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Total domination excellent trees ☆

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Abstract

A set S of vertices in a graph G is a total dominating set of G if every vertex of G is adjacent to some vertex in S (other than itself). The graph G is called total domination excellent if every vertex belongs to some total dominating set of G of minimum cardinality. We provide a constructive characterization of total domination excellent trees.

Keywords: Total domination; Tree

1. Introduction

Let G be a graph without isolated vertices, and let v be a vertex of G. A set $S \subseteq V(G)$ is a *total dominating set* if every vertex in $V(G)$ is adjacent to a vertex in S. Every graph without isolated vertices has a total dominating set, since $S = V(G)$ is such a set. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. A total dominating set of cardinality $\gamma_t(G)$ will be called a $\gamma_t(G)$-set.

Total domination in graphs was introduced by Cockayne et al. [1] and is now well studied in graph theory (see, for example, [2,7]). The literature on this subject has been surveyed and detailed in the two books by Haynes et al. [4,5].

Fricke et al. [3] defined a graph G to be γ_t-*excellent* if every vertex of G belongs to some $\gamma_t(G)$-set. They showed that the family of γ-excellent trees (trees where every vertex is in some minimum dominating set) is properly contained in the set of

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i-excellent trees (trees where every vertex is in some minimum independent dominating set). The γ-excellent trees have been characterized by Sumner [8], while the i-excellent trees have been characterized in [6] where it is shown that any such tree of order at least three can be constructed using a double-star as a base tree and recursively applying one of two operations.

In this paper, we provide a constructive characterization of γ_t-excellent trees. We use a similar technique to that employed in [6] (we use a different base tree and recursively apply one of four operations, instead of two operations, to build the γ_t-excellent trees). For this purpose, we introduce some additional notation.

We define the *total domination number of G relative to v*, denoted $\gamma_t^v(G)$, as the minimum cardinality of a total dominating set in G that contains v. A total dominating set of cardinality $\gamma_t^v(G)$ containing v we call a $\gamma_t^v(G)$-set. Hence, the graph G is γ_t-*excellent* if $\gamma_t^v(G)=\gamma_t(G)$ for every vertex v of G.

A vertex v is said to be *totally dominated* by a set $S\subseteq V(G)$ if it is adjacent to a vertex of S (other than itself). We define an *almost total dominating set of G relative to v* as a set of vertices of G that totally dominates all vertices of G, except possibly for v. The *almost total domination number of G relative to v*, denoted $\gamma_t^v(G;v)$, is the minimum cardinality of an almost total dominating set of G relative to v. An almost total dominating set of G relative to v of cardinality $\gamma_t^v(G;v)$ we call a $\gamma_t^v(G;v)$-set. (Note that it is possible for v to belong to a $\gamma_t^v(G;v)$-set although v itself may not be totally dominated.)

A subset $U\subseteq V(G)$ is *totally dominated* by a set $S\subseteq V(G)$ if every vertex of U is totally dominated by S. We define a *total dominating set of U in G* as a set of vertices in G that totally dominates U. The *total domination number of U in G*, denoted $\gamma_t(G;U)$, is the minimum cardinality of a total dominating set of U in G. A total dominating set of U in G of cardinality $\gamma_t(G;U)$ we call a $\gamma_t(G;U)$-set.

For notation and graph theory terminology we, in general, follow [4]. Specifically, let $G=(V,E)$ be a graph with vertex set V of order n and edge set E, and let v be a vertex in V. The open neighborhood of v is $N(v)=\{u\in V \mid uv\in E\}$ and the closed neighborhood of v is $N[v]=\{v\}\cup N(v)$. For a set $S\subseteq V$, its *open neighborhood* $N(S)=\bigcup_{v\in S}N(v)$ and its *closed neighborhood* $N[S]=N(S)\cup S$. The *private neighborhood* $\mathrm{pn}(v,S)$ of $v\in S$ is defined by $\mathrm{pn}(v,S)=N[v]-N[S-\{v\}]$.

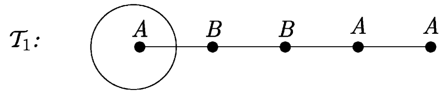
For ease of presentation, we mostly consider *rooted trees*. For a vertex v in a (rooted) tree T, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and we define $D[v]=D(v)\cup\{v\}$. The maximal subtree at v is the subtree of T induced by $D[v]$, and is denoted by T_v. A *leaf* of T is a vertex of degree 1, while a *support vertex* of T is a vertex adjacent to a leaf. A *strong support vertex* is adjacent to at least two leaves.

2. The family $\mathscr{T}$

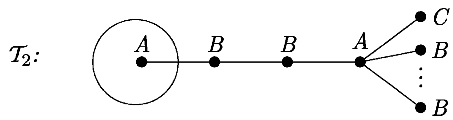
Let $\mathscr{T}$ be the family of trees T that can be obtained from a sequence $T_1,\ldots,T_j$ ($j\geqslant 1$) of trees such that T_1 is a star $K_{1,r}$ for $r\geqslant 1$ and $T=T_j$, and, if $j\geqslant 2$, T_{i+1} can be obtained recursively from T_i by one of the four operations $\mathscr{T}_1$, $\mathscr{T}_2$, $\mathscr{T}_3$ and $\mathscr{T}_4$ listed below.

