Existence of Coupled Quasi-solutions of Systems of Nonlinear Reaction–Diffusion Equations*

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1. INTRODUCTION

Systems of nonlinear parabolic initial boundary value problems arise in many applications, such as epidemics, ecology, biochemistry, biology, chemical and nuclear engineering. Constructive methods of proving existence results for such problems, which can also provide numerical procedures for the computation of solutions, are of greater value than theoretical existence results. The method of upper and lower solutions coupled with monotone iterative technique has been employed successfully to prove existence of multiple solutions of nonlinear reaction–diffusion equations, in special cases, by various authors [3–5, 10, 11, 15, 18]. Recently, in [6, 17] weakly coupled systems of reaction–diffusion equations, when the nonlinear terms are independent of gradient terms, are discussed and some special type of results are obtained.

We, in this paper, investigate general systems of nonlinear reaction–diffusion problems when the nonlinear terms possess a mixed quasi-monotone property. We discuss a very general situation and obtain coupled extremal quasi-solutions, which in special cases reduce to minimal and maximal solutions. We shall also indicate how one-step cyclic monotone iterative schemes can be generated which yield accelerated rate of convergence of iterates. This work is in the spirit of our recent paper [12] for elliptic systems.

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2. Definitions and Assumptions

Consider the second order nonlinear parabolic initial-boundary-value problem (IBVP for short)

\[ \mathcal{L}_k u_k = f_k(t, x, u, D_x u_k) \quad \text{in } Q_T, \]
\[ B_k u_k = \phi_k \quad \text{on } \Gamma_T, \]
\[ u_k(0, x) = \phi_{0k}(x) \quad \text{in } \bar{\Omega}, \]

(2.1)

where \( u \in \mathbb{R}^n \), \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with boundary \( \partial \Omega \) and closure \( \bar{\Omega} \), \( Q_T = (0, T) \times \Omega \), \( \Gamma_T = (0, T) \times \partial \Omega \), \( \bar{Q}_T = [0, T] \times \bar{\Omega} \), \( \bar{\Gamma}_T = [0, T] \times \partial \Omega \), \( T > 0 \). Here, for each \( k \in I = \{1, 2, \ldots, n\} \), \( \mathcal{L}_k \) is a second order differential operator defined by

\[ \mathcal{L} = \frac{\partial}{\partial t} - L_k, \]

(2.2)

where

\[ L_k = \sum_{i,j=1}^m a_{ij}^k(t, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^m b_i^k(t, x) \frac{\partial}{\partial x_i} + c^k(t, x), \]

(2.3)

and \( B_k \) is a boundary operator given by

\[ B_k u_k = p_k(t, x) u_k + q_k(t, x) \frac{du_k}{dv}, \]

(2.4)

where \( du_k/dv \) denotes the normal derivative of \( u_k \) and \( v(t, x) = (v_1(t, x), v_2(t, x), \ldots, v_m(t, x)) \) is the unit outward normal vector field on \( \partial \Omega \) for \( t \in [0, 1] \).

To define quasi-solutions of (2.1), for each \( k \in I \), let \( b_k, d_k \) be two non-negative integers such that \( b_k + d_k = n - 1 \). By splitting \( u \in \mathbb{R}^n \) into \( u = (u_k, [u]_{b_k}, [u]_{d_k}) \), we rewrite (2.1) in the following form:

\[ \mathcal{L}_k u_k = f_k(t, x, u_k, [u]_{b_k}, [u]_{d_k}, D_x u_k) \quad \text{in } Q_T, \]
\[ B_k u_k = \phi_k \quad \text{on } \Gamma_T, \]
\[ u_k(0, x) = \phi_{0k}(x) \quad \text{in } \bar{\Omega}. \]

(2.5)

Also, for any \( u, v \in \mathbb{R}^n \), we let \([u, v]_k\) denote an element of \( \mathbb{R}^n \) with the description \([u, v]_k = (u_k, [u]_{b_k}, [v]_{d_k})\). Without further mention, we assume that \( k \in I \) and all inequalities involving vectors are componentwise.
**Definition 2.1.** The functions $u, v \in C^{1,2}[\bar{Q}_T, \mathbb{R}^n]$ are said to be coupled quasi-solutions of (2.1) if

\begin{equation}
\begin{aligned}
\mathcal{L}_k u_k &= f_k(t, x, [u, v]_k, D_x u_k) \quad \text{in } Q_T, \\
\mathcal{L}_k v_k &= f_k(t, x, [v, u]_k, D_x v_k) \quad \text{in } Q_T, \\
B_k u_k &= \phi_k = B_k v_k \quad \text{on } \Gamma_T, \\
u_k(0, x) &= \phi_0(x) = v_{0k}(x) \quad \text{in } \bar{Q}.
\end{aligned}
\end{equation}

The functions $v, w \in C^{1,2}[\bar{Q}_T, \mathbb{R}^n]$ with $v(t, x) \leq w(t, x)$ on $\bar{Q}_T$ are said to be coupled lower and upper quasi-solutions, respectively, if

\begin{equation}
\begin{aligned}
\mathcal{L}_k v_k &\leq f_k(t, x, [v, w]_k, D_x v_k) \quad \text{in } Q_T, \\
B_k v_k &\leq \phi_k \quad \text{on } \Gamma_T, \\
v_k(0, x) &\leq \phi_0(x) \quad \text{in } \bar{Q},
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
\mathcal{L}_k w_k &\geq f_k(t, x, [w, v]_k, D_x w_k) \quad \text{in } Q_T, \\
B_k w_k &\geq \phi_k \quad \text{on } \Gamma_T, \\
w_k(0, x) &\geq \phi_0(x) \quad \text{on } \bar{Q}.
\end{aligned}
\end{equation}

**Definition 2.2.** The function $f$ is said to possess a mixed quasi-monotone property (mqmp for short) if for each $k \in I$, $f_k(t, x, u_k, [u]_{b_k}, [u]_{d_k}, D_x u_k)$ is monotone nondecreasing in $[u]_{b_k}$ and monotone nonincreasing in $[u]_{d_k}$.

