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Bijections on two variations of noncrossing partitions

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ABSTRACT

We find bijections on 2-distant noncrossing partitions, 12312-avoiding partitions, 3-Motzkin paths, UH-free Schröder paths and Schröder paths without peaks at even height. We also give a direct bijection between 2-distant noncrossing partitions and 12312-avoiding partitions.

1. Introduction

Noncrossing partitions were first introduced by Kreweras [6] in 1972. Recently, they have received great attention, and have been generalized in many different ways; for instance, see [1–3,5,7] and the references therein. In this paper, we consider two variations of noncrossing partitions: k-distant noncrossing partitions and $12\cdots r12$-avoiding partitions introduced by Drake and Kim [3], and Mansour and Severini [7] respectively, where they reduce to noncrossing partitions when $k = 1$ and $r = 2$.

A (set) *partition* of $[n] = \{1, 2, \ldots, n\}$ is a collection of mutually disjoint nonempty subsets, called *blocks*, of $[n]$ whose union is $[n]$. We will write a partition as a sequence of blocks $(B_1, B_2, \ldots, B_k)$ such that $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$. An *edge* of a partition is a pair (i, j) of integers contained in the same block that does not contain any integer t with $i < t < j$. The *standard representation* of a partition π of $[n]$ is the diagram having n vertices labeled with $1, 2, \ldots, n$, where i and j are connected by an arc if (i, j) is an edge of π; see Fig. 1. A *noncrossing partition* is a partition without any two crossing edges, i.e. (i_1, j_1) and (i_2, j_2) such that $i_1 < i_2 < j_1 < j_2$. It is well known that the number of noncrossing partitions of $[n]$ is the Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

For a positive integer k, a *k-distant noncrossing partition* is a partition without any two edges (i_1, j_1) and (i_2, j_2) satisfying $i_1 < i_2 < j_1 < j_2$ and $j_1 - i_2 \ge k$. Note that 1-distant noncrossing partitions are just noncrossing partitions. We denote by $\mathrm{NC}_k(n)$ the set of k-distant noncrossing partitions of $[n]$. Drake and Kim [3] found the following generating function for the number of 2-distant noncrossing partitions:

$$\sum_{n\ge 0} \#\mathrm{NC}_2(n)x^n = \frac{3 - 3x - \sqrt{1 - 6x + 5x^2}}{2(1 - x)}. \tag{1}$$

The *canonical word* of a partition $\pi = (B_1, B_2, \ldots, B_k)$ is the word $a_1a_2\cdots a_n$, where $a_i = j$ if $i \in B_j$. For instance, the canonical word of the partition in Fig. 1 is 123124412. In the literature canonical words are also called restricted growth

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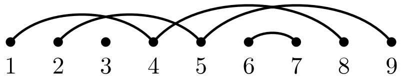

Fig. 1. The standard representation of $(\{1, 4, 8\}, \{2, 5, 9\}, \{3\}, \{6, 7\})$.

$$\begin{tikzcd}
\mathrm{NC}'_2(n) \arrow[d, "\psi"] & & \mathrm{SCH}'_{\mathrm{even}}(n-1) & P'_{12312}(n) \\
\mathrm{CH}'_2(n) \arrow[r, "f"] & \mathrm{MOT}_3(n-2) \arrow[r, "g"] & \mathrm{SCH}_{\mathrm{odd}}(n-1) \arrow[u, "\iota"] \arrow[r, "h"] & \mathrm{SCH}'_{\mathrm{UH}}(n-1) \arrow[u, "\phi"]
\end{tikzcd}$$

Fig. 2. Main bijections for $n \geq 2$.

functions. For a word τ, a partition is called *τ-avoiding* if its canonical word does not contain a subword which is order-isomorphic to τ. It is easy to see that a partition is noncrossing if and only if it is 1212-avoiding. We denote by $P_\tau(n)$ the set of τ-avoiding partitions of $[n]$.

Using the kernel method, Mansour and Severini [7] found the generating function for the number of $12 \cdots r12$-avoiding partitions of $[n]$. Interestingly, as a special case of their result, the generating function for the number of 12 312-avoiding partitions of $[n]$ is the same as (1), which implies $\#\mathrm{NC}_2(n) = \#P_{12312}(n)$. Moreover, this number also counts several kinds of lattice paths. The main purpose of this paper is to find bijections between $\mathrm{NC}_2(n)$ and $P_{12312}(n)$ together with some lattices paths described below.