We define the *status* of a vertex v, denoted sta(v), to be A, B or C where initially if $T_1 = K_2$, then sta(v) = A for each vertex v of T_1, and if $T_1 = K_{1,r}$ with $r \geqslant 2$, then sta(v) = A for the central vertex of T_1, sta(v) = B for every leaf v of T_1, except for one leaf, and sta(v) = C for the remaining leaf of T_1. Once a vertex is assigned a status, this status remains unchanged as the tree T is recursively constructed except possibly for a vertex of status C whose status may change to status A. (As soon as the neighbor of a vertex c of status C is no longer a strong support vertex, we change the status of c from status C to status A.) Intuitively, if a vertex v has status A or B in a γ_t-excellent tree, then using one of the four operations we construct a new γ_t-excellent tree by adding certain paths, stars, or subdivided stars and joining a specified vertex to v.

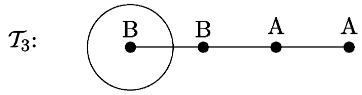
Operation $\mathscr{T}_1$**.** The tree T_{i+1} is obtained from T_i by adding a path u, w', w, z and the edge uy where $y \in V(T_i)$ and sta(y) = A, and letting sta(u) = sta(w') = B and sta(w) = sta(z) = A.



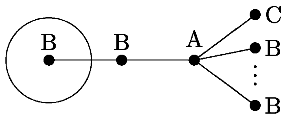
Operation $\mathscr{T}_2$**.** The tree T_{i+1} is obtained from T_i by adding a star $K_{1,t}$ for $t \geqslant 3$ with center w, subdivided one edge uw once, and then adding the edge uy where $y \in V(T_i)$ and sta(y) = A. Let sta(w) = A and let sta(z) = C for exactly one leaf z adjacent to w, and let sta(v) = B for each remaining vertex v that was added to T_i.



Operation $\mathscr{T}_3$**.** The tree T_{i+1} is obtained from T_i by adding a path u, w, z and the edge uy where $y \in V(T_i)$ and sta(y) = B, and letting sta(u) = B and sta(w) = sta(z) = A. If the vertex y' of status A adjacent to y is adjacent to a vertex c of status C, and if y' is not a strong support vertex in T_{i+1}, then we change the status of the vertex c from status C to status A (we remark that the existence and uniqueness of y' follows from Observation 2(ii)).



Operation $\mathscr{T}_4$**.** The tree T_{i+1} is obtained from T_i by adding a star $K_{1,t}$ for $t \geqslant 3$ with center w and adding the edge uy where $y \in V(T_i)$ and sta(y) = B and u is a vertex adjacent to w. Let sta(w) = A, let sta(z) = C for exactly one leaf z ($\neq u$) adjacent to w, and let sta(v) = B for each remaining vertex v that was added to T_i. If the vertex y' of status A adjacent to y is adjacent to a vertex c of status C, and if y' is not a strong support vertex in T_{i+1}, then we change the status of the vertex c from status C to status A.

$\mathcal{T}_4$: 

If $T \in \mathscr{T}$, and T is obtained from a sequence $T_1, \ldots, T_m$ of trees where T_1 is a star $K_{1,r}$ with $r \geqslant 1$ and $T = T_m$, and, if $m \geqslant 2$, T_{i+1} can be obtained from T_i by operation $\mathscr{T}_1$, $\mathscr{T}_2$, $\mathscr{T}_3$ or $\mathscr{T}_4$ for $i = 1, \ldots, m-1$, then we say that T has *length m in* $\mathscr{T}$. Since the length of T_{i+1} is one more than the length of T_i for $i = 1, \ldots, m-1$, and since T_{i+1} has exactly two additional vertices of status A or C than does T_i, we have the following observation.

Observation 1. *If $T \in \mathscr{T}$, then the total number of vertices of status A or C is twice the length of T.*

The following two observations follow readily from the way in which each tree in the family $\mathscr{T}$ is constructed.

Observation 2. *Let $T \in \mathscr{T}$ and let v be a vertex of T.*

(i) *If* $\text{sta}(v) = C$, *then v is a leaf of T and is adjacent to a strong support vertex of status A*;
(ii) *If* $\text{sta}(v) = B$, *then v is adjacent to a unique vertex of status A*;
(iii) *If* $\text{sta}(v) = A$, *then all but one neighbor of v has status B*;
(iv) *Every support vertex has status A.*

Observation 3. *If T is a nontrivial tree and v is a vertex of T, then*

$$\gamma_t^v(T; v) \leqslant \gamma_t(T) \leqslant \gamma_t^v(T; v) + 1.$$

Proof. Every $\gamma_t(T)$-set is an almost total dominating set of G relative to v, and so $\gamma_t^v(T; v) \leqslant \gamma_t(T)$. Let S be an $\gamma_t^v(T; v)$-set. If S is a total dominating set of T, then $\gamma_t(T) \leqslant |S|$. On the other hand, if v is not totally dominated by the set S, then, $S \cup \{v'\}$ is a total dominating set of T where v' is any neighbor of v, irrespective of whether $v \in S$ or $v \notin S$, and so $\gamma_t(T) \leqslant |S| + 1$. In any case, $\gamma_t(T) \leqslant |S| + 1 = \gamma_t^v(T; v) + 1$. $\square$

We now present our main result of this section.

Theorem 1. *Let $T \in \mathscr{T}$ have length m in $\mathscr{T}$ and let v be a vertex of T. Let U denote the set of vertices of T of status A or status C. Then*

(i) *T is a γ_t-excellent tree and* $\gamma_t(T) = 2m$;
(ii) *if* $\text{sta}(v) = A$, *then* $\gamma_t(T) = \gamma_t^v(T; v) + 1$;
(iii) $\gamma_t(T; U) = \gamma_t(T)$;

(iv) *if* sta$(v)=B$ *or* C, *then* $\gamma_t(T)=\gamma_t^v(T;v)$;
(v) *if* sta$(v)=A$, *then no leaf is at distance* 2 *or* 3 *from* v.