Let us list the following assumptions for convenience.

\begin{enumerate}
\item[(A_i)] (i) For each, $i, j=1, \ldots, m$ and $k \in I$, $a^k_{ij}$, $b^k_i$, and $c^k \in C^{\alpha/2,\alpha}[\bar{Q}_T, \mathbb{R}]$, $c^k(t, x) \leq 0$, and $\mathcal{L}_k$ is strictly uniformly parabolic in $\bar{Q}_T$;

(ii) for each $k \in I$, $p_k, q_k \in C^{(1+\alpha)/2,1+\alpha}[\bar{\Gamma}_T, \mathbb{R}]$, $p_k(t, x) > 0$ and $q_k(t, x) \geq 0$ on $\Gamma_T$;

(iii) $\partial \Omega$ belongs to class $C^{2+\alpha}$;

(iv) $f \in C^{\alpha/2,\alpha}[0, T] \times \bar{Q} \times \mathbb{R}^n \times \mathbb{R}^m$, that is, $f(t, x, u, y)$ is Hölder continuous in $t$ and $(x, u, y)$ with exponents $\alpha/2$ and $\alpha$, respectively, and $f$ satisfies a Nagumo condition, that is, there exists an increasing function $\psi_k: \mathbb{R}_+ \to \mathbb{R}_+$ such that for $(t, x, u, v) \in [0, T] \times \bar{Q} \times \mathbb{R}^n \times \mathbb{R}^m$,

\begin{equation}
|f_k(t, x, u, v)| \leq \psi_k(\|u\|)(1 + \|y\|^2);
\end{equation}

(v) $\phi \in C^{(1+\alpha)/2,1+\alpha}[\bar{\Gamma}_T, \mathbb{R}^n]$ and $\phi_0 \in C^{2+\alpha}[\bar{Q}, \mathbb{R}^n]$;

\end{enumerate}
(vi) the IBVP (2.5) satisfies the compatibility condition of order \([(x+1)/2]\).

\(\text{(A}_2\text{)}\) \(v, w\) are coupled lower and upper quasi-solutions relative to (2.5).

\(\text{(A}_3\text{)}\) For some \(M_k \geq 0\),
\[f_k(t, x, u, h(y)) - f_k(t, x, \bar{u}, h(y)) \geq -M_k(u_k - \bar{u}_k),\]
whenever \(v(t, x) \leq \bar{u} \leq u \leq w(t, x)\), with \(u_i = \bar{u}_i\) for \(i \neq k\), \((t, x) \in \bar{Q}_T\) and \(y \in \mathbb{R}^m\).

In \((\text{A}_3\text{)}\), the function \(h \in H = \{h \in C^1[\mathbb{R}^m, \mathbb{R}^m] : h(y) = y\text{ for }\|y\| < N, \|h(y)\| \leq \lambda \|y\| \text{ for } y \in \mathbb{R}^m, \text{ and } h(R^m) \text{ and } h_1(R^m) \text{ are bounded}\}\), where \(h, \lambda \), stands for the Jacobian matrix function of \(h, \lambda > 1; N\) is defined by
\[N > \max\{\bar{N}, \max_{(t, x) \in \bar{Q}_T} \|v_k(t, x)\|, \max_{(t, x) \in \bar{Q}_T} \|w_k(t, x)\|\},\]
\(\bar{N}\) being Nagumo constant relative to \(v, w\) and \(\psi_k(\mu) \lambda^2(1 + \|y\|^2), \mu = \max\{\|\bar{v}\|, \|\bar{w}\|\}\).

\[v_k = \min_{(t, x) \in \bar{Q}_T} \{v_k(t, x)\}, \quad \bar{v}_k = \max_{(t, x) \in \bar{Q}_T} \{w_k(t, x)\},\]
\[v = (v_1, v_2, ..., v_n), \quad \bar{v} = (\bar{v}_1, \bar{v}_2, ..., \bar{v}_n), k \in I.\]

3. \textit{Auxiliary Results}

Consider the modified second order nonlinear parabolic initial-boundary-value problem
\[
\mathcal{L}_k u_k = G_k(t, x, u_k, D_x u_k) - M_k u_k \quad \text{in } Q_T, \\
B_k u_k - \phi_k \quad \text{on } \Gamma, \\
u_k(0, x) = \phi_{0k}(x) \quad \text{in } \bar{Q}, \text{ for } k \in I, \tag{3.1}
\]
where \(M_k\) is as defined in assumption \((\text{A}_3)\), and
\[
G_k(t, x, u_k, D_x u_k) = f_k(t, x, [\eta_1(t, x), \eta_2(t, x)]_k, h(D_x u_k)) + M_k \eta_{1k}(t, x) \tag{3.2}
\]
and \(\eta_1, \eta_2 \in C^{(1+\alpha)/2, 1+\alpha}[\bar{Q}_T, \mathbb{R}^m] \) such that \(v(t, x) \leq \eta_1(t, x) \leq w(t, x), \nu(t, x) \leq \eta_2(t, x) \leq w(t, x) \) on \(\bar{Q}_T\).

Our first objective is to show that the IBVP (3.1) has a unique solution. The proof of existence and uniqueness of solutions of (3.1) is equivalent to the existence and uniqueness of solutions of
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\[ \mathcal{D}_k u_k = G_k(t, x, u_k, D_x u_k) \quad \text{in } Q_T, \]
\[ B_k u_k = \phi_k \text{ on } \Gamma_T, \]
\[ u_k(0, x) = \phi_{0k}(x) \quad \text{in } \Omega, \]

where \( \mathcal{D}_k = \partial/\partial t - L_k \) and \( L_k = L_k - M_k \) for \( k \in I \).

We note that (3.3) is completely decoupled system. Moreover, \( G_k \) in (3.3) is independent of \( u_k \). The proof of existence of solution of IBVP (3.3) follows from the verification of the hypotheses of Theorem 4.2.2 [13] relative to the IBVP (3.3). This is the objective of the following lemmas.

**Lemma 3.1.** Assume that \((A_1)\) holds. Then for each \( k \in I \), \( \mathcal{D}_k, B_k, \phi_k, \phi_{0k}, \Omega \) and \( G_k \) in (3.3) satisfy the assumption \((A_1)\).

**Proof.** From the definitions of \( \mathcal{L}_k, \mathcal{D}_k, M_k, G_k, \phi_k, \phi_{0k} \) and \( \Omega \) it is obvious that all conditions listed in \((A_1)\) except (iv) are satisfied by \( \mathcal{D}, B_k, \phi_k, \phi_{0k} \), and \( \Omega \). Therefore, it is enough to show that \( G_k \) in (3.3) satisfies condition (iv). For \( \eta_1, \eta_2 \in C^{(1+\alpha)/2, 1+\alpha}[\mathcal{Q}_T, R^m] \) with \( v(t, x) \leq \eta_1(t, x), \eta_2(t, x) \leq w(t, x) \), we have

\[ |G_k(t, x, u_k, y) - G_k(t, x, u_k, y)| \]
\[ \leq |f_k(t, x, [\eta_1(t, x), \eta_2(t, x)]_k, h(y))| + f_k(t, x, [\eta_1(t, x), \eta_2(t, x)]_k, h(y))| \]
\[ + M_k |\eta_1(t, x) - \eta_2(t, x)| \]
\[ \leq K(f_k)(|t - \tilde{t}|^{\alpha/2} + (K([\eta_1, \eta_2]_k) |t - \tilde{t}|^{(1+\alpha)/2} + M_k K(\eta_1) |t - \tilde{t}|^{(1+\alpha)/2} \]
\[ \leq K_G(t) |t - \tilde{t}|^{\alpha/2}, \]

