Proof. (a) It may be assumed that F is algebraically closed. $\forall i\, \exists c_i \in F$ $\forall m \in M_i\colon m\hat{K} = mc_i$ by Schur's lemma. $(\dim_F M_i) \cdot c_i = \chi_{M_i}(\hat{K}) = \sum_{g \in K} \chi_{M_i}(g) = \sum_{g \in K} \chi_{M_i}(1) = |K|\ (\dim_F M_i) \cdot 1_F$. Since

$$p \nmid \dim_F M_i\colon c_i = |K| \cdot 1_F$$
$$\Rightarrow x\hat{K} = |K|\, x \qquad \forall x \in \text{soc } FG$$
$$\Rightarrow \alpha\hat{K} = |K|\, \alpha.$$

Part (b) follows from Lemmas 3.1(b) and 3.2.

(c) $\chi_{\mathrm{soc}\, FG} = \chi_{(1_S)^G}$ (Lemma 3.9(b))

$= |N_G(S): S|\; T$ (Lemma 3.1(c))

$= |G: S|\; T$ (since $|G: N_G(S)| \equiv 1 \pmod p$).

(d) The multiplicity of $P(M_i)$ as a direct summand of FG is $\dim_F M_i$; therefore

$$\sum_{i=1}^{k} l(P(M_i)) \dim_F M_i$$

$$= l(FG) = \sum_i \#(M_i, FG) = \sum_i \dim_F P(M_i) \qquad \text{(Lemma 3.3(a))}$$

$$\leqslant \sum_i \dim_F (P(1_G) \otimes_F M_i) = \sum_i u \cdot \dim_F M_i.$$

It follows that $\sum_i [u - l(P(M_i))] \dim_F M_i \geqslant 0$ and

$$\text{“}=\text{”} \Leftrightarrow \forall i\colon P(M_i) \cong P(1_G) \otimes_F M_i$$

$$\Leftrightarrow \dim_F JFG = |G| - \frac{|G|}{u} \qquad \text{(Lemma 3.6(a))}.$$

(e) $\operatorname{rk} T_G = |G| - \dim_F JFG$ (Lemmas 3.2, 3.9(d))

$= |G: S|$ (Lemma 3.8(a)). Q.E.D.

LEMMA 3.11. *Let* $\dim_F JFG = |G| - |G|/u$ *and* $S \ntrianglelefteq G$ *and* $|G|$ *minimal having this property. Then*:

(a) $N \mathrel{\triangleleft\!\!\!\neq} G \Rightarrow p \mid |G: N|$.

(b) $O_p(G) = \langle 1 \rangle$.

Proof. (a) Assume

$$p \nmid |G: N| \Rightarrow \dim_F JFN = |G: N|^{-1} \dim_F JFG$$

(see Passman [17, p. 278])

$$= |G: N|^{-1} \left(|G| - \frac{|G|}{u} \right)$$

$$= |N| - \frac{|N|}{u}$$

$$= |N| - \frac{|N|}{\dim_F P(1_N)} \qquad \text{(Lemma 3.8(b))}$$

$\Rightarrow S \trianglelefteq N$ (by the minimality of $|G|$)

$\Rightarrow S \operatorname{char} N \trianglelefteq G \Rightarrow S \trianglelefteq G$, a contradiction.

$$\text{(b)} \qquad \dim_F JF[G/O_p(G)] = \dim_F JFG - |G| + |G\colon O_p(G)|$$

(see Brockhaus [3, Lemma 3(d)])

$$= |G| - \frac{|G|}{u} - |G| + |G\colon O_p(G)|$$

$$= |G/O_p(G)| - \frac{|G/O_p(G)|}{u\,|O_p(G)|^{-1}}$$

$$= |G/O_p(G)| - \frac{|G/O_p(G)|}{\dim_F P(1_{G/O_p(G)})}$$

(see Hamernik and Michler [8, p. 154]).

Assume $O_p(G) \neq \langle 1 \rangle$. Then, by minimality of $|G|$, $S/O_p(G) \trianglelefteq G/O_p(G)$ and therefore $S \trianglelefteq G$. Contradiction! It follows that $O_p(G) = \langle 1 \rangle$. Q.E.D.