Proof. Since T has length m in $\mathscr{T}$, T can be obtained from a sequence $T_1,\ldots,T_m$ of trees where T_1 is a star $K_{1,r}$ with $r\geqslant 1$ and $T=T_m$, and, if $m\geqslant 2$, T_{i+1} can be obtained from T_i by operation $\mathscr{T}_1$, $\mathscr{T}_2$, $\mathscr{T}_3$ or $\mathscr{T}_4$ for $i=1,\ldots,m-1$. To prove the desired result, we proceed by induction on the length m of the sequence of trees needed to construct the tree T.

Suppose $m=1$. Then T is a star $K_{1,r}$ for some $r\geqslant 1$. Thus, T is γ_t-excellent and $\gamma_t(T)=2$. Let v be a vertex of T with sta$(v)=A$. Then, $\gamma_t^v(T;v)=|\{v\}|$, and so $\gamma_t(T)=\gamma_t^v(T;v)+1$. If $r=1$, then T has two vertices of status A, while if $r\geqslant 2$, then T has one vertex of status A and one of status C. Hence, $|U|=2$ and $\gamma_t(T;U)=2=\gamma_t(T)$. Let v be a vertex of T with sta$(v)=B$ or C. Then, $r\geqslant 2$ and v is a leaf of T, and so $\gamma_t(T)=\gamma_t^v(T;v)=2$. If sta$(v)=A$, then no leaf is at distance 2 or 3 from v. Thus if $m=1$, then conditions (i)–(v) all hold.

Assume, then, that the result holds for all trees in $\mathscr{T}$ of length less than m in $\mathscr{T}$, where $m\geqslant 2$. Let T be a tree of length m in $\mathscr{T}$. Thus, $T\in\mathscr{T}$ can be obtained from a sequence $T_1,T_2,\ldots,T_m$ of m trees. For notational convenience, we denote T_{m-1} simply by T'. Applying the inductive hypothesis to $T'\in\mathscr{T}$, conditions (i)–(v) hold for the tree T'. We now consider four possibilities depending on whether T is obtained from T' by operation $\mathscr{T}_1$, $\mathscr{T}_2$, $\mathscr{T}_3$ or $\mathscr{T}_4$.

Case 1: T is obtained from T' by operation $\mathscr{T}_1$.

Suppose T is obtained from T' by adding a path u,w',w,z and the edge uy where $y\in V(T')$ and sta$(y)=A$. Hence, sta$(u)=$ sta$(w')=B$ and sta$(w)=$ sta$(z)=A$.

We show firstly that $\gamma_t(T)=\gamma_t(T')+2$. Any $\gamma_t(T')$-set can be extended to a total dominating set of T by adding the set $\{w',w\}$, and so $\gamma_t(T)\leqslant\gamma_t(T')+2$. Now let S be a $\gamma_t(T)$-set, and let $S'=S\cap V(T')$. We may assume that $w,w'\in S$. If $u\notin S$, then S' is a total dominating set of T', and so $\gamma_t(T')\leqslant|S'|=|S|-2=\gamma_t(T)-2$. On the other hand, suppose $u\in S$. Then, S' is an almost total dominating set of T relative to y, and so $\gamma_t^y(T';y)\leqslant|S'|=|S|-3$. Since T' satisfies condition (ii), $\gamma_t(T')=\gamma_t^y(T';y)+1\leqslant|S|-2=\gamma_t(T)-2$. Hence, irrespective of whether $u\in S$ or $u\notin S$, $\gamma_t(T')\leqslant\gamma_t(T)-2$. Consequently, $\gamma_t(T)=\gamma_t(T')+2$. Since T' satisfies condition (i), $\gamma_t(T')=2(m-1)$, and so $\gamma_t(T)=2m$.

Suppose $x\in V(T')$. Since T' is γ_t-excellent, $\gamma_t^x(T')=\gamma_t(T')$. Now, any $\gamma_t^x(T')$-set can be extended to a total dominating set of T by adding the set $\{w',w\}$, and so $\gamma_t^x(T)\leqslant\gamma_t^x(T')+2=\gamma_t(T')+2=\gamma_t(T)$. Suppose $x\in V(T)-V(T')$. Any $\gamma_t^y(T')$-set can be extended to a total dominating set of T by adding the vertex w and any neighbor of w, and so $\gamma_t^x(T)\leqslant\gamma_t^y(T')+2=\gamma_t(T')+2=\gamma_t(T)$ if $x\in N[w]$. Let S' be a $\gamma_t^y(T';y)$-set. Since sta$(y)=A$ and T' satisfies condition (ii), $|S'|=\gamma_t^y(T';y)=\gamma_t(T')-1$. Now, S' can be extended to a total dominating set of T by adding the set $\{u,w',w\}$, and so $\gamma_t^u(T)\leqslant|S'|+3=\gamma_t(T')+2=\gamma_t(T)$. It follows that $\gamma_t^x(T)\leqslant\gamma_t(T)$ for every vertex x of T. Consequently, $\gamma_t^x(T)=\gamma_t(T)$ for every vertex x of T. Hence, T is γ_t-excellent and $\gamma_t(T)=2m$, i.e., condition (i) holds for the tree T.

