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On the structure of the lattice of noncrossing partitions

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Abstract

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We show that the lattice of noncrossing (set) partitions is self-dual and that it admits a symmetric chain decomposition. The self-duality is proved via an order-reversing involution. Two proofs are given of the existence of the symmetric chain decomposition, one recursive and one constructive. Several identities involving Catalan numbers emerge from the construction of the symmetric chain decomposition.

Introduction and notation

We will examine some structural properties of the refinement order on the class of noncrossing partitions. First a few definitions; for definitions not given below see [1] or [16], and as a general reference see [5]. Consider the set $[n] := \{1, 2, \ldots, n\}$. A partition of $[n]$ is *noncrossing* if whenever $1 \leqslant a < b < c < d \leqslant n$ with a, c in the same block (which we will write as $a \sim c$), and b, d in the same block, then in fact all four elements are in the same block. Thus, with slashes separating the blocks, 138/2/4/57/6 is a noncrossing partition of [8], while 138/24/57/6 is crossing. There are various ways to represent a noncrossing partition. For our purposes we will use a *linear* and a *circular* representation. In the linear representation, $[n]$ appears as usual on the real line, and successive elements of in the same block are joined by an arc in the first quadrant; in the case of the circular representation, $[n]$ appears as n points around a circle, and two (cyclically) successive elements of the same block are joined by a chord. The noncrossing property of the partition corresponds to the fact the arcs (chords, respectively) do not intersect. See Fig. 1.

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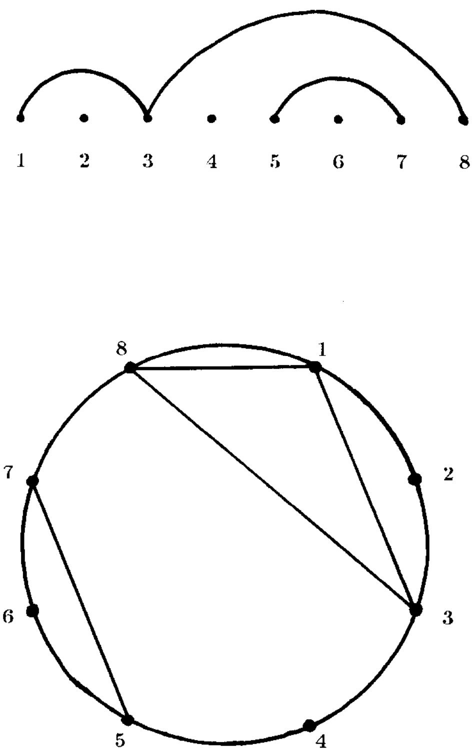

Fig. 1. The linear and the circular representations of 138/2/4/57/6.

Using either of these representations and induction it is easy to check that the number of noncrossing partitions of $[n]$ is the nth Catalan number, $C_n = 1/(n+1)\binom{2n}{n}$ (see, e.g., [5]).

The *refinement order* is defined on the set of all partitions of $[n]$. Under this ordering, two partitions π and π' satisfy $\pi \leqslant \pi'$ if every block of π is a subset of some block of π'. It is well known that the set $\Pi(n)$ of all partitions of $[n]$ forms a lattice under the refinement ordering (see, e.g., [1], [16]). It is also the case that the set NC(n) of noncrossing partitions of $[n]$ is a lattice under refinement [13]. Fig. 2 shows the Hasse diagram of NC(4).

A large number of papers, only some of which appear in our bibliography, deal specifically or by way of application with noncrossing partitions. To mention some known results which are related to the present paper, Kreweras [13] determined several enumeration formulae pertaining to noncrossing partitions as well as the Möbius function of NC(n); later, Edelman investigated multichain enumeration in NC(n) [7–8]; Björner [2] observed that Gessel's proof that $\Pi(n)$ is EL-shellable (see e.g. [16] for the definition) applies to NC(n) as well; very recently Edelman and Simion [9] investigated relations between chain enumeration in NC(n) and its EL-labeling.