A *lattice path of length n* is a sequence of points in $\mathbb{N} \times \mathbb{N}$ starting at $(0, 0)$ and ending at $(n, 0)$. For a lattice path $L = ((x_0, y_0), (x_1, y_1), \dots, (x_k, y_k))$, each $S_i = (x_i - x_{i-1}, y_i - y_{i-1})$ is called a *step* of L. The *height* of the step S_i is defined to be y_{i-1}. Sometimes we will identify a lattice path L with the word $S_1S_2 \dots S_k$ of its steps. Note that the number of steps is not necessarily equal to the length of the lattice path.

Let U, D and H denote an *up step*, a *down step* and a *horizontal step* respectively, i.e., $U = (1, 1)$, $D = (1, -1)$ and $H = (1, 0)$.

A *Schröder path* is a lattice path consisting of steps U, D and $H^2 = HH = (2, 0)$. Let $L = S_1S_2 \cdots S_k$ be a Schröder path. A *UH-pair* of L is a pair (S_i, S_{i+1}) of consecutive steps such that $S_i = U$ and $S_{i+1} = H^2$. We say that L is *UH-free* if it does not have a UH-pair. A *peak* of L is a pair (S_i, S_{i+1}) of consecutive steps such that $S_i = U$ and $S_{i+1} = D$. The *height* of a peak (S_i, S_{i+1}) is the height of $S_{i+1} = D$. We denote by $\mathrm{SCH}_{\mathrm{UH}}(n)$ the set of UH-free Schröder paths of length $2n$, and by $\mathrm{SCH}_{\mathrm{even}}(n)$ (resp. $\mathrm{SCH}_{\mathrm{odd}}(n)$) the set of Schröder paths of length $2n$ which have no peaks of even (resp. odd) height.

A *labeled step* is a step together with an integer label. Let D_i (resp. H_i) denote a labeled down step (resp. a labeled horizontal step) with label i. We denote by $\mathrm{CH}_2(n)$ the set of lattice paths $L = S_1S_2 \cdots S_n$ of length n consisting of U, D_1, D_2, H_0, H_1 and H_2 such that

- if $S_i = H_\ell$ or $S_i = D_\ell$, then S_i is of height at least ℓ,
- if $S_i = H_2$ or $S_i = D_2$, then $i \geq 2$ and $S_{i-1} \in \{U, H_1, H_2\}$.

A 3-*Motzkin path* is a lattice path consisting of U, D, H_0, H_1 and H_2. We denote by $\mathrm{MOT}_3(n)$ the set of 3-Motzkin paths of length n.

Drake and Kim [3] showed that the well-known bijection ψ between partitions and Charlier diagrams, see [4,5], yields a bijection $\psi : \mathrm{NC}_2(n) \to \mathrm{CH}_2(n)$. Yan [10] found a bijection $\phi : \mathrm{SCH}_{\mathrm{UH}}(n-1) \to P_{12312}(n)$ and a bijection between $\mathrm{SCH}_{\mathrm{UH}}(n)$ and $\mathrm{SCH}_{\mathrm{even}}(n)$. Thus all of $\mathrm{NC}_2(n)$, $\mathrm{CH}_2(n)$, $\mathrm{SCH}_{\mathrm{even}}(n-1)$, $\mathrm{SCH}_{\mathrm{UH}}(n-1)$ and $P_{12312}(n)$ have the same cardinality, which is counted by sequence A007317 from [8]. In order to find bijections between these objects, we introduce the following sets:

- $\mathrm{NC}'_2(n) = \{\pi \in \mathrm{NC}_2(n) : n \text{ is not a singleton}\}$
- $\mathrm{CH}'_2(n) = \{L \in \mathrm{CH}_2(n) : \text{the last step of } L \text{ is } D_1\}$
- $\mathrm{SCH}'_{\mathrm{even}}(n) = \{L \in \mathrm{SCH}_{\mathrm{even}}(n) : \text{the first step of } L \text{ is } U\}$
- $\mathrm{SCH}'_{\mathrm{UH}}(n) = \{L \in \mathrm{SCH}_{\mathrm{UH}}(n) : \text{the first step of } L \text{ is } U\}$
- $P'_{12312}(n) = \{\pi \in P_{12312}(n) : 1 \text{ and } 2 \text{ are not in the same block}\}$.