Suppose v is a vertex of T with sta$(v)=A$. Suppose $v\in V(T')$. Then any $\gamma_t^v(T';v)$-set can be extended to an almost total dominating set of T relative to v by adding the set $\{w', w\}$, and so $\gamma_t^v(T; v) \leqslant \gamma_t^v(T'; v) + 2 = \gamma_t(T') + 1 = \gamma_t(T) - 1$. Any $\gamma_t^y(T')$-set can be extended to an almost total dominating set of T relative to w by adding the vertex w, and so $\gamma_t^w(T; w) \leqslant \gamma_t^y(T') + 1 = \gamma_t(T') + 1 = \gamma_t(T) - 1$. Any $\gamma_t^y(T'; y)$-set can be extended to an almost total dominating set of T relative to z by adding the set $\{u, w'\}$, and so $\gamma_t^z(T; z) \leqslant \gamma_t^y(T'; y) + 2 = \gamma_t(T') + 1 = \gamma_t(T) - 1$. Hence, $\gamma_t^v(T; v) \leqslant \gamma_t(T) - 1$ for every vertex of T of status A. By Observation 3, $\gamma_t^v(T; v) \geqslant \gamma_t(T) - 1$ for every vertex v of T. Consequently, $\gamma_t^v(T; v) = \gamma_t(T) - 1$ for every vertex of T of status A. Hence, condition (ii) holds for the tree T.

Any $\gamma_t(T)$-set is a total dominating set of U in T, and so $\gamma_t(T; U) \leqslant \gamma_t(T) = 2m$. We show that $\gamma_t(T) \leqslant \gamma_t(T; U)$. Let S be a $\gamma_t(T; U)$-set. Since $\text{sta}(z) = A$, the vertex z must be totally dominated by S, and so $w \in S$. Since $\text{sta}(w) = A$, the vertex w must be totally dominated by S, and so we may assume that $w' \in S$. Let $S' = S \cap V(T')$. If $u \in S$, then replacing u by any neighbor of y in T' produces a total dominating set of U in T of cardinality S. Hence, we may assume that $u \notin S$. Let $U' = U - \{w, z\}$. Then, S' is a total dominating set of U' in T'. Since T' satisfies condition (iii), $2(m-1) = \gamma_t(T'; U') \leqslant |S'|$, and so $\gamma_t(T; U) = |S| = |S'| + 2 \geqslant 2m = \gamma_t(T)$. Consequently, $\gamma_t(T; U) = \gamma_t(T)$. Hence, condition (iii) holds for the tree T.

By Observation 3, $\gamma_t^v(T; v) \leqslant \gamma_t(T)$ for every vertex v of T. Suppose v is a vertex of T with $\text{sta}(v) = B$ or C. We show that $\gamma_t(T) \leqslant \gamma_t^v(T; v)$. Let S be a $\gamma_t^v(T; v)$-set.

Suppose $\text{sta}(v) = C$. Then, by Observation 2, v is a leaf of T and is adjacent to a strong support vertex v' of status A. Let z' be a leaf of v' different from v. Since z' is totally dominated by S, $v' \in S$. Thus, v is totally dominated by S. Hence, if $\text{sta}(v) = C$, then S is a total dominating set of T, and so $\gamma_t(T) \leqslant |S| = \gamma_t^v(T; v)$. Consequently, $\gamma_t(T) = \gamma_t^v(T; v)$ if $\text{sta}(v) = C$.

Suppose $\text{sta}(v) = B$. If S is a total dominating set of T, then $\gamma_t(T) \leqslant |S| = \gamma_t^v(T; v)$. Hence we may assume that S is an almost total dominating set of T relative to v and that v is not totally dominated by S. Since $\text{sta}(z) = A$, the vertex z must be totally dominated by S, and so $w \in S$. Since $\text{sta}(w) = A$, the vertex w must be totally dominated by S, and so we may assume that $w' \in S$. Hence both u and w' are totally dominated by S, and so $v \in V(T')$. Let $S' = S \cap V(T')$. If $u \in S$, then replacing u by the neighbor of y in T' of status A or C produces an almost total dominating set of T relative to v. Hence, we may assume that $u \notin S$. But then S' is an almost total dominating set of T' relative to v, and so $\gamma_t(T) - 2 = \gamma_t(T') = \gamma_t^v(T'; v) \leqslant |S'| = |S| - 2 = \gamma_t^v(T; v) - 2$. Thus, $\gamma_t(T) \leqslant \gamma_t^v(T; v)$. Consequently, $\gamma_t(T) = \gamma_t^v(T; v)$ if $\text{sta}(v) = B$. Hence, condition (iv) holds for the tree T.

Suppose $\text{sta}(v) = A$. If $v \in \{w, z\}$, then no leaf is at distance 2 or 3 from v. On the other hand, if $v \in V(T')$, then, by the inductive hypothesis, no leaf is at distance 2 or 3 from v in T' and therefore also in T. Hence, condition (v) holds for the tree T.

Case 2: T is obtained from T' by operation $\mathscr{T}_2$.
Suppose T is obtained from T' by adding a star $K_{1,t}$, $t \geqslant 3$, with center w, by subdividing one edge uw once, and then adding the edge uy where $y \in V(T')$ and $\text{sta}(y) = A$. Let w' denote the vertex adjacent to u and w, and let z denote the leaf adjacent to w with $\text{sta}(z) = C$.

Proceeding as in Case 1, we can show that $\gamma_t(T) = \gamma_t(T') + 2 = 2m$ and that conditions (i), (ii) and (v) hold for the tree T.

Let S be a $\gamma_t(T;U)$-set. Since $\text{sta}(z)=C$, the vertex z must be totally dominated by S, and so $w\in S$. Hence, proceeding as in Case 1, we can show that T satisfies condition (iii).