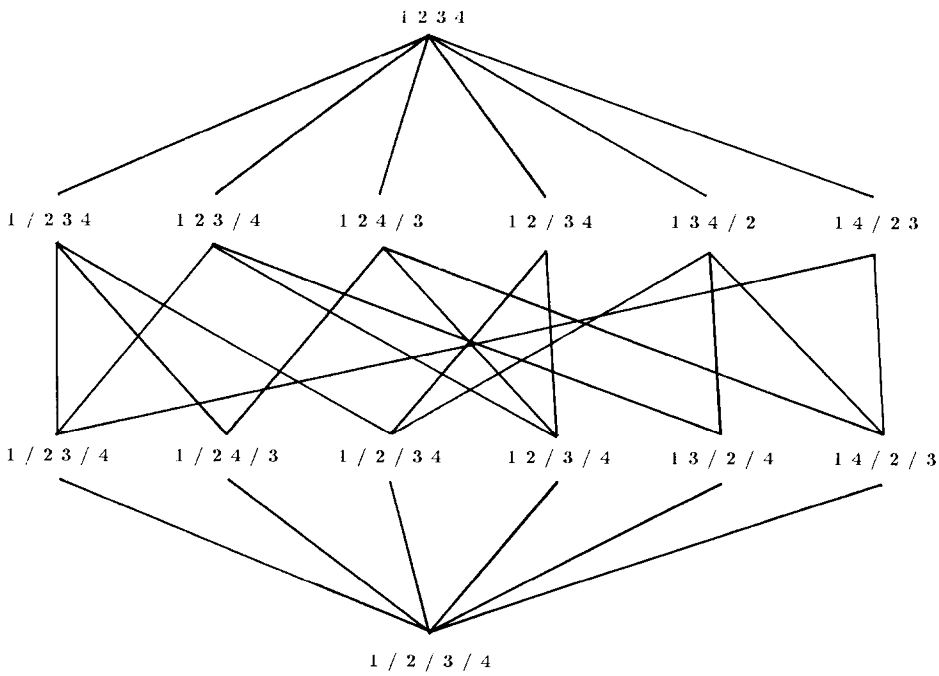

Fig. 2. The lattice NC(4).

The posets $\Pi(n)$ and $\mathrm{NC}(n)$ have several order-theoretic properties in common, in addition to the fact that both are lattices. Both are ranked by the function $\mathrm{rk}(\pi) = n - \mathrm{bk}(\pi)$, where $\mathrm{bk}(\pi)$ denotes the number of blocks of π, and both have height $n-1$. It is well known that the number of partitions of $[n]$ into k (non-empty) blocks is $S(n, k)$, the Stirling number of the second kind [5]. The number of noncrossing partitions of $[n]$ into k (non-empty) blocks is $W(n, k) := \binom{n}{k}\binom{n}{k-1}/n$ [7, 13]. A sequence $\{\alpha_k\}_{k=1}^{n}$ is logarithmically concave if for each k, $2 \leq k \leq n-1$, the following inequality is satisfied: $\alpha_k^2 \geq \alpha_{k-1}\alpha_{k+1}$. It is well known and easy to check that log-concavity implies unimodality, i.e., there exists m such that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m \geq \alpha_{m+1} \geq \cdots \geq \alpha_n$. Both sequences $\{S(n, k)\}_{k=1}^{n}$ and $\{W(n, k)\}_{k=1}^{n}$ are log-concave. (See [5] for the former claim, and direct calculation with binomial coefficients yields the latter.) Thus, both $\Pi(n)$ and $\mathrm{NC}(n)$ are rank unimodal.

This paper focuses on some properties enjoyed by $\mathrm{NC}(n)$ but not by $\Pi(n)$: rank symmetry, self-duality, and the existence of a symmetric chain decomposition (definitions are given below). It is immediate from the formula for $W(n, k)$ and the symmetry of the binomial coefficients that $\mathrm{NC}(n)$ is *rank symmetric*, that is, an equal number of elements have ranks k and $n-k-1$, for each $0 \leq k \leq n-1$. On the other hand, $\Pi(n)$ is not rank symmetric (a discussion of the location of the mode of $\{S(n, k)\}_{k=1}^{n}$ appears in [5], where additional references can be found). In Section 1 we make explicit the rank symmetry of $\mathrm{NC}(n)$ by means of an involution which matches the noncrossing partitions having k blocks with those having $n+1-k$ blocks. In fact this involution, α, is an order reversing map (i.e., $\pi \leqslant \pi'$ implies $\alpha(\pi) \geqslant \alpha(\pi')$), thus proving the stronger fact that $\mathrm{NC}(n)$ is a self-dual lattice.

In Section 2 we show that $\mathrm{NC}(n)$ admits a symmetric chain decomposition. A partially ordered set P with rank function rk and height h has a *symmetric chain decomposition* (SCD) if P is the union of disjoint saturated chains $\gamma_1, \gamma_2, \ldots, \gamma_q$, such that if x_i and y_i are the minimum and the maximum elements of γ_i, we have $\mathrm{rk}(x_i) + \mathrm{rk}(y_i) = h$. We give two proofs of the SCD property for $\mathrm{NC}(n)$. The first is a recursive existence proof. The second is an explicit construction of individual chains, in the spirit of the classical 'parenthesization' SCD for the Boolean lattice (see [11]). One of the ingredients in our construction is an idea used by Shapiro [15] to give a very short proof of an identity of Touchard's.

Obiously, a poset P with a SCD is necessarily rank symmetric and unimodal, but the SCD property also implies that P is k-*Sperner* for every k, i.e., if $A_1, A_2, \ldots, A_k$ are antichains in P, then the cardinality of the union $\bigcup_{i=1}^{k} A_i$ does not exceed the sum of the largest k Whitney numbers (i.e., rank sizes) of P (see, e.g., [1]). Consequently, our symmetric chain decomposition proves that $\mathrm{NC}(n)$ is k-Sperner. On the other hand, Canfield [4] disproved a conjecture of Rota by showing that $\Pi(n)$ is *not* 1-Sperner except for (relatively) small values of n.