The following lemma is due to R. Knörr and M. Lorenz.

LEMMA 3.12. *Let* $\dim_F JFG = |G| - |G|/u$ *and* $S \not\trianglelefteq G$ *and* $|G|$ *minimal having this property. Then* G *is simple.*

Proof. Assumption: $\exists N \trianglelefteq G\colon \langle 1 \rangle \neq N \neq |G|$.

(1) $\dim_F JF[G/N] = |G/N| - (G/N)/\dim_F P(1_{G/N})$.

Proof. Let V_1, V_2 be simple $F[G/N]$-modules. Define $vg := v(Ng)$ for $g \in G$, $v \in V_i$. Then V_i is also a simple FG-module with property $N \leqslant \ker V_i$ $(i \in \{1, 2\})$.

$$\#(1_{G/N}, V_1^* \otimes_F V_2) = \#(1_G, V_1^* \otimes_F V_2) = \begin{cases} 1 & \text{if } V_1 \cong_{FG} V_2 \\ 0 & \text{otherwise.} \end{cases}$$

Part (1) follows by Lemma 3.6(b), since $V_1 \cong_{FG} V_2 \Leftrightarrow V_1 \cong_{F[G/N]} V_2$. Therefore, by the minimality of $|G|$, $SN/N \trianglelefteq G/N \Rightarrow SN \trianglelefteq G \Rightarrow$

(2) $G = SN$ (by Lemma 3.11(a))

$$\Rightarrow |G\colon N| \quad \text{is a power of } p$$

$$\Rightarrow P(1_G) \cong (P(1_N))^G$$

by Green's theorem (see Puttaswamaiah and Dixon [18, p. 126]).

It follows that

(3) $u = |G\colon N| \dim_F P(1_N)$.

(4) $\dim_F JFN = |N| - |N|/\dim_F P(1_N)$.

Proof. $\varphi: FN/JFG \cap FN \to FG/JFG$, $x + JFG \cap FN \mapsto x + JFG$ defines an F-monomorphism. $JFG \cap FN \subseteq JFN$, since $JFG \cap FN$ is a nilpotent ideal of FN. It follows that

$$\begin{aligned}\dim_F JFN &\geqslant \dim_F JFG \cap FN = |N| - \dim_F FN/JFG \cap FN \\ &\geqslant |N| - \dim_F FG/JFG = |N| - |G| + \dim_F JFG \\ &= |N| - |G| + \left(|G| - \frac{|G|}{u}\right) \underset{(3)}{=} |N| - \frac{|N|}{\dim_F P(1_N)}.\end{aligned}$$

Therefore (4) holds.

It follows, by the minimality of $|G|$, that $S \cap N \trianglelefteq N$

$$\begin{aligned}&\Rightarrow S \cap N \text{ char } N \trianglelefteq G \Rightarrow S \cap N \trianglelefteq G \\ &\Rightarrow S \cap N = \langle 1 \rangle \qquad \text{(Lemma 3.11(b))} \\ &\underset{(2)}{\Rightarrow} G \text{ is } p\text{-nilpotent} \\ &\Rightarrow S \trianglelefteq G \qquad \text{(Lemma 1.3).}\end{aligned}$$

Contradiction! Q.E.D.

LEMMA 3.13. *Suppose* $u = |S|$. *Then*

(a) $S \leqslant U \leqslant G \Rightarrow O_{p'}(U) \leqslant O_{p'}(G)$.

(b) $|\mathrm{Bl}_p(G)| = 1 \Leftrightarrow O_{p'}(G) = \langle 1 \rangle$ *and each p-block of G has maximal defect.*

Proof. (a) $P(1_U) \cong P(1_G)|_U$, since $u = |S|$. Therefore $\ker P(1_U) \leqslant \ker P(1_G)$. By a theorem of H. Pahlings: $\ker P(1_U) = O_{p'}(U)$, $\ker P(1_G) = O_{p'}(G)$.

(b) "$\Rightarrow$": $O_{p'}(G) = \langle 1 \rangle$, since $[|O_{p'}(G)| \cdot 1_F]^{-1}\, \widehat{O_{p'}(G)}$ is a central idempotent.