Note that we can identify $\pi \in \mathrm{NC}_2(n)$ with $\pi' \in \mathrm{NC}'_2(k)$, where k is the integer such that j is a singleton for all $j \in \{k+1, k+2, \dots, n\}$ and k is not a singleton in π, and π' is the partition obtained from π by deleting integers greater than k. We can also identify $\pi \in P_{12312}(n)$ with $\overline{\pi} \in P'_{12312}(k)$, where k is the integer such that the number of consecutive 1's at the beginning of the canonical word of π is $n - k + 1$, and $\overline{\pi}$ is the partition whose canonical word is obtained from that of π by deleting the first $n - k$ 1's. Thus any bijection between $\mathrm{NC}'_2(n)$ and $P'_{12312}(n)$ naturally induces a bijection between $\mathrm{NC}_2(n)$ and $P_{12312}(n)$. Similarly, any bijection between $A'(n)$ and $B'(n)$ naturally induces a bijection between $A(n)$ and $B(n)$ where A and B are any two of NC_2, CH_2, $\mathrm{SCH}_{\mathrm{even}}$, $\mathrm{SCH}_{\mathrm{UH}}$, and P_{12312}. Thus in order to find a bijection between $\mathrm{NC}_2(n)$ and $P_{12312}(n)$, it is enough to find a bijection between $\mathrm{NC}'_2(n)$ and $P'_{12312}(n)$.

In this paper, we find bijections between these objects. For the overview of our bijections, see Fig. 2, where ψ is the known bijection between partitions and Charlier diagrams [4,5], and ϕ is Yan's bijection [10]. We note that our bijection g in Fig. 2 is also discovered by Shapiro and Wang [9]. We also provide a direct bijection between $\mathrm{NC}_2(n)$ and $P_{12312}(n)$ in Section 3.

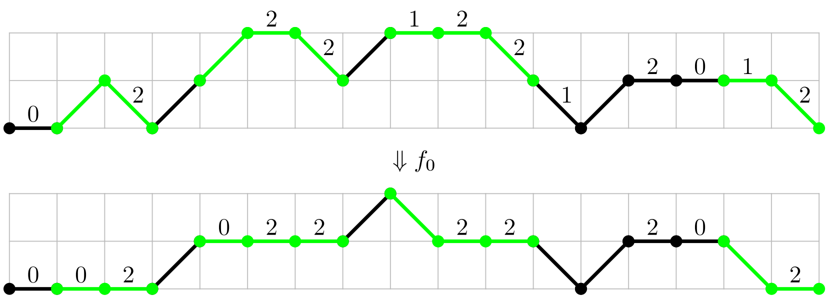

Fig. 3. An example of f_0.

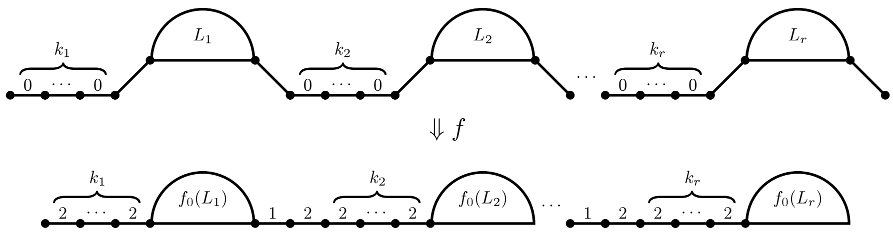

Fig. 4. Definition of f.

2. Bijections

In this section, we find the bijections f, g, h, and ι in Fig. 2.

2.1. The bijection $f : \mathrm{CH}_2'(n) \to \mathrm{MOT}_3(n-2)$

Recall that $\mathrm{CH}_2'(n)$ is the set of lattice paths $L = S_1S_2\cdots S_n$ of length n consisting of U, D_1, D_2, H_0, H_1 and H_2 such that

- if $S_i = H_\ell$ or $S_i = D_\ell$, then S_i is of height at least ℓ,
- if $S_i = H_2$ or $S_i = D_2$, then $i \geq 2$ and $S_{i-1} \in \{U, H_1, H_2\}$,
- $S_n = D_1$.