Let v be a vertex of T. If $\text{sta}(v)=C$, then, as in Case 1, we can show that $\gamma_t(T)=\gamma_t^v(T;v)$. Suppose that $\text{sta}(v)=B$. By Observation 3, $\gamma_t^v(T;v)\leqslant\gamma_t(T)$. We show that $\gamma_t(T)\leqslant\gamma_t^v(T;v)$. Let S be an $\gamma_t^v(T;v)$-set. If S is a total dominating set of T, then $\gamma_t(T)\leqslant|S|=\gamma_t^v(T;v)$. Hence we may assume that S is an almost total dominating set of T relative to v and that v is not totally dominated by S. Let z' be a leaf adjacent to w that is distinct from z. Since at least one of z and z' must be totally dominated by S, $w\in S$. Hence every leaf adjacent to w is totally dominated by S, and so v is not a leaf of T adjacent to w. Since $\text{sta}(w)=A$, the vertex w must be totally dominated by S, and so we may assume that $w'\in S$. Proceeding now as in Case 1, we can show that $\gamma_t(T)=\gamma_t^v(T;v)$. Hence, condition (iv) holds for the tree T.

Case 3: T is obtained from T' by operation $\mathscr{T}_3$.

Suppose T is obtained from T' by adding a path u,w,z and the edge uy where $y\in V(T')$ and $\text{sta}(y)=B$, and letting $\text{sta}(u)=B$ and $\text{sta}(w)=\text{sta}(z)=A$.

We show firstly that $\gamma_t(T)=\gamma_t(T')+2$. Any $\gamma_t(T')$-set can be extended to a total dominating set of T by adding the set $\{u,w\}$, and so $\gamma_t(T)\leqslant\gamma_t(T')+2$. Now let S be a $\gamma_t(T)$-set, and let $S'=S\cap V(T')$. We may assume that $u,w\in S$. Hence, S' is an almost total dominating set of T relative to y. Since $\text{sta}(y)=B$, and since condition (iv) holds for the tree T', $\gamma_t(T')=\gamma_t^y(T';y)\leqslant|S'|=|S|-2=\gamma_t(T)-2$. In any event, $\gamma_t(T')\leqslant\gamma_t(T)-2$. Consequently, $\gamma_t(T)=\gamma_t(T')+2$. Since T' satisfies condition (i), $\gamma_t(T')=2(m-1)$, and so $\gamma_t(T)=2m$.

Suppose $x\in V(T')$. Since T' is γ_t-excellent, $\gamma_t^x(T')=\gamma_t(T')$. Now, any $\gamma_t^x(T')$-set can be extended to a total dominating set of T by adding the set $\{u,w\}$, and so $\gamma_t^x(T)\leqslant\gamma_t^x(T')+2=\gamma_t(T')+2=\gamma_t(T)$. Any $\gamma_t(T')$-set can be extended to a total dominating set of T by adding the vertex w and any neighbor of w, and so $\gamma_t^x(T)\leqslant\gamma_t(T')+2=\gamma_t(T)$ if $x\in N[w]$. Consequently, $\gamma_t^x(T)=\gamma_t(T)$ for every vertex x of T. Hence, T is γ_t-excellent and $\gamma_t(T)=2m$., i.e., condition (i) holds for the tree T.

Suppose v is a vertex of T with $\text{sta}(v)=A$. Suppose $v\in V(T')$. Then any $\gamma_t^v(T';v)$-set can be extended to an almost total dominating set of T relative to v by adding the set $\{u,w\}$, and so $\gamma_t^v(T;v)\leqslant\gamma_t^v(T';v)+2=\gamma_t(T')+1=\gamma_t(T)-1$. Any $\gamma_t^y(T')$-set can be extended to an almost total dominating set of T relative to w by adding the vertex w, and so $\gamma_t^w(T;w)\leqslant\gamma_t^y(T')+1=\gamma_t(T')+1=\gamma_t(T)-1$. Any $\gamma_t^y(T')$-set can be extended to an almost total dominating set of T relative to z by adding the vertex u, and so $\gamma_t^z(T;z)\leqslant\gamma_t^y(T')+1=\gamma_t(T)-1$. Hence, $\gamma_t^v(T;v)\leqslant\gamma_t(T)-1$ for every vertex of T of status A. By Observation 3, $\gamma_t^v(T;v)\geqslant\gamma_t(T)-1$ for every vertex v of T. Consequently, $\gamma_t^v(T;v)=\gamma_t(T)-1$ for every vertex of T of status A. Hence, condition (ii) holds for the tree T.

Any $\gamma_t(T)$-set is a total dominating set of U in T, and so $\gamma_t(T;U)\leqslant\gamma_t(T)=2m$. We show that $\gamma_t(T)\leqslant\gamma_t(T;U)$. Let S be a $\gamma_t(T;U)$-set. Since $\text{sta}(z)=A$, the vertex z must be totally dominated by S, and so $w\in S$. Since $\text{sta}(w)=A$, the vertex w must be totally dominated by S, and so we may assume that $u\in S$. Let $S'=S\cap V(T')$ and let $U'=U-\{w,z\}$. Since $\text{sta}(y)=B$, S' is a total dominating set of U' in T'. Since T' satisfies condition (iii), $2(m-1)=\gamma_t(T';U')\leqslant|S'|$, and so $\gamma_t(T;U)=|S|=$

$|S'| + 2 \geq 2m = \gamma_t(T)$. Consequently, $\gamma_t(T;U) = \gamma_t(T)$. Hence, condition (iii) holds for the tree T.