"$\Leftarrow$": Apply (a) in case $U = N_G(S)$. It follows that $O_{p'}(U) = \langle 1 \rangle$. U is p-solvable; therefore $|\mathrm{Bl}_p(U)| = 1$ by Fong [6]. By Brauer's first main theorem (see Puttaswamaiah and Dixon [18, p. 151]) G has only one p-block of maximal defect. So $|\mathrm{Bl}_p(G)| = 1$. Q.E.D.

COROLLARY 3.14. *Suppose* $\dim_F JFG = |G| - |G|/u$ *and* $S \ntrianglelefteq G$ *and* $|G|$ *minimal having this property. Then G is simple, non-cyclic and has only one p-block.*

Proof. Lemmas 3.6(c), 3.8(a), 3.12, and 3.13. Q.E.D.

LEMMA 3.15. $\zeta \in \mathrm{Irr}(G)$, $x \in Z(S)$, $\zeta(x)=0 \Rightarrow \zeta$ does not belong to the principal p-block.

Proof. Trivial. Q.E.D.

4. GROUPS HAVING A SINGLE p-BLOCK

Assume $F := R/I$, where R is the ring of algebraic integers in $\mathbb{C}$ and I is a fixed maximal ideal of R containing pR. Then F is an algebraically closed field with characteristic p, algebraic over its prime field (see Isaacs [10, pp. 262–263]).

LEMMA 4.1. (a) *Let* $|\mathrm{Bl}_p(G)|=1$, $g \in G$. *Then*: $p \nmid |G : C_G(g)|$ $(=|g^G|) \Leftrightarrow g$ *lies in the centre of a Sylow p-subgroup.*

(b) *The following statements are equivalent*:

(i) $|\mathrm{Bl}_p(G)|=1$.

(ii) $\forall \zeta \in \mathrm{Irr}(G)$: $|g^G| \equiv |g^G|\ (\zeta(g)/\zeta(1)) \pmod I$.

(iii) $\hat{K} - |K| \cdot 1_F \in JFG$ *for each conjugacy class K.*

(iv) $\alpha\hat{K} = |K|\,\alpha$ *for each conjugacy class K.*

(v) $\hat{K}^{|S|}=0$ *for each p'-conjugacy class* $K \neq \{1\}$.

(vi) $C_G(S) \subseteq S$ *and for each p'-element* $g \neq 1$:

$$|g^G|\,\frac{\zeta(g)}{\zeta(1)} \equiv 0 \pmod I.$$

(vii) *Let K be a conjugacy class and* $g \in G$. *Then*

$$(\textit{number of p-elements in } gK) \equiv \begin{cases} |K| & \textit{if } g \textit{ is a p-element} \\ 0 & \textit{otherwise} \end{cases} \pmod p.$$

Proof. It holds that

(1) $|\mathrm{Bl}_p(G)| = \dim_F ZFG/J(ZFG)$ (Clarke [5, Lemma 2]).

(2) $J(ZFG) = ZFG \cap \mathrm{Ann}_{FG}(\alpha)$ (Iizuka and Watanabe [9]).

(3) $[J(ZFG)]^{|S|}=0$ (Okuyama [15]).

(4) $J(ZFG) \subseteq JFG$.

Proof. $x \in J(ZFG) \Rightarrow xFG$ is a nilpotent ideal of $FG \Rightarrow x \in xFG \subseteq JFG$.

(5) $|\mathrm{Bl}_p(G)|=1 \Rightarrow \alpha\hat{K} = |K|\,\alpha$ for each conjugacy class K.

Proof. By (1)

$$ZFG = F \oplus J(ZFG)$$

$$\Rightarrow \exists c_K \in F\colon \hat{K} - c_K \in J(ZFG) \underset{(4)}{\subseteq} JFG \underset{\text{Lemma 3.2}}{\subseteq} \mathrm{Ann}_{FG}(\alpha)$$

$$\Rightarrow \alpha\hat{K} = \alpha c_K \text{ and } \hat{K} - c_K \in \mathrm{Ann}_{FG}(\hat{G})$$

$$(\text{since } JFG \subseteq \mathrm{Ann}_{FG}(1_G) = \mathrm{Ann}_{FG}(\hat{G}))$$

$$\Rightarrow c_K\hat{G} = \hat{G}\hat{K} = |K|\,\hat{G} \Rightarrow c_K = |K| \cdot 1_F.$$

So (5) holds.