In the third section, through counting arguments based on our symmetric chain decomposition, we derive several identities which involve the Catalan numbers. In particular, in (3.4) we recover an identity previously proved via enumeration of shuffles and Baxter permutations by Cori, Dulucq and Viennot [6]. We also obtain a closely related identity, (3.5), and an identity involving noncrossing partitions with side conditions, (3.6) and (3.7).

1. Self-duality

As mentioned in the introduction, it is clear from the formula for $W(n, k)$ that $\mathrm{NC}(n)$ is rank symmetric. The following theorem asserts a stronger property and makes the rank symmetry explicit.

Theorem 1.1. *For each $n \geqslant 1$, the lattice* $\mathrm{NC}(n)$ *is self-dual.*

Proof. Let $\pi \in \mathrm{NC}(n)$ and $\mathrm{bk}(\pi) = k$. Set $\alpha(\pi)$ equal to the partition of $[n]$ in which i and j $(i<j)$ satisfy $i \sim j$ if and only if no block of π contains two elements k and l with either $i \leqslant n-k < j \leqslant n-l$ or $n-k < i \leqslant n-l < j$. This amounts to the following: represent π circularly as described in the introduction with the points labeled $1, 2, \ldots, n$ clockwise; subdivide each of the n arcs by a new point; label the point which subdivides the arc $(n-1, n)$ with 1, and then the other

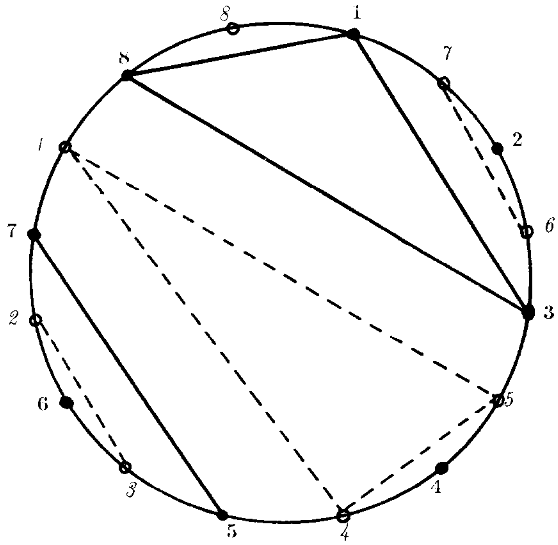

Fig. 3. 138/2/4/57/6 (solid lines) and 145/23/67/8 (dashed lines).

division points with $2, 3, \ldots, n$ in *counter*-clockwise order; now define the map $\alpha: \mathrm{NC}(n) \to \mathrm{NC}(n)$ by setting $\alpha(\pi)$ to be the coarsest noncrossing partition of the division points whose chords do *not* cross the chords in the representation of π. Thus, if $\pi = 138/2/4/57/6$, then $\alpha(\pi) = 145/23/67/8$. This is illustrated in Fig. 3.

We claim that α is an order-reversing involution on $\mathrm{NC}(n)$. It is immediate from its definition that α is an involution. To see that this involution is order-reversing, suppose $\pi \leqslant \pi'$ in $\mathrm{NC}(n)$. Then every chord in the circular representation of $\alpha(\pi')$ avoids crossing the chords of π, hence $\alpha(\pi')$ refines $\alpha(\pi)$, that is, $\alpha(\pi') \leqslant \alpha(\pi)$. □

Our map α is closely related to Kreweras' *complementation* map [13], which can be verified to be order reversing on $\mathrm{NC}(n)$ but is not an involution. We hope to address the size and structure of the orbits of Kreweras's map in a future paper.

Proposition 1.2. *If $n = 2m + 1$, then the involution $\alpha: \mathrm{NC}(n) \to \mathrm{NC}(n)$ described in the proof of Theorem* 1.1 *has C_m fixed points.*

Proof. Consider $\pi \in \mathrm{NC}(n)$, $n = 2m + 1$, and its circular representation. Let us label the division points (of which $\alpha(\pi)$ is a partition) as $1', 2', \ldots, (2m+1)'$. Observe the symmetry of the $2(2m + 1)$ points and their labels with respect to the diameter D passing through the midpoint of the arc $(m + 1), (m + 1)'$.