By Observation 3, $\gamma_t^v(T;v) \leq \gamma_t(T)$ for every vertex v of T. Suppose v is a vertex of T with $\text{sta}(v) = B$ or C. We show that $\gamma_t(T) \leq \gamma_t^v(T;v)$. Let S be an $\gamma_t^v(T;v)$-set. If $\text{sta}(v) = C$, then, as in Case 1, we can show that $\gamma_t(T) = \gamma_t^v(T;v)$. Hence we may assume that $\text{sta}(v) = B$. If S is a total dominating set of T, then $\gamma_t(T) \leq |S| = \gamma_t^v(T;v)$. Hence we may assume that S is an almost total dominating set of T relative to v and that v is not totally dominated by S. Since $\text{sta}(z) = A$, the vertex z must be totally dominated by S, and so $w \in S$. Since $\text{sta}(w) = A$, the vertex w must be totally dominated by S, and so we may assume that $u \in S$. Let $S' = S \cap V(T')$ and let $U' = U - \{w, z\}$. Since $\text{sta}(y) = B$, S' is a total dominating set of U' in T'. Since T' satisfies condition (iii), $2(m-1) = \gamma_t(T';U') \leq |S'|$, and so $\gamma_t^v(T;v) = |S| = |S'| + 2 \geq 2m = \gamma_t(T)$. Consequently, $\gamma_t(T) = \gamma_t^v(T;v)$. Hence, condition (iv) holds for the tree T.

Suppose $\text{sta}(v) = A$. By Observation 2, the vertex y is not a support vertex of T'. Hence, if $v = w$ or if $v = z$, then no leaf is at distance 2 or 3 from v. On the other hand, if $v \in V(T')$, then, by the inductive hypothesis, no leaf is at distance 2 or 3 from v in T' and therefore also in T. Hence, condition (v) holds for the tree T.

Case 4: T is obtained from T' by operation $\mathscr{T}_4$.

Suppose T is obtained from T' by adding a star $K_{1,t}$ for $t \geq 3$ with center w and the edge uy where $y \in V(T')$ and $\text{sta}(y) = B$ and u is a vertex adjacent to w. Let z denote the leaf adjacent to w with $\text{sta}(z) = C$. Then, $\text{sta}(w) = A$ and $\text{sta}(v) = B$ for each remaining vertex v that was added to T'.

Proceeding as in Case 3, we can show that $\gamma_t(T) = \gamma_t(T') + 2 = 2m$ and that conditions (i), (ii) and (v) hold for the tree T.

Let S be a $\gamma_t(T;U)$-set. Since $\text{sta}(z) = C$, the vertex z must be totally dominated by S, and so $w \in S$. Hence, proceeding as in Case 1, we can show that T satisfies condition (iii).

It remains to show that T satisfies condition (iv). Let v be a vertex of T. If $\text{sta}(v) = C$, then, as in Case 1, we can show that $\gamma_t(T) = \gamma_t^v(T;v)$. Suppose that $\text{sta}(v) = B$. By Observation 3, $\gamma_t^v(T;v) \leq \gamma_t(T)$. We show that $\gamma_t(T) \leq \gamma_t^v(T;v)$. Let S be an $\gamma_t^v(T;v)$-set. If S is a total dominating set of T, then $\gamma_t(T) \leq |S| = \gamma_t^v(T;v)$. Hence we may assume that S is an almost total dominating set of T relative to v and that v is not totally dominated by S. Let z' be a leaf adjacent to w that is distinct from z. Since at least one of z and z' must be totally dominated by S, we must have $w \in S$. Hence every leaf adjacent to w is totally dominated by S, and so v is not a leaf of T adjacent to w. Since $\text{sta}(w) = A$, the vertex w must be totally dominated by S, and so we may assume that $u \in S$. Proceeding now as in Case 3, we can show that $\gamma_t(T) = \gamma_t^v(T;v)$. Hence, condition (iv) holds for the tree T.

This completes the proof of Theorem 1. □

As an immediate consequence of Theorem 1, we have the following results.

Corollary 2. *If* $T \in \mathscr{T}$, *then* $\text{sta}(v) = A$ *if and only if* $\gamma_t(T) = \gamma_t^v(T;v) + 1$.

Corollary 3. *If* $T \in \mathscr{T}$ *and* v *is a vertex of* T *at distance* 2 *or* 3 *from a leaf satisfying* $\deg v \geqslant 2$, *then* $\mathrm{sta}(v) = B$.

3. Main result

In this section, we provide a constructive characterization of γ_t-excellent trees. We shall prove:

Theorem 4. *A nontrivial tree* T *is* γ_t*-excellent if and only if* $T \in \mathscr{T}$.

Proof. The sufficiency follows from Theorem 1. To prove the necessity, we proceed by induction on the order n of a γ_t-excellent tree T. If $\mathrm{diam}(T) = 1$, then $T = K_2 \in \mathscr{T}$. If $\mathrm{diam}(T) = 2$, then T is a star $K_{1,r}$ with $r \geqslant 2$, and so $T \in \mathscr{T}$. Hence we may assume that $\mathrm{diam}(T) \geqslant 3$. Since no double-star is γ_t-excellent, $\mathrm{diam}(T) \geqslant 4$. Let T be rooted at an end-vertex r of a longest path. Let u be a vertex at distance $\mathrm{diam}(T) - 2$ from r on a longest path starting at r, and let v be the child of u on this path. Let w denote the parent of u, and let y denote the parent of w. Before proceeding further, we list three observations. □

Observation 4. *No child of* u *is a leaf.*

Proof. Suppose u has a child z which is a leaf. Since T is a γ_t-excellent tree, $\gamma_t^z(T) = \gamma_t(T)$. Let S be a $\gamma_t^z(T)$-set. Then, $\{u, v\} \subset S$, and so $S - \{z\}$ is a total dominating set of T. Hence, $\gamma_t(T) \leqslant |S| - 1 < \gamma_t^z(T)$, a contradiction. □

Observation 5. $\deg u = 2$.