The second condition above is equivalent to the condition that the lattice path consists of the following combined steps for any $k \geq 0$:

$$UH_2^k,\ UH_2^kD_2,\ H_1H_2^k,\ H_1H_2^kD_2,\ H_0,\ D_1. \tag{2}$$

Let $A(n)$ denote the set of lattice paths of length n consisting of the combined steps in Eq. (2) such that H_2 does not touch the x-axis. Let $B(n)$ denote the set of 3-Motzkin paths of length n such that each H_2 touching the x-axis must occur after D, H_0 or H_2.

We define $f_0 : A(n) \to B(n)$ as follows. Let $L \in A(n)$. Then $f_0(L)$ is defined to be the lattice path obtained from L by changing $UH_2^kD_2$ to $H_0H_2^{k+1}$, $H_1H_2^kD_2$ to DH_2^{k+1} and D_1 to D. It is easy to see that $f_0(L) \in B$ and f_0 is invertible; see Fig. 3.

Now we define $f : \mathrm{CH}_2'(n) \to \mathrm{MOT}_3(n-2)$ as follows. Let $L \in \mathrm{CH}_2'(n)$. Then L is decomposed uniquely as

$$H_0^{k_1}(UL_1D_1)H_0^{k_2}(UL_2D_1)\cdots H_0^{k_r}(UL_rD_1),$$

where $L_i \in A(n_i)$ for some $k_i, n_i \geq 0$ and $r \geq 1$. Then define $f(L)$ to be

$$H_2^{k_1}f_0(L_1)(H_1H_2^{k_2+1}f_0(L_2))(H_1H_2^{k_3+1}f_0(L_3))\cdots(H_1H_2^{k_r+1}f_0(L_r)).$$

See Fig. 4.

Theorem 2.1. *The map $f : \mathrm{CH}_2'(n) \to \mathrm{MOT}_3(n-2)$ is a bijection.*

Proof. Each $L \in \mathrm{MOT}_3(n-2)$ is uniquely decomposed as

$$H_2^{k_1}L_1(H_1H_2^{k_2+1}L_2)(H_1H_2^{k_3+1}L_3)\cdots(H_1H_2^{k_r+1}L_r),$$

where $L_i \in B(n_i)$ for some $k_i, n_i \geq 0$ and $r \geq 1$. Thus we have the inverse $f^{-1}(L)$ which is decomposed as

$$H_0^{k_1}(Uf_0^{-1}(L_1)D_1)H_0^{k_2}(Uf_0^{-1}(L_2)D_1)\cdots H_0^{k_r}(Uf_0^{-1}(L_r)D_1). \quad \square$$

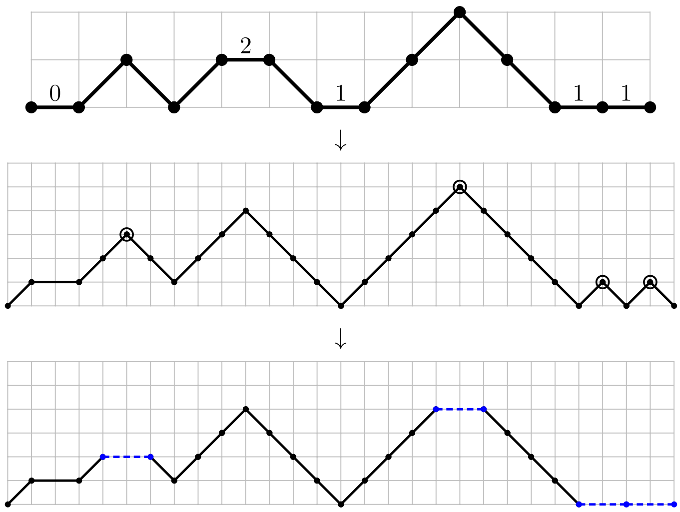

Fig. 5. An example of g. Odd peaks are circled. The horizontal steps of even height are dashed and colored blue. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