(a) Let $g \in G$.

Case 1. g lies in the centre of a Sylow p-subgroup $\Rightarrow p \nmid |G\colon C_G(g)|$.

Case 2. g does not lie in the centre of a Sylow p-subgroup

Case 2a. g is a p-element $\Rightarrow p \mid |G\colon C_G(g)|$.

Case 2b. g is not a p-element. $K := g^G$. $\alpha\hat{K} = |K|\,\alpha$ by (5). $1 \notin \mathrm{Supp}(\alpha\hat{K})$ (since K does not contain any p-element). Therefore $p \mid |K| = |G\colon C_G(g)|$.

(b) (i) $\Leftrightarrow$ (ii): Follows from Isaacs [10, p. 271].

(i) $\Rightarrow$ (iii): $\hat{K} - |K| \cdot 1_F \in_{(5)} ZFG \cap \mathrm{Ann}_{FG}(\alpha) =_{(2)} J(ZFG) \subseteq_{(4)} JFG$.

(iii) $\Rightarrow$ (iv): Follows from $JFG \subseteq \mathrm{Ann}_{FG}(\alpha)$.

(iv) $\Rightarrow$ (v): Let K be a p'-class $\neq \{1\}$. By assumption $\alpha\hat{K} = |K|\,\alpha$. $1 \notin \mathrm{Supp}(\alpha\hat{K})$, since K does not contain any p-element $\Rightarrow p \mid |K| \Rightarrow \alpha\hat{K} = 0 \Rightarrow \hat{K} \in ZFG \cap \mathrm{Ann}_{FG}(\alpha) =_{(2)} J(ZFG) \Rightarrow_{(3)} \hat{K}^{|S|} = 0$.

(v) $\Rightarrow$ (i): Let e be an idempotent in ZFG. By Osima's theorem (see Michler [12, p. 467])

$$e = \sum_{K\ p'\text{-class}} c_K\hat{K}, \qquad \text{where} \quad c_K \in F,$$

$$e = e^{|S|} = c_{\{1\}}^{|S|} + \sum_{K\ p'\text{-class} \neq \{1\}} c_K^{|S|}\hat{K}^{|S|} = c_{\{1\}}^{|S|}$$

by assumption. $\Rightarrow e \in F \Rightarrow e = 1$. Therefore $|\mathrm{Bl}_p(G)| = 1$.

(i) $\Rightarrow$ (vi): $g \in C_G(S) \Rightarrow S \subseteq C_G(g) \Rightarrow p \nmid |G\colon C_G(g)| \Rightarrow_{(a)} g$ is a p-element $\Rightarrow S\langle g\rangle$ p-group $\Rightarrow g \in S$. It follows that $C_G(S) \subseteq S$. Furthermore (ii) holds.

(vi) $\Rightarrow$ (ii): It may be assumed that g is a p'-element $\neq 1$. $p \mid |G\colon C_G(g)| = |g^G|$ (since $C_G(S) \subseteq S$). So $|g^G|\,(\zeta(g)/\zeta(1)) \equiv 0 \equiv |g^G|$ (mod p).

(iv) $\Leftrightarrow$ (vii): Let K be a conjugacy class. Choose an arrangement $g_1, ..., g_n$ for the elements of G such that $\{g_1, ..., g_{|K|}\} = K$ and define the matrix $T_G = (t_{ij})$ with respect to this arrangement.

Define an F-linear map $\varphi: FG \to FG$ by $\varphi(g_j) := \sum_i t_{ij} g_i$. Then $\varphi(g) = \alpha g \; \forall g \in G$ (see proof of Lemma 3.2).

$$\alpha\hat{K} = \sum_{g \in K} \varphi(g) = \sum_{j=1}^{|K|} \varphi(g_j) = \sum_{j=1}^{|K|} \sum_{i=1}^{n} t_{ij} g_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{|K|} t_{ij} \right) g_i.$$

Therefore $\alpha\hat{K} = |K| \, \alpha \Leftrightarrow \forall i$:

$$\sum_{j=1}^{|K|} t_{ij} = |K| \quad \text{if } g_i \text{ is a } p\text{-element} \qquad (\text{mod } p).$$
$$= 0 \quad \text{otherwise.}$$

Since $\sum_{j=1}^{|K|} t_{ij} =$ (number of p-elements in $g_i^{-1}K$), the result follows. Q.E.D.