where \( K_G(t) = K(f_k)(1 + (K([\eta_1, \eta_2]_k) T^{1/2} + M_k K(\eta_1) T^{1/2} \). This shows that \( G_k(t, x, u_k) \) is Hölder continuous in \( t \) with exponent \( \alpha/2 \). The proof of the Hölder continuity of \( G_k(t, x, u_k, y) \) in \( (x, u_k, y) \) with exponent \( \alpha \) can be formulated analogous to the proof that is given in Lemma 3.1 [12]. This together with the fact that \( G_k \) is independent of \( u_k \), implies that \( G_k \in C^{\alpha, 2, 2}[0, T] \times \Omega \times R \times R^m, R \). Furthermore,

\[ |G_k(t, x, u_k, x)| \leq |f_k(t, x, [\eta_1(t, x), \eta_2(t, x)]_k, h(y))| + M_k |\eta_1(t, x)| \]
\[ \leq \psi_k(|H_1(t, x, \eta_2(t, x)|_k)|1 + \|h(y)|^2 + M_k |\eta_1(t, x)| \]
\[ \leq \psi_k(y)(1 + \|y\|^2), \]
where

$$\Psi_k(v, w) = \max_{z \in [v, w]} \{\psi_k(\|z\|) \lambda^2, M_k \|z\|\}.$$  

This completes the proof of the lemma.

**Lemma 3.2.** Let the assumptions (A₂) and (A₃) hold. Suppose that $f$ possesses mqmp. Then, for each $k \in I$, the components $v_k$ and $w_k$ of $v$ and $w$, respectively, are lower and upper solutions of the IBVP (3.3).

**Proof.** First we shall show that each component $v_k$ of $v$ is a lower solution of (3.3). For this purpose, it is enough to show that

$$f_k(t, x, [v, w], D_x v_k) + M_k v_k \leq G_k(t, x, v_k, D_x v_k) \quad \text{in } Q_T. \tag{3.4}$$

In fact, from (A₂) and the definition of $\bar{G}_k$, we get

$$\bar{G}_k v_k \leq G_k(t, x, v_k, D_x v_k) \quad \text{in } Q_T,$$

$$B_k v_k \leq \phi_k \quad \text{on } \Gamma_T,$$

$$u_k(0, x) \leq \phi_{0k}(x) \quad \text{in } \bar{Q}.$$

To prove (3.4), we note that $\|v_x(t, x)\|, \|w_x(t, x)\| \leq N$ for $(t, x) \in \bar{Q}_T$, and hence $h(D_x v_k) = D_x v_k$. From (A₂), (A₃), the definition of $G_k$ and the mixed monotone property of $f$, we get

$$f_k(t, x, [v, w], D_x v_k) - G_k(t, x, v_k, D_x v_k) + M_k v_k \leq f_k(t, x, [v, w], D_x v_k) - f_k(t, x, [\eta_1, \eta_2], D_x v_k) + M_k(v_k - \eta_{1k})$$

$$\leq M_k(\eta_{1k} - v_k) + M_k(v_k - \eta_{1k}) = 0.$$

This establishes the inequality (3.4). A similar argument shows that each component of $w$ is an upper solution of (3.3). The proof is complete.

Now, we are ready to prove existence and uniqueness of solutions of (3.3).

**Theorem 3.1.** Let assumptions (A₁), (A₂) and (A₃) hold. Assume further that $f$ in (2.1) possesses mqmp. Then there exists a unique solution $u \in \mathcal{C}^{1+\alpha/2,2+\alpha}[\bar{Q}_T, \mathbb{R}^n]$ to the IBVP (3.3) such that $u(t, x) \leq u(t, x) \leq w(t, x)$ on $\bar{Q}_T$. Moreover, there exists an $N_0 > 0$ such that $\|D_x u(t, x)\| \leq N_0$ on $\bar{Q}_T$, where $N_0$ depends on $\mathcal{P}$, $h, f, v, w, \phi$ and $\phi_0$.

**Proof.** By Lemmas 3.1 and 3.2, it is clear by Theorem 4.2.2 [13] that the IBVP (3.3) has a solution $u \in \mathcal{C}^{1+\alpha/2,2+\alpha}[\bar{Q}_T, \mathbb{R}^n]$ such that $v(t, x) \leq u(t, x) \leq w(t, x)$ on $\bar{Q}_T$. Furthermore, by modification of
Theorem 2.2 [3] (Theorem A.4.6 [13]), \( D_x u_k(t, x) \) satisfies the relation 
\[ \| D_x u_k(t, x) \| \leq N_0, \]
where \( N_0 \) depends on \( \mathcal{W}_k, f_k, h, v, w, \phi_k \) and \( \phi_{0k} \). This implies that \( \| D_x u(t, x) \| \leq N_0, \quad N_0 = (\sum_{k=1}^{\infty} N_{0k})^{1/2} \).

Now, it remains to prove that the solution \( u(t, x) \) of (3.3) is unique. We assume that it is false. Then there exists another solution \( z \in C^{1+a/2,2+a}[\tilde{Q}_T, R^n] \) of (3.3) such that \( v(t, x) \leq z(t, x) \leq w(t, x) \), and 
\[ \| D_x z(t, x) \| \leq N_0 \] on \( \tilde{Q}_T \). This implies that there exists at least one component \( u_k(t, x) \) of \( u(t, x) - z(t, x) \) such that either \( u_k(t, x) - z_k(t, x) \) or \( z_k(t, x) - u_k(t, x) \) will attain its positive maximum at some \( (t_0, x_0) \in \tilde{Q}_T \) with \( t_0 > 0 \). Without loss of generality, we assume that \( u_k(t, x) - z_k(t, x) \) attains its positive maximum at \( (t_0, x_0) \in \tilde{Q}_T \) with \( t_0 > 0 \).

First, we assume that \( (t_0, x_0) \in \tilde{Q}_T \). Then \( (\partial/\partial t)(u_k(t_0, x_0) - z_k(t_0, x_0)) \geq 0, \ D_x(u_k(t_0, x_0) - z_k(t_0, x_0)) = 0, \) and the Hessian matrix 
\[ (\partial^2/\partial x_i \partial x_j)(u_k(t_0, x_0) - z_k(t_0, x_0)) \] is negative semi-definite. Since 
\[ (a_j^k(t_0, x_0)) \] is positive-definite, we have 
\[ \sum_{j=1}^{m} a_j^k(t_0, x_0)(\partial^2/\partial x_i \partial x_j)(u_k(t_0, x_0) - z_k(t_0, x_0)) \leq 0. \]
From this and the nature of \( c^k(t_0, x_0) \) and \( M_k \), we get
\[ \mathcal{D}_k(u_k(t_0, x_0) - z_k(t_0, x_0)) > 0. \tag{3.5} \]