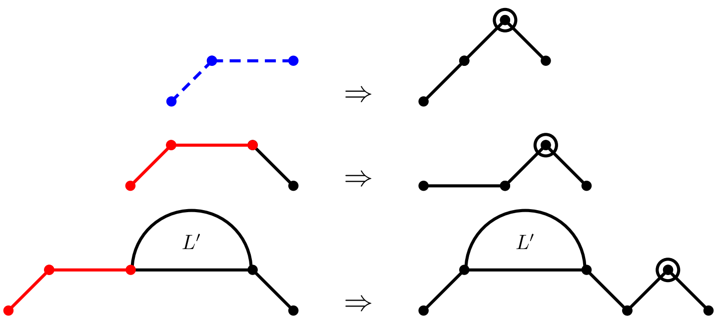

Fig. 6. The essence of h_0. Red (resp. dashed blue) color is for UH-pairs whose horizontal step is of odd (resp. even) height. Odd peaks are circled. The lattice path L' is not empty. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

2.2. The bijection $g : \mathrm{MOT}_3(n) \to \mathrm{SCH}_{\mathrm{odd}}(n+1)$

We define $g : \mathrm{MOT}_3(n) \to \mathrm{SCH}_{\mathrm{odd}}(n+1)$ as follows. Let $L \in \mathrm{MOT}_3(n)$. Then $g(L)$ is the lattice path obtained from L by doing the following.

1. Change U to UU, D to DD, H_0 to H^2, H_1 to DU, and H_2 to UD.
2. Add U at the beginning and D at the end.
3. Change all the consecutive steps UD which form a peak of odd height to H^2.

See Fig. 5 for an example of g.

Theorem 2.2. *The map* $g : \mathrm{MOT}_3(n) \to \mathrm{SCH}_{\mathrm{odd}}(n+1)$ *is a bijection.*

Proof. Clearly the first and the second steps in the construction of g are invertible. The third step is also invertible because every step H^2 of even height always comes from a peak of odd height. Thus g is invertible. □

2.3. The bijection $h : \mathrm{SCH}_{\mathrm{odd}}(n) \to \mathrm{SCH}'_{\mathrm{UH}}(n)$

Let $L = S_1S_2\cdots S_k$ be a Schröder path. For any up step $S_i = U$ of L, there is a unique down step $S_j = D$ such that $i < j$ and $S_{i+1}S_{i+2}\cdots S_{j-1}$ is a (possibly empty) lattice path. We call such S_j *the down step corresponding* to S_i. We also call S_i *the up step corresponding* to S_j.

For a UH-pair (S_i, S_{i+1}), i.e. $S_i = U$ and $S_{i+1} = H^2$, we define the function ξ as follows.

$$\xi(S_i, S_{i+1}) = \begin{cases} i, & \text{if } S_{i+1} \text{ is of even height;} \\ j, & \text{if } S_{i+1} \text{ is of odd height,} \end{cases}$$

where j is the integer such that S_j is the down step corresponding to S_i. If L is not UH-free, we define the ξ-*maximal* UH-pair of L to be the UH-pair (S_i, S_{i+1}) with the largest ξ value.