5. Proof of Theorem 1.1

Let S_n denote the symmetric group on n symbols and A_n the corresponding alternating group.

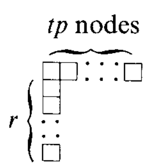
Let D be a Young diagram. The associated Young diagram D' arises by interchanging the rows and columns (see Kerber [11, p. 20]). The uniquely determined Young diagram $\tilde{D}$, which is obtained by removing as many p-hooks as possible, is called the p-core of D. $|D|$ denotes the number of nodes of D.

R. Knörr told me the following

Lemma 5.1. *Let $n \in \mathbb{N}$, $n \geqslant 4$, $n \geqslant p \neq 2$. Then there is a Young diagram D with $|D| = n$ and $|\tilde{D}| \geqslant p$.*

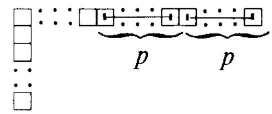
Proof. $\exists r, t \in \mathbb{N}$, $r < p$, $t \geqslant 1$: $n = tp + r$.

Case 1. $r \neq 0$. Consider the diagram

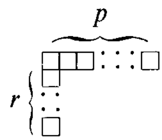


$|D| = n$

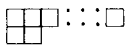
Remove successively p-hooks in the first row from the right:



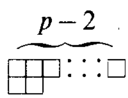
Then there remains the p-core $\tilde{D}$ with $|\tilde{D}| = p + r \geqslant p$:



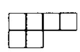
Case 2. $r = 0$. Consider D:



p-core, if $p \neq 3$:



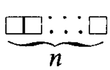
p-core for $p = 3$:



In both cases $|\tilde{D}| \geqslant p$. Q.E.D.

LEMMA 5.2. *$n \in \mathbb{N}$, $n \geqslant 4$, $2 \neq p \mid |A_n| \Rightarrow$ not every p-block of A_n has maximal defect.*

Proof. Let $|S_n| = p^a m$, $m \in \mathbb{N}$, $p \nmid m$. If $\zeta \in \mathrm{Irr}(S_n)$, let D_ζ denote its corresponding Young diagram and $d(\zeta)$ the defect of the p-block, which contains ζ. It follows from Kerber [11, p. 132, 7.4] that $d(\zeta) = a \Leftrightarrow |\tilde{D}_\zeta| = \min\{\tilde{D}_\xi \mid \xi \in \mathrm{Irr}(S_n)\}$. The diagram

D: 

has $|\tilde{D}| < p$. Therefore

(1) $\forall \zeta \in \mathrm{Irr}(S_n)$: $d(\zeta) = a \Leftrightarrow |\tilde{D}_\zeta| < p$. Since $p \mid |A_n| \Leftrightarrow p \mid |S_n| \Leftrightarrow p \leqslant n$ it follows from Lemma 5.1 that

(2) $\exists \zeta_1 \in \mathrm{Irr}(S_n)$: $|\tilde{D}_{\zeta_1}| \geqslant p$. Let φ_1 be an irreducible constituent of

$\zeta_{1|A_n}$ and B the p-block of A_n which contains φ_1. Assumption: B has maximal defect. Therefore

(3) $\exists \varphi_2 \in B$: $p \nmid \varphi_2(1)$. By Kerber [11, p. 133, 7.6] there exists $\zeta_2 \in \mathrm{Irr}(S_n)$: φ_2 is a constituent of $\zeta_{2|A_n}$ and $\tilde{D}_{\zeta_2} = \tilde{D}_{\zeta_1}$ or $\tilde{D}_{\zeta_2} = \widetilde{(D'_{\zeta_1})}$. In each case $|\tilde{D}_{\zeta_2}| = |\tilde{D}_{\zeta_1}| \geqslant_{(2)} p$, since $|\widetilde{(D'_{\zeta_1})}| = |\tilde{D}_{\zeta_1}|$. $\Rightarrow_{(1)} d(\zeta_2) \neq a$. It follows that