If $\alpha(\pi) = \pi$, then we cannot have $(2m+1) \sim_\pi i$ for any $i \leqslant m$, otherwise we would have to have $(2m+1)' \sim_{\alpha(\pi)} i'$, leading to crossing chords in the representations of π and $\alpha(\pi)$. On the other hand, we must have $(2m+1) \sim_\pi (m+1)$, otherwise we would have a chord from $(2m+1)'$ to some $i', i \leqslant m$, contradicting our previous remark. Thus, a fixed point π of α must have $m+1$ and $2m+1$ in the same block and is completely determined by its restriction to a noncrossing partition of $\{m+1, m+2, \ldots, 2m\}$. Indeed, each of the C_m such restrictions induces a partition on $1', 2', \ldots, m'$ via the map α whose reflection in D yields the rest of the blocks of π. Then the circular representation of $\alpha(\pi)$ is the reflection in D of the circular representation of π, and $\alpha(\pi) = \pi$. □

For example, 12/35/4 and 1/2/345 are the two fixed points of α in NC(5). Observe that the argument above does not hold for even n, as it should not, when α has no fixed points.

Remark 1.3. An immediate consequence of proposition 1.2 is that $C_{2m+1} \equiv C_m (\text{mod}\, 2)$, while Theorem 1.1 implies $C_{2m} \equiv 0\ (\text{mod}\, 2)$. Therefore, $C_n \equiv 0\ (\text{mod}\, 2)$ except when $n = 2^p - 1$, for some $p \geqslant 1$; in that case $C_{2^p-1} \equiv C_1 = 1\ (\text{mod}\, 2)$. A different proof of this parity property of the Catalan numbers, based on lattice paths, appears in [10]. Yet another proof of these relations can be obtained by reducing modulo 2 and iterating the relation in 3.1. below.

2. Symmetric chain decomposition

We now turn to our second structural property for which we will give two different proofs.

Theorem 2. *For each $n \geqslant 1$, the lattice* NC(n) *admits a symmetric chain decomposition.*

Existence proof 2.1. It is trivial to check that the SCD property holds for small n. Assume that it holds for each noncrossing partition lattice NC(k) with $k < n$. Now decompose NC(n) as $\bigcup_{i=1}^n R_i$, where $R_1 = \{\pi \in \text{NC}(n)\colon \{1\} \text{ is a block of } \pi\}$, and $R_i = \{\pi \in \text{NC}(n)\colon i = \min\{j\colon j \neq 1, 1 \sim j\}\}$ for $i \geqslant 2$. Observe that the posets R_1 and R_2 are isomorphic to NC($n-1$) and moreover, that $R_1 \cup R_2$ is isomorphic to the product of NC($n-1$) and a 2-element chain, since each partition in R_1 is covered by only one partition from R_2, namely the partition obtained from it by merging the block $\{1\}$ with the block containing the element 2. Observe further that for $i \geqslant 3$, R_i is isomorphic to the product of NC($i-2$) and NC($n-i+1$), realized as noncrossing partitions of $\{2, 3, \ldots, i-1\}$ and $\{i, i+1, \ldots, n\}$, respectively. Now apply induction and [1, p. 434, Prop. 8.64], which states that the SCD property is preserved under the poset product operation. (This is an immediate consequence of the fact that the product of two chains has a SCD. See [3].) We infer that $R_1 \cup R_2$ and each of the R_i for $i \geqslant 3$ has an SCD. Now note that $R_1 \cup R_2$ has height $n-1$ and that each of R_i, $i \geqslant 3$, is an interval in NC(n). Furthermore, the minimum and maximum elements of $R_1 \cup R_2$ are $\hat{0} := 1/2/\cdots/n$ and $\hat{1} := 12\cdots n$, while the minimum element of R_i for $i \geqslant 3$ is $\hat{0}_i := 1i/2/\cdots/i-1/i+1/\cdots/n$ and the maximum element of R_i for $i \geqslant 3$ is $\hat{1}_i := 1i(i+1)(i+2)\cdots n/23\cdots(i-1)$. Thus, in each case the ranks of the minimum and of the maximum have sum $n-1$, and so $R_1 \cup R_2$ and each R_i with $i \geqslant 3$ are symmetrically embedded in NC(n). This implies that the symmetric chains in $R_1 \cup R_2$ and in each R_i for $i \geqslant 3$ are symmetric chains in NC(n), and completes the existence proof. □

Let us remark that there are known sufficient conditions for the existence of a symmetric chain decomposition, such as [3, 12] and [1, Cor. 8.66]. These, however, either do not apply directly to the poset NC(n), or do not apply to it at all. For example, Aigner's result assumes that the poset under consideration is a modular geometric lattice, while NC(n) is not even semi-modular if $n \geqslant 4$, since, for instance, the atoms $13/2/4/5/\cdots/n$ and $1/24/3/5/\cdots/n$ both cover their infimum $\hat{0}$, but neither is covered by their supremum $1234/5/\cdots/n$.

The second proof gives a greedy algorithm for constructing individual chains which form a SCD of NC(n).