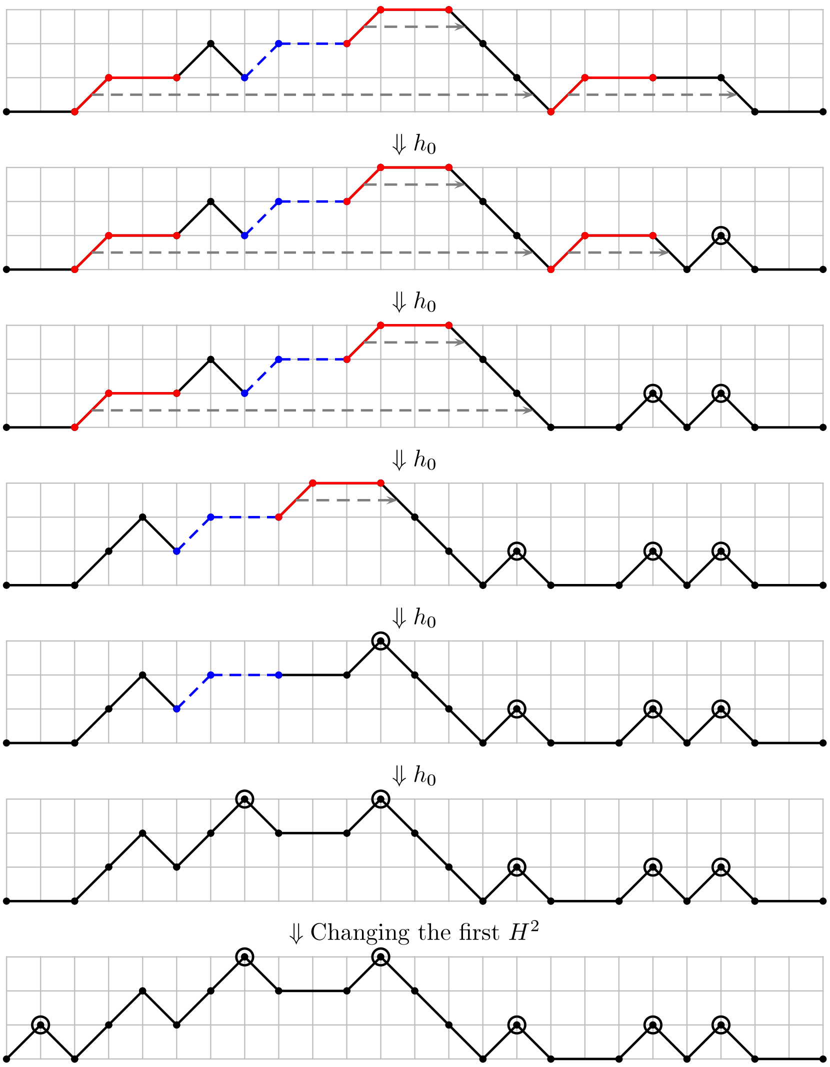

Fig. 7. An example of h. Red (resp. dashed blue) color is for UH-pairs whose horizontal step is of odd (resp. even) height. Odd peaks are circled. Dashed arrows indicate the down steps corresponding to the up steps. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Now let $L = S_1 S_2 \cdots S_k \in \mathrm{SCH}_{\mathrm{odd}}(n)$. If L is not UH-free, we define $h_0(L)$ as follows. Suppose (S_i, S_{i+1}) is the ξ-maximal UH-pair of L, and S_j is the down step corresponding to S_i.

1. If S_{i+1} is of even height, then $h_0(L)$ is the lattice path obtained from L by replacing $S_i S_{i+1}$ with UUD.
2. If S_{i+1} is of odd height, then let $L' = S_{i+2} S_{i+3} \cdots S_{j-1}$.
   (a) If L' is empty, i.e., $j = i + 2$, then $h_0(L)$ is the lattice path obtained from L by replacing $S_i S_{i+1} S_{i+2}$ with $H^2 UD$.
   (b) If L' is not empty, then $h_0(L)$ is the lattice path obtained from L by replacing $S_i S_{i+1} \cdots S_j$ with $UL'DUD$.

See Fig. 6.

Now we define $h : \mathrm{SCH}_{\mathrm{odd}}(n) \to \mathrm{SCH}'_{\mathrm{UH}}(n)$ as follows. Let $L \in \mathrm{SCH}_{\mathrm{odd}}(n)$ and $L_0 = L$. Then we define $L_i = h_0(L_{i-1})$ for $i \geq 1$ if L_{i-1} is not UH-free. Since the number of UH-free pairs of L_i is one less than that of L_{i-1}, or they are the same and

$$\xi(\text{the maximal UH-pair of } L_i) < \xi(\text{the maximal UH-pair of } L_{i-1}),$$

we always get L_r which is UH-free for some r. We define $h(L)$ to be L_r if L_r does not start with H^2, and the lattice path obtained from L_r by replacing H^2 with UD otherwise. For an example, see Fig. 7.

Theorem 2.3. *The map $h : \mathrm{SCH}_{\mathrm{odd}}(n) \to \mathrm{SCH}'_{\mathrm{UH}}(n)$ is a bijection.*

Proof. In the procedure of h, the odd peaks are constructed from right to left. Since h_0 is invertible, so is h.

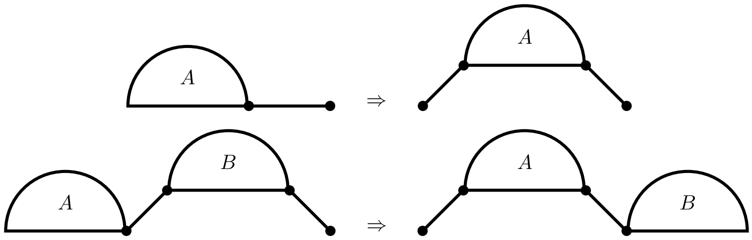

Fig. 8. The map ι.