On the other hand, using the definition of \( G_k \), we obtain
\[ \mathcal{D}_k(u_k(t_0, x_0) - z_k(t_0, x_0)) \]
\[ = G_k(t_0, x_0, u_k(t_0, x_0), D_x u_k(t_0, x_0)) - G_k(t_0, x_0, z_k(t_0, x_0), D_x z_k(t_0, x_0)) \]
\[ = f_k(t_0, x_0, [\eta_1(t_0, x_0), \eta_2(t_0, x_0)], h(D_x u_k(t_0, x_0))) \]
\[ - f_k(t_0, x_0, [\eta_1(t_0, x_0), \eta_2(t_0, x_0)], h(D_x z_k(t_0, x_0))) \]
\[ = 0. \]
This contradicts with (3.5). Hence, there is no component of \( u(t, x) - z(t, x) \) that attains its positive maximum on \( \tilde{Q}_T \). This means that \( (t_0, x_0) \in \tilde{Q}_T \). In this case \( (d/dv)(u_k(t_0, x_0) - z_k(t_0, x_0)) \geq 0, \) that is 
\[ (d/dv) u_k(t_0, x_0) \geq (d/dv) z_k(t_0, x_0), \] which shows that
\[ p_k(t_0, x_0) u_k(t_0, x_0) + q_k(t_0, x_0) \frac{d}{dv} u_k(t_0, x_0) > p_k(t_0, x_0) z_k(t_0, x_0) \]
\[ + q_k(t_0, x_0) \frac{d}{dv} z_k(t_0, x_0) \]
and hence \( B_k u_k(t_0, x_0) > B_k z_k(t_0, x_0) \). This contradicts the fact that 
\[ B_k u_k(t_0, x_0) = \phi_k(t_0, x_0) = B_k z_k(t_0, x_0) \] on \( \Gamma_T \). It follows that there is no component of \( u(t, x) - z(t, x) \) which attains its positive maximum on \( \Gamma_T \). Hence \( u(t, x) - z(t, x) \leq 0 \) on \( \tilde{Q}_T \). Similarly, one can prove that
Thus we have \( u(t, x) = z(t, x) \) on \( \overline{Q}_T \), proving the uniqueness of (3.3). The proof of the theorem is complete.

For each \( \eta_1, \eta_2 \in C^{1+1/2,1+2}[\overline{Q}_T, R^n] \) such that \( v(t, x) \leq \eta_1(t, x), \eta_1(t, x) \leq w(t, x) \) on \( \overline{Q}_T \), we define a mapping \( A \) by

\[
A(\eta_1, \eta_2) = u(3.6)
\]

where \( u \in C^{1+1/2,1+2}[\overline{Q}_T, R^n] \) is the unique solution of the IBVP (3.3). The following result characterizes the properties of \( A \).

**Lemma 3.3.** Under the assumptions of Theorem 3.1, the mapping defined by (3.6) possesses the following properties:

1. \( v \leq A(v, w), w \geq A(w, v) \);
2. \( A \) is a mixed-monotone operator on the segment \( \langle v, w \rangle \), where \( \langle v, w \rangle = \{ u \in C^{1,2}[\overline{Q}_T, R^n]: v(t, x) \leq u(t, x) \leq w(t, x) \text{ on } Q_T \} \).

**Proof.** The boundedness of \( \Omega \) together with the fact that its boundary belongs to \( C^{2+\alpha} \) shows that if \( \eta \in C^{1,2}[\overline{Q}_T, R^n] \), then \( \eta \in W^{1,2}[\overline{Q}_T, R^n] \) for \( q > 1 \). This, in view of imbedding Theorem \[13, 14\], yields that \( \eta \in C^{1+1/2,2}[\overline{Q}_T, R^n] \). Thus for any \( v, w \in \langle v, w \rangle \), we conclude by Theorem 3.1 that \( A(\eta_1, \eta_2) \) belongs to \( C^{1+1/2,2}[\overline{Q}_T, R^n] \) and satisfies the inequality \( v(t, x) \leq A(\eta_1, \eta_2)(t, x) \leq w(t, x) \) on \( \overline{Q}_T \). But we know that \( C^{1+1/2,2}[\overline{Q}_T, R^n] \) is completely imbedded in \( C^{1,2}[\overline{Q}_T, R^n] \). From this, one can conclude that \( A(\eta_1, \eta_2) \in \langle v, w \rangle \). It is immediate that the proof of (i) follows from the choices of \( \eta_1 = v, \eta_2 = w \), and \( \eta_1 = w, \eta_2 = v \), respectively. In fact, we have proved that \( A \) maps \( \langle v, w \rangle \) into itself.

To prove (ii), let \( \eta_1, \eta_2, \xi \in \langle v, w \rangle \) and \( \eta_1(t, x) \leq \eta_2(t, x) \) on \( \overline{Q}_T \). Let \( A(\eta_1, \xi) = z \) and \( A(\eta_2, \xi) = u \). We will show that \( A(\eta_1, \xi)(t, x) \leq A(\eta_2, \xi)(t, x) \) on \( \overline{Q}_T \). Assume that this is false. Then there exists at least one component \( z_k(t, x) - u_k(t, x) \) of \( z(t, x) - u(t, x) \) such that \( z_k(t, x) - u_k(t, x) \) attains its positive maximum at \( (t_0, x_0) \in \overline{Q}_T \) with \( t_0 > 0 \). If \( (t_0, x_0) \in Q_T \), it follows from the argument that is used in the proof of uniqueness of solution in Theorem 3.1, that

\[
\mathcal{K}_k(z_k(t_0, x_0) - u_k(t_0, x_0)) > 0.
\]

On the other hand, using the definition \( G_k \) and \( qmmp \) of \( f \), we have

\[
\mathcal{K}_k(z_k - u_k)(t_0, x_0) = G_k(t_0, x_0, z_k(t_0, x_0), D_xz_k(t_0, x_0))
\]

\[
- G_k(t_0, x_0, u_k(t_0, x_0), D_xu_k(t_0, x_0))
\]

\[
\leq f_k(t_0, x_0, [\eta_1(t_0, x_0), \xi(t_0, x_0)]_k, h(D_xz_k(t_0, x_0))
\]

\[
- f_k(t_0, x_0, [\eta_2(t_0, x_0), \xi(t_0, x_0)]_k, h(D_xu_k(t_0, x_0))
\]

\[
\leq 0.
\]
This contradiction proves that there is no component \( z_k(t, x) - u_k(t, x) \) of \( z(t, x) - u(t, x) \) such that \( z_k(t, x) - u_k(t, x) \) attains its positive maximum on \( Q_T \). Now, we assume that \((t_0, x_0) \in \Gamma_T\). In this case, as in the proof of Theorem 3.1, we arrive at \( B_k z_k(t_0, x_0) > B_k u_k(t_0, x_0) \) which contradicts \( B_k z_k(t_0, x_0) = \phi_k(t_0, x_0) = B_k u_k(t_0, x_0) \) for \((t_0, x_0) \in \Gamma_T\). It follows that \( z(t, x) - u(t, x) \leq 0 \) on \( Q_T \), which establishes that \( A(\eta_1, \xi)(t, x) \leq A(\eta_2, \xi)(t, x) \) on \( Q_T \). Similarly, one can prove that \( A(\eta, \xi_1)(t, x) \geq A(\eta, \xi_2)(t, x) \) on \( Q_T \) for any \( \eta, \xi_1, \xi_2 \in (v, w) \) and \( \xi_1(t, x) \leq \xi_2(t, x) \) in \( Q_T \). This proves that for \( \eta, \xi \in (v, w) \), \( A(\eta, \xi) \) is monotone nondecreasing in \( \eta \) for each fixed \( \xi \) and nonincreasing in \( \xi \) for each fixed \( \eta \). Consequently, the mixed-monotone property of \( A \) is true on the segment \( (v, w) \). This completes the proof of the lemma.