Proof. Suppose $\deg u \geqslant 3$. Let $v_1 \in C(u) - \{v\}$. By Observation 4, v_1 is not a leaf and is therefore a support vertex. Let z be a child of v, and let S be a $\gamma_t^z(T)$-set. Since every support vertex belongs to S, $C(u) \subset S$. In particular, $v_1 \in S$. We may assume that $u \in S$ (otherwise we replace the child of v_1 in S with u.) But then $S - \{z\}$ is a total dominating set of T. Hence, $\gamma_t(T) \leqslant |S| - 1 < \gamma_t^z(T)$, a contradiction. □

Observation 6. *No child of* w *is a leaf.*

Proof. Suppose w has a child z which is a leaf. Let S be a $\gamma_t^z(T)$-set. Since every support vertex is in S, $\{v, w\} \subset S$. We may assume that $u \in S$ (otherwise we replace the child of v in S with u). But then $S - \{z\}$ is a total dominating set of T. Hence, $\gamma_t(T) \leqslant |S| - 1 < \gamma_t^z(T)$, a contradiction. □

We now consider two possibilities depending on whether or not w has a child that is a support vertex.

Case 1: Suppose a child of w is a support vertex.

Let $T' = T - V(T_u)$, i.e., $T' = T - N[v]$.

Claim 1. $\gamma_t^x(T') = \gamma_t(T) - 2$ *for every* $x \in V(T')$.

Proof. Let $x \in V(T')$. Any $\gamma_t^x(T')$-set can be extended to a total dominating set containing x by adding the set $\{u, v\}$, and so $\gamma_t(T) \leqslant \gamma_t^x(T') + 2$. Now let S_x be an $\gamma_t^x(T)$-set, and let $S'_x = S_x \cap V(T')$. Since T is γ_t-excellent, $|S_x| = \gamma_t(T)$. We may assume that $\{u, v\} \subset S_x$. Since S_x must contain every support vertex of T, and since w has a child that is a support vertex, it follows that S'_x is a total dominating set of T' containing x. Hence, $\gamma_t^x(T') \leqslant |S'_x| = |S_x| - 2 = \gamma_t^x(T) - 2$. Consequently, $\gamma_t^x(T') = \gamma_t(T) - 2$. □

By Claim 1 applied to a vertex in a minimum total dominating set of T', T' is a γ_t-excellent tree. Applying the inductive hypothesis to $T', T' \in \mathscr{T}$. Hence, T' can be obtained from a sequence $T_1, \ldots, T_m$ of trees where T_1 is a star $K_{1,r}$ with $r \geqslant 1$ and $T' = T_m$, and, if $m \geqslant 2$, T_{i+1} can be obtained from T_i by operation $\mathscr{T}_1$, $\mathscr{T}_2$, $\mathscr{T}_3$ or $\mathscr{T}_4$ for $i = 1, \ldots, m-1$.

Since $\text{diam}(T) \geqslant 4$, we know that w cannot be the root of T, and so $\deg_{T'} w \geqslant 2$. By assumption, w is at distance 2 from a leaf in T'. Hence, by Corollary 3, $\text{sta}(w) = B$.

Now let $T = T_{m+1}$ be the tree obtained from $T' \cup T_u$ by adding the edge uw. Then, T can be obtained from T' by operation $\mathscr{T}_3$ or $\mathscr{T}_4$. Hence, $T \in \mathscr{T}$.

Case 2: No child of w is a support vertex and $\deg w \geqslant 3$.

As shown in Observation 5, each child of w has degree 2. Let u_1 be a child of w distinct from u, and let v_1 the child of u_1. Let $T' = T - V(T_u)$, i.e., $T' = T - N[v]$.

Claim 2. $\gamma_t(T') = \gamma_t(T) - 2$.

Proof. Any $\gamma_t(T')$-set can be extended to a total dominating set of T by adding the set $\{u, v\}$, and so $\gamma_t(T) \leqslant \gamma_t(T') + 2$. Now let S be a $\gamma_t(T)$-set, and let $S' = S \cap V(T')$. Since T is γ_t-excellent, $|S| = \gamma_t(T)$. Since every support vertex of T belongs to S, all descendants at distance 2 from w belong to S. We may assume that every child of w belongs to S. Hence, S' is a total dominating set of T', and so $\gamma_t(T') \leqslant |S'| = |S| - 2 = \gamma_t(T) - 2$. Consequently, $\gamma_t(T') = \gamma_t(T) - 2$. □

Claim 3. $\gamma_t^x(T') = \gamma_t(T) - 2$ *for every* $x \in V(T')$.

Proof. Let $x \in V(T')$. Any $\gamma_t^x(T')$-set can be extended to a total dominating set containing x by adding the set $\{u, v\}$, and so $\gamma_t(T) \leqslant \gamma_t^x(T') + 2$. For each $x \in V(T')$, we let S_x be a $\gamma_t^x(T)$-set, and let $S'_x = S_x \cap V(T')$. Since T is γ_t-excellent, $|S_x| = \gamma_t(T)$. Since every support vertex belongs to S_x, all descendants at distance 2 from w belong to S_x. In particular, $\{v, v_1\} \subset S_x$. We may assume that $u \in S_x$.

Suppose x is a child of v_1. Consider the set S_y, where y is the parent of w. We may assume that $x \in S_y$ (if $u_1 \in S_y$, then simply replace u_1 by x), and so S_y is a total dominating set of T' containing x and y. Hence, $\gamma_t^x(T') \leqslant \gamma_t^y(T') \leqslant |S'_y| = |S_y| - 2 = \gamma_t^y(T) - 2 = \gamma_t(T) - 2$. Consequently, $\gamma_t^x(T') = \gamma_t(T) - 2$.