2.4. The bijection $\iota : \mathrm{SCH}_{\mathrm{odd}}(n) \to \mathrm{SCH}'_{\mathrm{even}}(n)$

For $L = S_1S_2 \cdots S_k \in \mathrm{SCH}_{\mathrm{odd}}(n)$, we define $\iota(L)$ as follows.

1. If $S_k = H^2$, then $\iota(L) = US_1 \cdots S_{k-1}D$.
2. If $S_k = D$, then let S_i be the up step corresponding to S_k and we define $\iota(L) = US_1 \cdots S_{i-1}DS_{i+1} \cdots S_{k-1}$.

See Fig. 8.

Then $\iota(L) \in \mathrm{SCH}'_{\mathrm{even}}(n)$. Clearly, $\iota : \mathrm{SCH}_{\mathrm{odd}}(n) \to \mathrm{SCH}'_{\mathrm{even}}(n)$ is a bijection.

3. A direct bijection between $\mathrm{NC}_2(n)$ and $P_{12312}(n)$

Now we have a bijection $\phi \circ h \circ g \circ f \circ \psi : \mathrm{NC}'_2(n) \to P'_{12312}(n)$; see Fig. 2. As noted in the introduction, this induces a bijection between $\mathrm{NC}_2(n)$ and $P_{12312}(n)$. Since both $\mathrm{NC}_2(n)$ and $P_{12312}(n)$ are partitions with some conditions, it is natural to ask a direct bijection between them. In this section we find such a direct bijection.

From now on, we will identify a partition in $P_{12312}(n)$ with its canonical word.

A *marked* partition is a partition in which each part may be marked. Similarly a *marked* word is a word in which each letter may be marked.

Let $\pi \in \mathrm{NC}_2(n)$. For $i \in [n]$, let T_i be the marked partition of $[i]$ obtained from π by removing all the integers greater than i and by marking integers which are connected to an integer greater than i. Using the sequence $\emptyset = T_0, T_1, T_2, \ldots, T_n = \pi$ of marked partitions, we define a sequence of marked words $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ as follows. Here, if (i, j) is an edge we say that *j is connected to i.*

Let $\mathbf{w}_0$ be the empty word. For $1 \le i \le n$, $\mathbf{w}_i$ is defined as follows.

1. If i is not connected to any integer in T_i, then $\mathbf{w}_i = \mathbf{w}_{i-1}m$, where $m = \max(\mathbf{w}_{i-1}) + 1$. Otherwise, i is connected to either the largest marked integer or the second largest marked integer of T_{i-1}.
   - If i is connected to the largest marked integer of T_{i-1}, then let $\mathbf{w}_i = \mathbf{w}_{i-1}a_1$, where a_1 is the rightmost marked letter of $\mathbf{w}_{i-1}$. And then we make the marked letter a_1 unmarked.
   - If i is connected to the second largest marked integer of T_{i-1}, then let $\mathbf{w}_i = \mathbf{w}_{i-1}a_2$, where a_2 is the second rightmost marked letter of $\mathbf{w}_{i-1}$. The second rightmost marked letter of $\mathbf{w}_{i-1}$ remains marked; however, we make the rightmost marked letter of $\mathbf{w}_{i-1}$ unmarked in $\mathbf{w}_i$.
2. If i is marked in T_i, then we find the largest letters in $\mathbf{w}_i$ and make the leftmost letter among them marked.

For an example, see Fig. 9.

Lemma 3.1. *The word $\mathbf{w}_n$ obtained above is* 12312*-avoiding.*

Proof. Suppose $\mathbf{w}_n$ has a subsequence $abcab$ where $a < b < c$. When the second b is added the first b must have been marked. Moreover, the first b must have been marked before adding the second a because an unmarked integer becomes marked only if it is the largest integer (in this case at least c) in the sequence. Thus when the second a is added, the first a and b have been marked. Since the first a is the second rightmost marked integer at this moment, we must unmark the rightmost marked integer, the first b, and mark the largest integer which is at least c. Thus after this process, b cannot be marked and we cannot have the second b, which is a contradiction. $\square$

If we know $\mathbf{w}_n$, we can reverse this procedure. For $1 \le i \le n$, $\mathbf{w}_{i-1}$ is obtained from $\mathbf{w}_i$ as follows. Suppose $m = \max(\mathbf{w}_i)$ and t is the last letter of $\mathbf{w}_i$.