**Remark 3.1.** From the mixed-monotone property of \( A \), we can conclude that for \( \eta_1, \eta_2 \in (v, w) \)

\[
A(\eta_1, \eta_2)(t, x) \leq A(\eta_2, \eta_1)(t, x)
\]

on \( Q_T \) (3.7) whenever \( \eta_1(t, x) \leq \eta_2(t, x) \) on \( Q_T \).

**Remark 3.2.** From the proof of Lemma 3.3, one can easily show that

\[
v \leq A(v, \xi), \quad w \geq A(w, \xi)
\]

(3.8) for any \( \xi \in (v, w) \). This remark is useful in constructing extremal quasi-solutions of (2.1).

### 4. Monotone Iterative Technique

In this section, we present our main theorem concerning the existence of the coupled quasi-maximal and minimal solutions of (2.1).

Because of Lemma 3.3 and Remark 3.1, we can define the sequences

\[
v_i = A(v_{i-1}, w_{i-1}), \quad w_i = A(w_{i-1}, v_{i-1})
\]

(4.1) with \( v_0 = v, w_0 = w \) and conclude that

\[
v \leq v_{i-1} \leq v_i \leq w, \quad v \leq w_i \leq w_{i-1} \leq w
\]

(4.2) and \( v_i, w_i \in C^{1+\alpha/2, 2+\alpha}[\bar{Q}_T, R^n] \) for \( i = 1, 2, \ldots \).

**Theorem 4.1.** Suppose that assumptions \((A_1), (A_2)\) and \((A_3)\) hold. Further assume that \( f \) in (2.1) possesses mmp. Then there exist sequences \( \{v_i\} \) and \( \{w_i\} \) which converge in \( C^{1,2}[\bar{Q}_T, R^n] \) to \( \rho \) and \( r \), respectively. Moreover, \( \rho \) and \( r \) are coupled quasi-minimal and maximal solutions of (2.1).
Proof. Because of Theorem 3.1 and the definition of $A$ in (3.6), the existence of sequences $\{v_i\}$ and $\{w_i\}$ defined by (4.1) in the space $C^{1+\alpha/2,2}[^{\Omega_T}, R^n]$ is guaranteed. Moreover, by Lemma 3.3 and Remark 3.1, these sequences satisfy the relation (4.2).

Now, we will center our discussion about the sequence $\{v_i\}$. We note that $C^{1+\alpha/2,2}[^{\Omega_T}, R^n] \subset W^{1,q}[^{\Omega_T}, R^n]$ for $q \geq (m+2)/(1-\alpha)$. By $L^q$-estimates [14, 19], $v_i$ satisfies

$$
\|v_i\|_{\mathcal{W}^{1,q}_q[^{\Omega_T}, R^n]} \leq C(\|F_i\|_{L^q[^{\Omega_T}, R^n]} + \|\phi\|_{\mathcal{W}^{1/2-1/\alpha}[^{\Omega_T}, R^n]} + \|\phi_0\|_{W^{2,2}[^{\Omega_T}, R^n]})
$$

(4.3)

where

$$
F_{ki}(t, x) = G_k(t, x, v_{ki-1}(t, x), D_x v_{ki-1}(t, x)) \quad \text{for } k \in I, i = 1, 2, \ldots,
$$

and $F_i(t, x) = (F_1(t, x), F_2(t, x), \ldots, F_{nl}(t, x))$. Furthermore, by Theorem 3.1, $v_i$ satisfies

$$
\|D_x v_i(t, x)\| \leq N_0 \quad \text{for } i = 1, 2, \ldots
$$

(4.4)

where $N_0$ is independent of $i$. The estimate (4.4), the definition $G_k$ and the fact that $v_i \in \langle v, w \rangle$ for all $i$, and the continuity of $F_i$ imply that the sequence $\{F_i\}$ is uniformly bounded in $C^{\Omega_T}, R^n]$. Since $C^{\Omega_T}, R^n]$ is dense in $L^q[^{\Omega_T}, R^n]$, it follows that $\{F_i\}$ is the bounded sequence in $L^q[^{\Omega_T}, R^n]$. From this and (4.3), we conclude that the sequence $\{v_i\}$ is uniformly bounded in $W^{1,q}_q[^{\Omega_T}, R^n]$. Hence by the application of imbedding Theorem [14], we obtain

$$
\|v_i\|_{C^{1+\alpha/2,1}[^{\Omega_T}, R^n]} \leq C(\|v_i\|_{\mathcal{W}^{1,q}_q[^{\Omega_T}, R^n]})
$$

(4.5)

for all $i = 1, 2, \ldots$, where $C$ is a constant which is independent of any element of $W^{1,q}_q[^{\Omega_T}, R^n]$. From (4.5), we can conclude that every uniformly bounded sequence in $W^{1,q}_q[^{\Omega_T}, R^n]$ is also uniformly bounded in $C^{1+\alpha/2,1+\alpha}[^{\Omega_T}, R^n]$. Therefore, by the application of Lemma A.4.5 [13], $\{F_i\}$ is uniformly bounded in $C^{1+\alpha}[^{\Omega_T}, R^n]$, and hence, by Schauder type estimates [13], we have

$$
\|v_i\|_{C^{1+\alpha/2,2+\alpha}[^{\Omega_T}, R^n]} \leq C(\|F_i\|_{C^{1+\alpha/2,2+\alpha}[^{\Omega_T}, R^n]} + \|\phi\|_{C^{1+\alpha/2,2+\alpha}[^{\Omega_T}, R^n]} + \|\phi_0\|_{C^{2+\alpha}[^{\Omega_T}, R^n]})
$$