Constructive proof 2.2. Take $\pi \in \mathrm{NC}(n)$ and consider its linear representation. With π we associate a word w of length $n-1$ over the alphabet $\{b, e, l, r\}$ as follows: $w(\pi) = w_1 w_2 \cdots w_{n-1}$, where

$$w_i = \begin{cases} b, & \text{if } i \not\sim i+1 \text{ and } i \text{ is } \textit{not} \text{ the largest element in its block;} \\ e, & \text{if } i \not\sim 1 \text{ and } i+1 \text{ is } \textit{not} \text{ the smallest element in its block;} \\ l, & \text{if } i \not\sim i+1,\ i \text{ is the largest element in its block,} \\ & \qquad \text{and } i+1 \text{ is the smallest element in its block;} \\ r, & \text{if } i \sim i+1. \end{cases}$$

Let B(E, L, R, respectively) be the set $\{i\colon w_i = b\}$ ($w_i = e, l, r$, respectively). Now think of l and r as left and right parentheses, respectively, and match all possible parentheses. We define the *core* of π to be the quadruple $c(\pi) = (B, E, ML, MR)$, where ML and MR are the subsets of L and R, respectively, consisting of the matched l's and r's. For example, if $n = 20$ and $\pi = 1\,2\,9\,12/3\,4/5\,7\,8/6/10/11/13\,14\,18\,19\,20/15/16/17$, then

$$w(\pi) = rbrlbereblelrbllerr,$$

and $c(\pi) = (\{2, 5, 9, 14\}, \{6, 8, 11, 17\}, \{4, 12, 15, 16\}, \{7, 13, 18, 19\})$. See also Fig. 4.

Before describing the construction of symmetric chains, let us make some observations which will be used repeatedly in what follows.

Observation 2.3. *For any noncrossing partition π we have $|B| + |E| + |L| + |R| = n-1$, $\mathrm{bk}(\pi) = 1 + |B| + |L| = 1 + |E| + |L|$, and so $|B| = |E|$.*

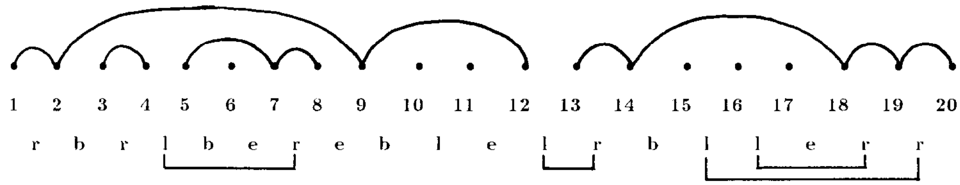

Fig. 4.

In fact more is true, as stated in the following.

Observation 2.4. *If w is the word associated with a partition $\pi \in \mathrm{NC}(n)$ as above, then there is a complete parenthesization of the b's and e's in w.*

Indeed, $w_i = b$ indicates that a nested block *begins* at $i+1$, while $w_i = e$ indicates that a nested block ends at i. Thus, if $w_i = b$, then there is a block in π which contains i *and* elements larger than i. Let $j = \min\{k: k > i, k \sim i\}$. Then necessarily $w_{j-1} = e$ and now match (or parenthesize) w_i with w_{j-1}.

Now we can describe the construction of the chain γ which contains a given noncrossing partition π; the chain is determined by the core of the partition. Let the core of π be $c = (B, E, ML, MR)$ and form the word $w(c)$ whose ith letter is equal to b (e, r, respectively) if $i \in B$ ($i \in E$, MR, respectively) and equal to l otherwise. The word $w(c)$ so constructed gives the minimum element of a chain $\gamma = \gamma(c)$, and successive partitions on γ correspond to the words obtained from $w(c)$ by changing the l's in $L - ML$ to r's in order, from left to right (in terms of partitions, we merge *certain* 'adjacent' blocks.)

Clearly, the resulting chain is saturated. The maximum element of $\gamma(c)$ has the word $W(c)$ in which the core is c and no l is unmatched. Fig. 5 shows the successive words and the chain of noncrossing partitions with core $c(\pi)$ from the previous example.

Observe that the core of a partition is well defined and that all partitions with the same core lie on one chain. Using the numerical relations from Observation 2.3, the number of blocks of the minimum element of $\gamma(c)$ equals $1 + |B| + ((n-1) - |B| - |E| - |MR|) = n - |E| - |MR|$, while the number of blocks of the maximum element of $\gamma(c)$ equals $1 + |B| + |ML|$. Since $|E| = |B|$ and $|MR| = |ML|$, it is now clear that γ is a symmetric chain. □

Remark 2.5. Although $\mathrm{NC}(n)$ admits both an order-reversing involution and an SCD, for $n > 3$ it does not admit an SCD together with an order-reversing involution that maps each chain to itself. To see this for $n = 2m + 1$, observe that such an involution would have to fix every partition π into $m + 1$ blocks and map bijectively the set of partitions which cover π to the set of partitions which are covered by π. No such bijection and hence no such involution exists for odd $n > 3$ since, for example, the partition $\pi = 12/34/\cdots/(2m-3)(2m-2)/(2m-$