1. If the leftmost m is marked in $\mathbf{w}_i$, then make it unmarked.
2. If t appears only once in $\mathbf{w}_i$ (equivalently t is greater than any other letters in $\mathbf{w}_i$), then we simply remove t. Otherwise, find the leftmost t in $\mathbf{w}_i$.
   - If the leftmost t is unmarked, then we remove the last letter t and make the leftmost t marked.
   - If the leftmost t is marked, then we must have $t < m$ since we have made the leftmost m unmarked. In this case we remove the last t, and make the leftmost t still marked and the leftmost m marked.

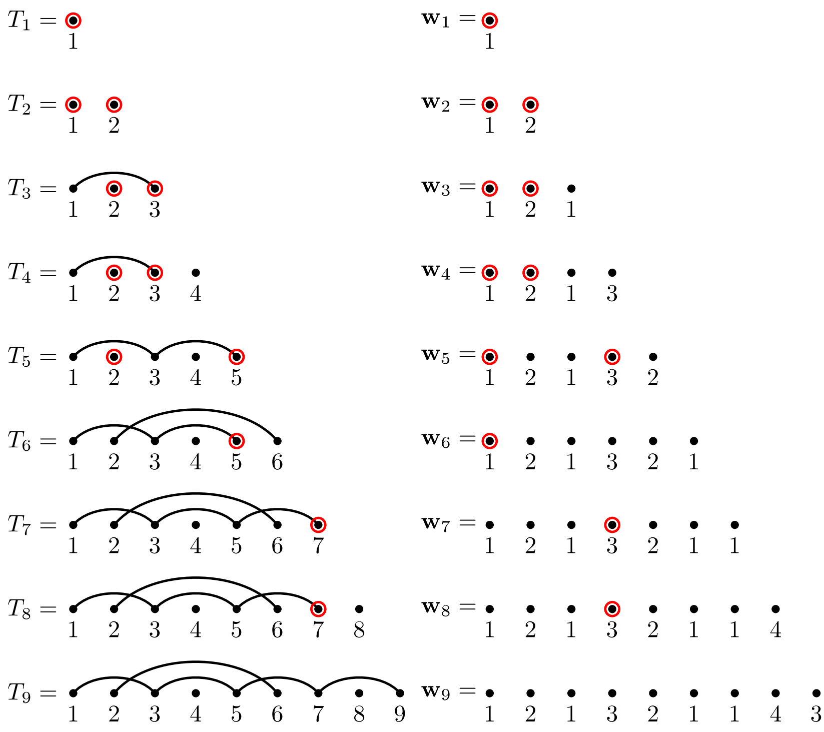

Fig. 9. T_i's and corresponding $\mathbf{w}_i$'s. Marked integers and marked letters are circled.

Now we construct $T_0, T_1, \dots, T_n$ as follows. Let $T_0 = \emptyset$. For $1 \le i \le n$, T_i is obtained as follows.

1. First, let T_i be the marked partition obtained from T_{i-1} by adding i.
2. If the last letter of $\mathbf{w}_i$ is equal to the rightmost (resp. the second rightmost) marked letter of $\mathbf{w}_{i-1}$, then connect i to the largest (resp. the second largest) marked integer, say j, of T_{i-1}, and make j unmarked.
3. Let $m = \max(\mathbf{w}_i)$. If the leftmost m is marked in $\mathbf{w}_i$, then make i marked in T_i.

It is easy to check that this is the inverse map. Thus we get the following theorem.

Theorem 3.2. *For $\pi \in \mathrm{NC}_2(n)$, the map $\pi \mapsto \mathbf{w}_n$ is a bijection from $\mathrm{NC}_2(n)$ to $P_{12312}(n)$.*

The bijection $\pi \mapsto \mathbf{w}_n$ is different from the composition $\phi \circ h \circ g \circ f \circ \psi$. For instance, if $\pi = (\{1, 3\}, \{2\})$, then $\mathbf{w}_3 = 121$ but $(\phi \circ h \circ g \circ f \circ \psi)(\pi) = 112$.

Note that both $\mathrm{NC}_2(n)$ and $P_{12312}(n)$ contain noncrossing partitions. It would be interesting to find a bijection between $\mathrm{NC}_2(n)$ and $P_{12312}(n)$ which sends noncrossing partitions to noncrossing partitions.

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