(4.6)

for all $i = 1, 2, \ldots$, which implies the uniform boundedness of $\{v_i\}$ in $C^{1+\alpha/2,2+\alpha}[^{\Omega_T}, R^n]$. Therefore, by the natural compact imbedding of $C^{1+\alpha/2,2+\alpha}[^{\Omega_T}, R^n]$ into $C^{1,2}[^{\Omega_T}, R^n]$, the sequence $\{v_i\}$ is relatively compact in $C^{1,2}[^{\Omega_T}, R^n]$. This implies that there exists a subsequence of $\{v_i\}$
which converges in $C^{1,2}[\bar{Q}_T, R^n]$. Let $v^* \in C^{1,2}[\bar{Q}_T, R^n]$ be a limit of this subsequence. On the other hand, by monotonicity of $\{v_i\}$, the sequence $\{v_i(t, x)\}$ converges pointwise to $\rho(t, x)$ on $\bar{Q}_T$. But the convergence of subsequence of $\{v_i\}$ in $C^{1,2}[\bar{Q}_T, R^n]$ implies the pointwise convergence. Therefore, $\rho(t, x) = v^*(t, x)$ on $\bar{Q}_T$. This shows that the whole sequence $\{v_i\}$ converges in $C^{1,2}[\bar{Q}_T, R^n]$ to $\rho$, that is, $\lim_{i \to \infty} v_i = \rho$ in $C^{1,2}[\bar{Q}_T, R^n]$ and $v(t, x) \leq \rho(t, x) \leq w(t, x)$ on $\bar{Q}_T$. Similarly by imitating the preceding argument relative to the sequence $\{w_i\}$, one can conclude that the sequence $\{w_i\}$ converges in $C^{1,2}[\bar{Q}_T, R^n]$ and its limit is denoted by $r$, which belongs to $C^{1,2}[\bar{Q}_T, R^n]$ and satisfies the relation $v(t, x) \leq r(t, x) \leq w(t, x)$ on $\bar{Q}_T$. Thus the limits

$$
\lim_{i \to \infty} \mathcal{L}_k v_{ki}(t, x) = \mathcal{L}_k \rho_k(t, x), \quad \lim_{i \to \infty} \mathcal{L}_k w_{ki}(t, x) = \mathcal{L}_k r_k(t, x),
$$

$$
\lim_{i \to \infty} B_k v_{ki}(t, x) = B_k \rho_k(t, x), \quad \lim_{i \to \infty} B_k w_{ki}(t, x) = B_k r_k(t, x),
$$

$$
\lim_{i \to \infty} \left[ f_k(t, x, [v_{i-1}(t, x), w_{i-1}(t, x)], h(D_x v_{ki}(t, x))) \right] = f_k(t, x, [\rho(t, x), r(t, x)], h(D_x \rho_k(t, x)))
$$

and

$$
\lim_{i \to \infty} \left[ f_k(t, x, [w_{i-1}(t, x), v_{i-1}(t, x)], h(D_x w_{ki}(t, x))) \right] = f_k(t, x, [r(t, x), \rho(t, x)], h(D_x r_k(t, x)))
$$

east exist uniformly on $\bar{Q}_T$ for all $k \in I$. Thus from (3.1), (3.2), and the facts that $\mathcal{L}_k = \partial/\partial t - \bar{L}_k$, $\bar{L}_k = L_k - M_k$, we conclude that $\rho(t, x)$ and $r(t, x)$ are coupled quasi-solutions of the IBVP

$$
\mathcal{L}_k u_k = f_k(t, x, u, h(D_x u_k)) \quad \text{in } Q_T,
$$

$$
B_k u_k = \phi_k \quad \text{on } \Gamma_T,
$$

$$
u_k(0, x) = \phi_{0k}(x) \quad \text{in } \bar{Q}, \text{ for all } k \in I,
$$

where $f_k$, $\mathcal{L}_k$, $B_k$, $\phi_k$, $\phi_{0k}$, and $h$ are defined in (A1), (A2) and (A3), that is, $\rho(t, x)$ and $r(t, x)$ satisfy

$$
\mathcal{L}_k \rho_k(t, x) = f_k(t, x, [\rho(t, x), r(t, x)], h(D_x \rho_k(t, x))) \quad \text{in } Q_T,
$$

$$
\mathcal{L}_k r_k(t, x) = f_k(t, x, [r(t, x), \rho(t, x)], h(D_x r_k(t, x))) \quad \text{in } Q_T,
$$

$$
B_k \rho_k(t, x) = \phi_k(t, x) = B_k r_k(t, x) \quad \text{on } \Gamma_T,
$$

$$\rho_k(0, x) = \phi_{0k}(x) = r_k(0, x) \quad \text{in } \bar{Q}.$$
Now, we need to show that \( p(t, x) \) and \( r(t, x) \) are coupled quasi-minimal and maximal solutions of (4.7). Let \( u \) and \( z \) be any coupled quasi-solutions of (4.7) such that \( v \leq u \leq w \) and \( v \leq z \leq w \) on \( \bar{Q}_T \). Since \( v_0 \leq u, z \leq w_0 \), \( A(u, z) = u, A(z, u) = z \), it follows by Lemma 3.3(ii) and Remark 3.1 that

\[
v_1 = A(v_0, w_0) \leq A(u, w_0) \leq A(u, z) = u \quad \text{on } \bar{Q}_T,
\]

and

\[
z = A(z, u) \leq A(w_0, u) \leq A(w_0, v_0) = w_1 \quad \text{on } \bar{Q}_T.
\]

Let us assume that for some integer \( j > 1 \), \( v_j \leq u, z \leq w_j \) are valid on \( \bar{Q}_T \). Then we shall show that \( v_{j+1} \leq u, z \leq w_{j+1} \) on \( \bar{Q}_T \). Again from Lemma 3.3(ii), Remark 3.1, and the definition of \( v_j \) and \( w_j \) in (4.1), we arrive at

\[
v_{j+1} = A(v_j, w_j) \leq A(u, w_j) \leq A(u, z) = u \quad \text{on } \bar{Q}_T,
\]

and

\[
z = A(z, u) \leq A(w_j, u) \leq A(w_j, v_j) = w_{j+1} \quad \text{on } \bar{Q}_T.
\]

Thus, it follows by mathematical induction that

\[
v_i(t, x) \leq u(t, x), z(t, x) \leq w_i(t, x) \quad \text{on } \bar{Q}_T \quad (4.8)
\]

for all \( i = 1, 2, \ldots \). Hence we have

\[
\rho(t, x) \leq u(t, x), z(t, x) \leq r(t, x) \quad \text{on } \bar{Q}_T,
\]

proving that \( \rho \) and \( r \) are coupled quasi-minimal and maximal solutions of (4.7).