1 2 9 12 / 3 4 / 5 7 8 / 6 / 10 11 / 13 14 18 19 20 / 15 / 16 / 17

$W(c) = r\ b\ r\ l\ b\ c\ r\ c\ b\ r\ c\ l\ r\ b\ l\ l\ c\ r\ r$

1 2 9 12 / 3 4 / 5 7 8 / 6 / 10 / 11 / 13 14 18 19 20 / 15 / 16 / 17

1 2 9 12 / 3 / 4 / 5 7 8 / 6 / 10 / 11 / 13 14 18 19 20 / 15 / 16 / 17

1 / 2 9 12 / 3 / 4 / 5 7 8 / 6 / 10 / 11 / 13 14 18 19 20 / 15 / 16 / 17

$w(c) = l\ b\ l\ l\ b\ c\ r\ c\ b\ l\ c\ l\ r\ b\ l\ l\ c\ r\ r$

Fig. 5. The chain $\gamma(c)$ with core $c = (\{2, 5, 9, 14\}, \{6, 8, 11, 17\}, \{4, 12, 15, 16\}, \{7, 13, 18, 19\})$.

$1)2m/(2m+1)$ into $m+1$ blocks, is covered by $\binom{m+1}{2}$ partitions but covers only m partitions. Similarly, if $n = 2m \geqslant 4$, consider the partition $\pi = 12/34/\cdots/(2m-3)(2m-2)/(2m-1)2m$ into m blocks and let S be the set of partitions covered by π. Then $|S| = m$ and every partition in S is covered by $\binom{m+1}{2}$ partitions. A chain-preserving order-reversing involution α would have to map π to an element of S and S onto the set of partitions which cover $\alpha(\pi)$. Yet $|S| = m$, $|\alpha(S)| = \binom{m+1}{2}$, while α must be a bijection. Since $m > 1$, we have reached a contradiction.

Remark 2.6. How do the SCD's provided by 2.1 and 2.2 compare? Proof 2.1 leads to one of many different SCD's of NC(n), depending on how each product of two chains is decomposed into symmetric chains. However, if we choose to decompose the product of $x_1 \lessdot x_2 \lessdot \cdots \lessdot x_m$ and $y_1 \lessdot y_2 \lessdot \cdots \lessdot y_p$ so that the jth symmetric chain in the decomposition is $(x_1, y_j) \lessdot (x_2, y_j) \lessdot \cdots \lessdot (x_{m-j+1}, y_j) \lessdot$

$(x_{m-j+1}, y_{j+1}) \lessdot \cdots \lessdot (x_{m-j+1}, y_p)$, then the Proofs 2.1 and 2.2 yield the same SCD of NC(n). Indeed, if π is a partition in R_i, for $i \geqslant 3$ as in 2.1, then let π' and π'' be the corresponding partitions in NC($i-2$) and NC($n-i+1$). In NC($i-2$), π' lies on a symmetric chain γ', and in NC($n-i+1$), π'' lies on a symmetric chain γ''; let the cores of these chains be (B', E', ML', MR') and (B'', E'', ML'', MR''), respectively. Observe that if the core of π is (B, E, ML, MR), then then $B = \{1\} \cup B' \cup B''$, slightly abusing notation since before taking the unions, all integers in B' must be augmented by one unit and all integers in B'' must be augmented by $i-1$ units. Similar relations hold for E, L, and R. Let j be the number of unmatched l's from π' which become matched in π, necessarily with j unmatched r's from π''. Then π lies on the $(j+1)$st chain in the SCD of the product of γ' with γ'', if the product of these two chains is decomposed as described above. Thus, for $i \geqslant 3$, each $\pi \in R_i$ occurs in the symmetric chain corresponding to its core, exactly as in the Proof 2.2. If $\pi \in R_1 \cup R_2 \simeq \mathbf{2} \times \mathrm{NC}(n-1)$, let π'' be the corresponding partition from NC($n-1$), and let γ'' be the symmetric chain of NC($n-1$) on which π'' lies. Then either $1 \sim 2$ in π, and then π lies on the first chain of the SCD of $\mathbf{2} \times \gamma''$, with a core satisfying $1 \notin \mathrm{ML}$, or else $1 \nsim 2$ in π, and then π lies on the second chain of the SCD of $\mathbf{2} \times \gamma''$ if and only if $1 \in \mathrm{ML}$. The chain containing π obtained in this way is the same as the chain containing π in 2.2, because of the 'greedy' approach used in 2.2, where at each stage, we move upward along a chain by turning the leftmost unmatched l into an r.

3. Identities

Based on the construction of the symmetric chain decomposition, we shall see and use the fact that the elements of NC(n) can be partitioned into boolean lattices. Indeed, all partitions having prescribed sets B and E in their core form a copy of the boolean lattice $B_{n-1-2|B|}$, whose elements can be identified with the subsets $R \subseteq [n-1] - (B \cup E)$ as in (2.2).