Suppose x is not a child of v_1. Then we may assume that $u_1 \in S_x$ (if S_x contains a child of v_1, then replace this child with u_1), and so S_x is a total dominating set of

T' containing x. Hence, $\gamma_t^x(T') \leqslant |S'_x| = |S_x| - 2 = \gamma_t^x(T) - 2 = \gamma_t(T) - 2$. Consequently, $\gamma_t^x(T') = \gamma_t(T) - 2$. □

By Claim 3, T' is a γ_t-excellent tree. Applying the inductive hypothesis to T', $T' \in \mathscr{T}$. Hence, T' can be obtained from a sequence $T_1, \ldots, T_m$ of trees where T_1 is a star $K_{1,r}$ with $r \geqslant 1$ and $T' = T_m$, and, if $m \geqslant 2$, T_{i+1} can be obtained from T_i by operation $\mathscr{T}_1$, $\mathscr{T}_2$, $\mathscr{T}_3$ or $\mathscr{T}_4$ for $i = 1, \ldots, m-1$.

Since $\deg_{T'} w \geqslant 2$ and w is at distance 3 from a leaf in T', it follows from Corollary 3 that $\mathrm{sta}(w) = B$.

Now let $T = T_{m+1}$ be the tree obtained from $T' \cup T_u$ by adding the edge uw. Then, T can be obtained from T' by operation $\mathscr{T}_3$ or $\mathscr{T}_4$. Hence, $T \in \mathscr{T}$.

Case 3: $\deg w = 2$.

Let $T' = T - V(T_w)$, i.e., $T' = T - N[v] - w$. Since T is γ_t-excellent, y cannot be the root of T, and so T' is a nontrivial tree.

Claim 4. $\gamma_t(T') = \gamma_t(T) - 2$.

Proof. Any $\gamma_t(T')$-set can be extended to a total dominating set of T by adding the set $\{u, v\}$, and so $\gamma_t(T) \leqslant \gamma_t(T') + 2$. Now let S be a $\gamma_t(T)$-set, and let $S' = S \cap V(T')$. Since T is γ_t-excellent, $|S| = \gamma_t(T)$. We may assume that $\{u, v\} \subset S$. If $w \in S$, then $\mathrm{pn}(w, S) = \{y\}$ and $(S - \{w\}) \cup \{y'\}$ is a $\gamma_t(T)$-set, where $y' \in N(y) - \{w\}$. Thus we may assume that $w \notin S$. Hence, S' is a total dominating set of T', and so $\gamma_t(T') \leqslant |S'| = |S| - 2 = \gamma_t(T) - 2$. Consequently, $\gamma_t(T') = \gamma_t(T) - 2$. □

Claim 5. $\gamma_t^x(T') = \gamma_t(T) - 2$ *for every* $x \in V(T')$.

Proof. Let $x \in V(T')$. Any $\gamma_t^x(T')$-set can be extended to a total dominating set containing x by adding the set $\{u, v\}$, and so $\gamma_t(T) \leqslant \gamma_t^x(T') + 2$. For each $x \in V(T')$, we let S_x be a $\gamma_t^x(T)$-set, and let $S'_x = S_x \cap V(T')$. Since T is γ_t-excellent, $|S_x| = \gamma_t(T)$. We may assume that $\{u, v\} \subset S_x$. If $w \in S$, then $\mathrm{pn}(w, S) = \{y\}$ and $(S - \{w\}) \cup \{y'\}$ is a $\gamma_t(T)$-set, where $y' \in N(y) - \{w\}$. Thus we may assume that $w \notin S$. Hence, S'_x is a total dominating set of T' containing x, and so $\gamma_t^x(T') \leqslant |S'_x| = |S_x| - 2 = \gamma_t(T) - 2$. Consequently, $\gamma_t^x(T') = \gamma_t(T) - 2$. □

Claim 6. $\gamma_t(T') = \gamma_t^y(T'; y) + 1$.

Proof. Let S_w be a $\gamma_t^w(T)$-set, and let $S'_w = S_w \cap V(T')$. Since T is γ_t-excellent, $|S_w| = \gamma_t(T)$. We may assume that $\{u, v\} \subset S_w$. Now S'_w is an almost total dominating set of T' relative to y, and so $\gamma_t^y(T'; y) \leqslant |S'_w| = |S_w| - 3 = \gamma_t(T) - 3 = \gamma_t(T') - 1$. However, by Observation 3, $\gamma_t(T') \leqslant \gamma_t^y(T'; y) + 1$. Consequently, $\gamma_t(T') = \gamma_t^y(T'; y) + 1$. □

By Claim 5, T' is a γ_t-excellent tree. Applying the inductive hypothesis to T', $T' \in \mathscr{T}$. Hence, T' can be obtained from a sequence $T_1, \ldots, T_m$ of trees where T_1 is a star $K_{1,r}$ with $r \geqslant 1$ and $T' = T_m$, and, if $m \geqslant 2$, T_{i+1} can be obtained from T_i by operation $\mathscr{T}_1$, $\mathscr{T}_2$, $\mathscr{T}_3$ or $\mathscr{T}_4$ for $i = 1, \ldots, m-1$.

Since $T' \in \mathcal{T}$, it follows from Corollary 2 and Claim 6 that $\text{sta}(y) = A$. Hence, T can be obtained from $T' \cup T_w$ by adding the edge wy. Thus, T can be obtained from T' by operation $\mathcal{T}_1$ or $\mathcal{T}_2$. Hence, $T \in \mathcal{T}$.

This completes the proof of Theorem 4. □

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