Finally, we will show that \( \rho(t, x) \) and \( r(t, x) \) that are defined above, are coupled quasi-minimal and maximal solutions of (2.1). To prove this, it is enough to verify that for every \( k \in I \),

\[
h(D, \rho_k(t, x)) = D \cdot \rho_k(t, x) \quad \text{and} \quad h(D, r_k(t, x)) = D \cdot r_k(t, x) \quad \text{on } \bar{Q}_T. \quad (4.9)
\]

We observe that \( \rho \) is a quasi-solution of (4.7) relative to \( r \), and vice-versa. From (A_1) and (A_3), we have

\[
|f_k(t, x, u, h(y))| \leq \psi_k(\|u\|)(1 + \|h(y)\|^2)
\leq \psi_k(\|u\|)(1 + \lambda^2 \|y\|^2)
\leq \psi_k(\mu) \lambda^2 (1 + \|y\|^2) \quad (4.10)
\]

whenever \( v(t, x) \leq u \leq w(t, x) \) on \( \bar{Q}_T \). Further, we recall that \( v(t, x) \leq \rho(t, x) \leq w(t, x) \) and \( v(t, x) \leq r(t, x) \leq w(t, x) \) on \( \bar{Q}_T \). Now, by the
application of the modified version (Theorem A.4.6 [13]) of Theorem 2.2 [3], we conclude that there exists a positive number $\bar{N}$ as defined in (A3) such that $\|D_{x}\rho(t, x)\| \leq \bar{N}$ and $\|D_{x}r(t, x)\| \leq \bar{N}$ on $\bar{Q}_{T}$. This together with the definition of $h$ establishes the relation (4.9), completing the proof of the theorem.

Remark 4.1. By following the discussion in [12], the scope of the notion of coupled quasi-solutions of (2.1) can be illustrated, analogously.

5. ONE-STEP CYCLIC MONOTONE ITERATIONS

In this section, we present an iterative scheme which accelerates the rate of convergence of the sequence of iterates defined in the previous section. A method that is useful for the solution of IBVP (2.1) is the one-step cyclic iterative procedure [12]. We note that the procedure (4.1) is equivalent to

$$v_{ki} = A_{k}(v_{i-1}, w_{i-1}), \quad w_{ki} = A_{k}(w_{i-1}, v_{i-1}), \quad i = 1, 2, ... \quad (5.1)$$

with $v_{0} = (v_{i0}, v_{02}, ..., v_{n0}) = v$ and $w_{0} = (w_{10}, w_{20}, ..., w_{n0}) = w$, where $v_{ki}$ and $w_{ki}$ are the unique solutions of (3.3) relative to $[v_{i-1}, w_{i-1}]_{k}$ and $[w_{i-1}, v_{i-1}]_{k}$ for $k \in I$, respectively. In this case, the above-mentioned procedure is modified as follows:

$$v_{ki}^{*} = A_{k}(v_{*i-1}, w_{*i-1}), \quad w_{ki}^{*} = A_{k}(w_{*i-1}, v_{*i-1}), \quad i = 1, 2, ... \quad (5.2)$$

with $v_{0}^{*} = v_{0} = v$ and $w_{0}^{*} = w_{0} = w$, where $v_{ki}^{*}$ and $w_{ki}^{*}$ are the unique solutions of (3.3) relative to $[v_{*i-1}, w_{*i-1}]_{k}$ and $[w_{*i-1}, v_{*i-1}]_{k}$ for $k \in I$, respectively. For fixed $i = 1, 2, ...$, vectors $v_{i-1}^{*k-1}$, $w_{i-1}^{*k-1}$ are defined by

$$v_{i-1}^{*k-1} = [v_{*i-1}, v_{*i-1}, v_{*i-1}, v_{*i-1}, v_{*i-1}],$$
$$w_{i-1}^{*k-1} = [w_{i-1}, w_{i-1}, w_{i-1}, w_{i-1}, w_{i-1}], \quad for \ k \in I, \quad (5.3)$$

with

$$v_{i-1}^{*0} = v_{i-1} = [v_{*i-1}, v_{*i-1}, ..., v_{*i-1}],$$

and

$$w_{i-1}^{*0} = w_{i-1} = [w_{*i-1}, w_{*i-1}, ..., w_{*i-1}].$$

Theorem 5.1. Assume that the hypotheses of Theorem 4.1 hold. Then the sequences $\{v_{i}^{*}\}$ and $\{w_{i}^{*}\}$ defined by (5.2) converge in $C^{1,2}[\bar{Q}_{T}, \mathbb{R}^{*}]$. Moreover, the limits $\rho(t, x)$ and $r(t, x)$ of $\{v_{i}^{*}\}$ and $\{w_{i}^{*}\}$, respectively, are coupled quasi-minimal and maximal solutions of (2.1), and $\rho, r \in \langle v, w \rangle$. 
Proof. We note that \( v_i^*, w_i^* \in C^{1+\alpha/2,2+\alpha}[\bar{Q}_T, R^n] \). By imitating the proof of Theorem 4.1, the proof of this theorem follows immediately, provided we show that sequences \( \{v_i^*\} \) and \( \{w_i^*\} \) satisfy the following relation

\[
v_0(t, x) \leq v_i^*(t, x) \leq v_{i+1}^*(t, x) \leq w_{i+1}^*(t, x) \leq w_i^*(t, x) \leq w_0(t, x) \quad \text{on} \quad \bar{Q}_T
\]  

(5.4)

for \( i = 1, 2, 3, \ldots \). For this purpose, we first observe from Lemma 3.3(ii) that for each \( k \in I \)

\[
A_k(\eta_1, \xi)(t, x) \leq A_k(\eta_2, \xi)(t, x) \quad \text{on} \quad \bar{Q}_T
\]

(5.5)

whenever

\[
\eta_1, \eta_2, \xi \in \langle v, w \rangle \quad \text{and} \quad \eta_1(t, x) \leq \eta_2(t, x) \quad \text{on} \quad \bar{Q}_T
\]

and

\[
A_k(\eta, \xi_1)(t, x) \geq A_k(\eta, \xi_2)(t, x) \quad \text{on} \quad \bar{Q}_T
\]

(5.6)

whenever \( \eta, \xi_1, \xi_2 \in \langle v, w \rangle \) and \( \xi_1(t, x) \leq \xi_2(t, x) \) on \( \bar{Q}_T \). In the light of (5.4), (5.5) and (5.6), we need to show that for each \( k \in I \)

\[
v_{k+1}^*(t, x) \leq v_{k+1}^*(t, x) \leq w_{k+1}^*(t, x) \leq w_k^*(t, x) \quad \text{on} \quad \bar{Q}_T,
\]

(5.7)

for \( i = 1, 2, \ldots \). For \( i = 1 \), we have

\[
v_{k1}^* = A_k(v_0^{*k-1}, w_0^{*k-1}), \quad w_{k1}^* = A_k(v_0^{*k-1}, w_0^{*k-1}).
\]

(5.8)

From this, we obtain

\[
v_{11}^* = A_1(v_0, w_0), \quad w_{11}^* = A_1(w_0, v_0).
\]

By Lemma 3.3(ii), (5.5) and (5.6), we have

\[
v_{10}(t, x) \leq v_1^*(t, x) \leq w_1^*(t, x) \leq w_{10}(t, x) \quad \text{on} \quad \bar{Q}_T
\]

(5.9)

and hence by (5.3), we get

\[
v_0(t, x) \leq v_0^*(t, x) \leq w_0^*(t, x) \leq w_0(t, x) \quad \text{on} \quad \bar{Q}_T.
\]