The corollaries below are enumerative consequences of the fact that NC(n) can be decomposed into intervals isomorphic to boolean lattices. First we obtain the following identity due to Touchard, a short combinatorial proof of which appears in [15]. (Our sets L and R are the green and red points in [15].)

Corollary 3.1. *For every $n \geqslant 1$,*

$$\sum_{k \geqslant 0} \binom{n-1}{2k} C_k 2^{n-2k-1} = C_n.$$

Proof. Recall that the b's and e's are always matched, and that the partitions having prescribed sets B and E in their core form a boolean lattice $B_{n-1-2|B|}$ in

NC(n). We can now enumerate all noncrossing partitions of $[n]$: for each $k < n/2$, choose a subset X of $[n-1]$ or cardinality $2k$ together with a noncrossing partition of X into k blocks each of cardinality 2. Let XB (XE, respectively) be the set of minimum (maximum, respectively) elements of the k blocks of the partition chosen for X. Then there are 2^{n-2k-1} noncrossing partitions of $[n]$ whose core has prescribed $B = \mathrm{XB}$ and $E = \mathrm{XE}$. Summing over k yields Touchard's identity. □

A refinement of this identity arises immediately from counting the words w of Proof (2.2) which correspond to the noncrossing partitions of a fixed rank in NC(n).

Corollary 3.2. *For all $n \geq 1$ and $i \in [n]$,*

$$\sum_{k \geq 0} \binom{n-1}{2k} \binom{n-2k-1}{n-k-i} C_k = \frac{\binom{n}{i}\binom{n}{i-1}}{n}.$$

Proof. This identity can be verified by straightforward calculation, but its derivation comes from the enumeration of the partitions in NC(n) such that $\mathrm{bk}(\pi) = i$ using ideas from (2.2). Recall from observation 2.3 that $\mathrm{bk}(\pi) = 1 + |B| + |L| = n - |B| - |R|$ and proceed as in the proof of (3.1). We have $\binom{n-1}{2k} C_k$ choices for B and E, each of cardinality k, and, if $\mathrm{bk}(\pi) = i$, we have $\binom{n-1-2k}{n-k-i}$ choices for R. This yields the left hand side, while the right hand side is simply the formula mentioned in the introduction for the number $W(n, i)$ of noncrossing partitions into i blocks. □

Of course, Corollary 3.1 follows from 3.2 by summing over i. The next identity follows from the construction of the SCD as well.

Proposition 3.3. *For all $n \geq 1$ and $(n+1)/2 \leq i \leq n$,*

$$\sum_{k \geq 0} \binom{n-1}{2k} C_k \left[\binom{n-2k-1}{i-k-1} - \binom{n-2k-1}{i-k} \right] = \frac{\binom{n+1}{i}\binom{n-1}{i-1}}{\binom{i+1}{2}} \left(i - \frac{n}{2} \right).$$

Proof. Because $i \geq (n+1)/2$, we have $W(n, i+1) \leq W(n, i)$. Let us calculate the excess of non-crossing partitions into i blocks over those into $i+1$ blocks. One approach is to evaluate directly the difference $W(n, i) - W(n, i+1)$ and this leads to the right hand side of the identity. The other approach, using the SCD property of NC(n), is to count the symmetric chains whose minimum element is a partition into i blocks. In our construction of the symmetric chains, the minimum partition on a chain corresponds to a word w in which all the r's are matched, i.e., $R = \mathrm{MR}$. As in the case of any word from our constructive proof, $|B| = |E|$, $|B| + |E| + |L| + |R| = n - 1$, and there is a noncrossing pairing of the e's with the b's (see Observations 2.4 and 2.5). Since the number of blocks must be i, we must have $|B| + |L| = i - 1$. Therefore, to obtain w, we must choose the positions for the b's and e's and pair them; if, say $|E| = |B| = k$, this can be done in $\binom{n-1}{2k}C_k$ ways, and there are $n - 1 - 2k$ letters in w to be filled with l's and r's. The preceding relations force $|L| = i - k - 1$; finally, since all r's must be matched, no prefix of w may contain more r's than l's. Thus, by a standard reflection argument (see, e.g., [M]), the number of ways to complete w is $\binom{n-2k-1}{i-k-1} - \binom{n-2k-1}{i-k}$. Upon summing over $k = |B| = |E| \geqslant 0$, we obtain the left hand side of the identity. □

We point out two special cases of the preceding proposition; the first special case is an identity proved in a different setting in [6].

Corollary 3.4. *If $n = 2m + 1$, then the number of single-element chains in any symmetric chain decomposition of* $\mathrm{NC}(n)$ *is*

$$\sum_{k\geqslant 0} \binom{2m}{2k} C_k C_{m-k} = C_m C_{m+1}.$$

Proof. Simply set $n = 2m + 1$ and $i = m + 1$ in Proposition 3.3. The left hand side can be interpreted directly: a partition constitutes a single-element chain if and only if its word w has no unmatched l's and no unmatched r's. Thus, in w there is not only a complete parenthesization/matching of the b's and e's as usual, but also a complete parenthesization of the l's and r's. If we range over the possibilities for $k = |B| = |E|$, we obtain the left-hand side. □

We obtain a related identity in the case when n is even.