(4) $p \mid \zeta_2(1)$. By Kerber [11, p. 85, 4.54]: $\zeta_{2|A_n} = \varphi_2$ or $\zeta_{2|A_n} = \varphi_2 + \varphi_3$, where $\varphi_3 \in \mathrm{Irr}(A_n)$ and φ_2, φ_3 are conjugate $\Rightarrow \varphi_2(1) = \zeta_2(1)$ or $\varphi_2(1) = \zeta_2(1)/2 \Rightarrow_{(4)} p \mid \varphi_2(1)$, since $p \neq 2$. This contradicts (3). Therefore B does not have maximal defect. Q.E.D.

LEMMA 5.3. The 26 known sporadic simple groups have at least two p-blocks, if $p \neq 2$.

Remark. The Mathieu groups M_{22} and M_{24} have only one p-block. (For M_{24} see Brauer [1, p. 162].)

Proof of Lemma 5.3. Follows from the character tables, using Lemma 3.15. Q.E.D.

Since by the classification theorem every non-cyclic finite simple group is an alternating group or a group of Lie type or one of the 26 known sporadic groups, Theorem 1.1 now follows from Okuyama's result for $p = 2$ (see Introduction) ("$\Rightarrow$" follows from Lemma 1.3).

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REFERENCES

1. R. BRAUER, Some applications of the theory of blocks of characters of finite groups, I, *J. Algebra* **1** (1964), 152–167.
2. R. BRAUER AND C. J. NESBITT, On the modular characters of groups, *Ann. of Math. (2)* **42** (1941), 556–590.
3. P. BROCKHAUS, A remark on the radical of a group algebra, *Proc. Edinburgh Math. Soc.* **25** (1982), 69–71.
4. P. BROCKHAUS AND G. MICHLER, Finite simple groups of Lie type have non-principal p-blocks, $p \neq 2$, to appear.
5. R. J. CLARKE, On the radical of the centre of a group algebra, *J. London Math. Soc. (2)* **1** (1969), 565–572.
6. P. FONG, On the characters of p-solvable groups, *Trans. Amer. Math. Soc.* **98** (1961), 263–284.

7. E. Formanek and R. L. Snider, The Jacobson radical of the group algebra of a p-solvable group (1977), unpublished.
8. W. Hamernik and G. Michler, On vertices of simple modules in p-solvable groups, *Mitt. Math. Inst. Univ. Gießen* **121** (1976).
9. K. Iizuka and A. Watanabe, On the number of blocks of irreducible characters of a finite group with a given defect group, *Kumamoto J. Sci. Math.* **9** (1973), 55–61.
10. I. M. Isaacs, "Character Theory of Finite Groups," Academic Press, New York, 1976.
11. A. Kerber, "Representations of Permutation Groups, I," Lecture Notes in Mathematics No. 240, Springer-Verlag, Berlin, 1971.
12. G. Michler, Blocks and centers of group algebras, *in* "Lecture Notes in Mathematics No. 246," pp. 430–465, Springer-Verlag, Berlin, 1973.
13. K. Motose, On radicals of principal blocks, *Hokkaido Math. J.* **6** (1977), 255–259.
14. T. Okuyama, Some studies on group algebras, *Hokkaido Math. J.* **9** (1980), 217–221.
15. T. Okuyama, On the radical of the center of a group algebra, to appear.
16. T. Okuyama, On a problem of Wallace, to appear.
17. D. S. Passman, "The Algebraic Structure of Group Rings," Wiley, New York, 1977.
18. B. M. Puttaswamaiah and J. D. Dixon, "Modular Representations of Finite Groups," Academic Press, New York, 1977.
19. Y. Tsushima, On the annihilator ideals of the radical of a group algebra, *Osaka J. Math.* **8** (1971), 91–97.
20. Y. Tsushima, Some notes on the radical of a finite group ring, *Osaka J. Math.* **15** (1978), 647–653.
21. D. A. R. Wallace, On the radical of a group algebra, *Proc. Amer. Math. Soc.* **12** (1961), 133–137.