From (5.8), we have

\[
v_{21}^* = A_2(v_0^{*1}, w_0^{*1}), \quad w_{21}^* = A_2(w_0^{*1}, v_0^{*1}).
\]
Again, by Lemma 3.3(ii), (5.5) and (5.6), we have

\[ A_2(v_0, w_0) \leq A_2(v_0^*, w_0^*) \leq A_2(w_0, v_0^*) \]

\[ \leq A_2(w_0, v_0^*) \leq A_2(w_0, v_0) \]

and consequently

\[ v_{20}(t, x) \leq v_{21}^*(t, x) \leq w_{21}(t, x) \leq w_{20}(t, x) \quad \text{on } \bar{Q}_T. \] (5.10)

This, together with (5.3), yields

\[ v_0(t, x) \leq v_0^*2(t, x) \leq w_0^2(x) \leq w_0(t, x) \quad \text{on } \bar{Q}_T. \]

By continuing this process \( n \) number of times we arrive at

\[ v_0(t, x) \leq v_1^*(t, x) \leq w_1^*(t, x) \leq w_0(t, x) \quad \text{on } \bar{Q}_T. \]

If we assume that

\[ v_0(t, x) \leq v_j^*(t, x) \leq \cdots \leq v_{j-1}^*(t, x) \leq v_j^*(t, x) \leq w_j^*(t, x) \leq w_{j-1}(t, x) \]

\[ \leq \cdots \leq w_1^*(t, x) \leq w_0(t, x) \quad \text{on } \bar{Q}_T, \] (5.11)

for some \( j > 1 \), and then prove that

\[ v_0(t, x) \leq v_j^*(t, x) \leq \cdots \leq v_j^*(t, x) \leq v_{j+1}^*(t, x) \leq w_{j+1}^*(t, x) \leq w_j^*(t, x) \]

\[ \leq \cdots \leq w_1^*(t, x) \leq w_0(t, x) \quad \text{on } \bar{Q}_T, \]

by induction the proof of (5.4) results. For \( i = j + 1 \), we have

\[ v_{k+1}^* = A_k(v_j^{*k-1}, w_j^{*k-1}), \quad w_{k+1}^* = A_k(w_j^{*k-1}, w_j^{*k-1}). \]

From this, we see that

\[ v_{1j+1} = A_1(v_j^*, w_j^*), \quad w_{1j+1} = A_1(w_j^*, v_j^*). \]

From (5.5), (5.6), (5.11), and the facts that \( v_{ij} = A_1(v_{i-1}, w_{j-1}) \) and \( v_{ij} = A_1(w_{i-1}, v_{j-1}) \), we obtain

\[ A_1(v_j^*, v_{j-1}) \leq A_1(v_j^*, w_j^*) \leq A_1(v_j^*, w_j^*) \]

\[ \leq A_1(w_j^*, v_j^*) \leq A_1(w_j^*, v_j^*) \]

Hence

\[ v_j^*(t, x) \leq v_{ij+1}^*(t, x) \leq w_{ij+1}(t, x) \leq w_j^*(t, x) \quad \text{on } \bar{Q}_T, \]
which implies
\[ v_j^*(t, x) \leq v_j^{*+1}(t, x) \leq w_j^{*+1}(t, x) \leq w_j^*(t, x) \quad \text{on } \bar{Q}_T. \] 

For \( k = 2 \), we have
\[ v_{2j+1}^* = A_2(v_j^{*+1}, w_j^{*+1}), \quad w_{2j+1}^* = A_2(w_j^{*+1}, v_j^{*+1}). \] 

From (5.3), (5.12) and (5.11), we observe that
\[ v_{j-1}^*(t, x) \leq v_{j-1}^{*+1}(t, x) \leq v_j^{*+1}(t, x) \leq w_j^{*+1}(t, x) \leq w_{j-1}^{*+1}(t, x) \leq w_{j-1}^*(t, x) \quad \text{on } \bar{Q}_T. \]

This, together with (5.5), (5.6), (5.13),
\[ v_{2j}^* = A_2(v_{2j-1}^{*+1}, w_{2j-1}^{*+1}) \quad \text{and} \quad w_{2j}^* = A_2(w_{2j-1}^{*+1}, v_{2j-1}^{*+1}), \]

yields
\[ A_2(v_{2j-1}^{*+1}, w_{2j-1}^{*+1}) \leq A_2(v_j^{*+1}, w_j^{*+1}) \leq A_2(v_j^{*+1}, w_j^*) \leq A_2(w_j^{*+1}, v_j^*). \]

Hence
\[ v_{2j+1}^*(t, x) \leq v_{2j+1}^{*+1}(t, x) \leq w_{2j+1}^{*+1}(t, x) \leq w_{2j}^*(t, x) \quad \text{on } \bar{Q}_T. \]

This, together with (5.12) and (5.3), gives
\[ v_j^*(t, x) \leq v_j^{*+2}(t, x) \leq w_j^{*+2}(t, x) \leq w_j^*(t, x) \quad \text{on } \bar{Q}_T. \] 

From (5.11) and (5.14), we have
\[ v_{j-1}^*(t, x) \leq v_{j-1}^{*+2}(t, x) \leq v_j^{*+2}(t, x) \leq w_j^{*+2}(t, x) \leq w_{j-1}^{*+2}(t, x) \leq w_{j-1}^*(t, x) \quad \text{on } \bar{Q}_T. \]

By following the previous argument, we can conclude that
\[ v_{2j}^*(t, x) \leq v_{2j-1}^{*+2}(t, x) \leq w_{2j-1}^{*+2}(t, x) \leq w_{2j}^*(t, x) \quad \text{on } \bar{Q}_T. \]

By continuing this process for \( n \) number of times, we arrive at
\[ v_j^*(t, x) \leq v_j^{*+n}(t, x) \leq v_j^{*+1}(t, x) \leq w_j^{*+1}(t, x) \leq w_j^*(t, x) \quad \text{on } \bar{Q}_T. \]
This, together with (5.11), establishes the result

\[ v_0(t, x) \leq v_1^*(t, x) \leq \cdots \leq v_{j-1}^*(t, x) \leq v_j^*(t, x) \leq v_{j+1}^*(t, x) \leq w_j^*(t, x) \leq \cdots \leq w_1(t, x) \leq w_0(t, x), \]

completing the proof of the theorem.

**Remark 5.1.** If \( f \) possesses the quasi-monotone nondecreasing property, then the splitting of \( u \) will be given by \( u = (u_k, [u]_{bb}) \), where \( b_k = n - 1 \) for all \( k \in I \). In this case, the one-step cyclic procedure (5.2) reduces to

\[ v_{ki}^* = A_k(v_{j-1}^*), \quad w_{ki}^* = A_k(w_{j-1}^*), \quad (5.15) \]

for \( k \in I \) and \( i = 1, 2, \ldots \), where \( v_0^*, w_0^*, v_{j-1}^* \) and \( w_{j-1}^* \) are as defined in (5.2) and (5.3).

**REFERENCES**