Corollary 3.5. *If $n = 2m$, then the number of two-element chains in any symmetric chain decomposition of* $\mathrm{NC}(n)$ *is*

$$\sum_{k\geqslant 0} \binom{2m-1}{2k} C_k C_{m-k} = \frac{1}{2} C_m C_{m+1}.$$

Proof. Set $n = 2m$ and $i = m + 1$ in Proposition 3.3. Alternatively, the left hand side can be obtained by noticing that there is precisely one unmatched l in the word w of the minimum element of a 2-element symmetric chain. □

We derive now a combinatorial identity involving noncrossing partitions which is in the spirit of many integer partition identities. We give a bijective proof for our identity.

Proposition 3.6. *Let* $\overline{\mathrm{NC}}(n)$ *be the set of noncrossing partitions of* $[n]$ *with no consecutive intergers in the same block, and let* $\mathrm{NC}_2(n)$ *be the set of noncrossing partitions of* $[n]$ *into blocks of cardinality at most* 2. *Then for all* $n \geqslant 1$ *we have* $|\overline{\mathrm{NC}}(n)| = |\mathrm{NC}_2(n-1)|$.

Proof. Recall that $\mathrm{NC}(n)$ can be expressed as the disjoint union of boolean lattices. Our identity arises by counting these boolean lattices in two ways. On one hand, each boolean lattice corresponds to prescribed sets B and E (containined in $[n-1]$) in the core of the noncrossing partitions. By Observation 2.4, the b's and e's are well parenthesized (or matched), and thus, each boolean lattice corresponds bijectively to a partition in $\mathrm{NC}_2(n-1)$ (each element in $[n-1]-(B \cup E)$ is in a block by itself). On the other hand, count the boolean lattices into which $\mathrm{NC}(n)$ decomposes by counting their minimum elements: $\pi \in \mathrm{NC}(n)$ is such a minimum element if and only if $R = \phi$ in $w(\pi)$. But $R = \phi$ in turn is equivalent to $i \not\sim_\pi (i+1)$ for all $i \in [n-1]$. Thus, each boolean lattice corresponds bijectively to a partition in $\overline{\mathrm{NC}}(n)$. Therefore, the number of noncrossing partitions in each of the two classes is equal to the number of boolean lattices into which $\mathrm{NC}(n)$ is decomposed under our SCD.

A direct correspondence between the above two types of partitions can be described as follows: Let $\pi \in \overline{\mathrm{NC}}(n)$. Write $w = w(\pi)$ as in (2.2) and determine the matching of B and E. Let π^* be the partition of $[n-1]$ in which $i \sim j$, $i < j$, if and only if $w_i = b$, $w_j = e$, and these b and e are matched in $w(\pi)$. Clearly, $\pi^* \in \mathrm{NC}_2(n-1)$. Conversely, if $\pi^* \in \mathrm{NC}_2(n-1)$, construct the word $w \in \{b, e, l, r\}^{n-1}$ by letting $w_i = b(w_i = e$, respectively) iff i is the minimum (maximum, respectively) element of a nontrivial block of π^*, and $w_i = l$ otherwise. Then the partition π whose word is w is in $\overline{\mathrm{NC}}(n)$, and corresponds bijectively to π^*. $\square$

It is a routine exercise to establish the recurrence

$$|\overline{\mathrm{NC}}(n)| = \sum_{m=0}^{n-2} |\overline{\mathrm{NC}}(m)| \cdot |\overline{\mathrm{NC}}(n-m-1)|,$$

valid for $n \geqslant 2$, by counting the partitions according to the minimum element which is in the same block as 1. This recurrence together with the initial values $|\overline{\mathrm{NC}}(0)| = 1$, $|\overline{\mathrm{NC}}(1)| = 1$, leads to the generating function

$$\sum_{n=0}^{\infty} |\overline{\mathrm{NC}}(n)|\, x^n = \frac{1 + x - \sqrt{1-(2x+3x^2)}}{2x}.$$

An immediate extension of Proposition 3.6, whose proof is similar and we omit, is the following.

Proposition 3.7. *Fix* $k \in \{1, 2, \ldots, n\}$. *Let* $\overline{\mathrm{NC}}(n, k)$ *denote the set of noncrossing partitions of* $[n]$ *in which* $i \neq j$, $i \sim j$ *imply* $|i-j| > k$. *Let also* $\mathrm{NC}_2(n, k)$

denote the set of noncrossing partitions of $[n]$ *in which each block has cardinality at most* 2 *and* $i \neq j$, $i \sim j$ *imply* $|i-j| > k$. *Then*

$$|\overline{\mathrm{NC}}(n, k)| = |\mathrm{NC}_2(n-1, k-1)|.$$

The case $k = 1$ gives Proposition 3.6.

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