Journal of Approximation Theory **114**, 147–151 (2002)
doi:10.1006/jath.2001.3630, available online at http://www.idealibrary.com on IDEAL®

Prediction Operators in Banach Ideal Spaces[1]

Tingfu Wang

Department of Mathematics, Harbin University of Science and Technology, Harbin 150080, People's Republic of China

and

Zheng Liu

Department of Mathematics and Physics, Anshan Institute of Iron and Steel Technology, Anshan 114002, People's Republic of China

Communicated by Will Light

Received April 14, 1999; accepted in revised form September 12, 2001;

A necessary and sufficient condition for an operator in a Banach ideal space to be a prediction operator is given.

Key Words: prediction operator; Banach ideal space.

Let X be a Banach space, C a set in X, and $x \in X$. An element $y \in C$ is called an element of best approximation of x in C if we have

$$\|x-y\| = \inf\{\|x-z\| : z \in C\}.$$

If the element of best approximation of x in C is unique, we denote it by $\pi(x \mid C)$. The operator $\pi(\cdot \mid C)$ is called the best approximation operator of X onto C.

A space X is said to be strictly convex if for each pair $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$ we have $\|x+y\| < 1$. The space X is said to be smooth if at each point $0 \neq x \in X$, there is only one support functional f_x, i.e., $f_x \in X^*$, $\|f_x\|_{X^*} = 1$, and $f_x(x) = \|x\|$.

Let (G, Σ, μ) be a σ-finite measure space and denote by $S = S(G, \Sigma, \mu)$ the set of all equivalence classes of Σ-measurable real-valued functions on G with algebraic operations and order defined in a natural way. A linear subset X of S endowed with some norm $\|\cdot\|$ is called a Banach ideal space if $(X, \|\cdot\|)$ is complete and it satisfies the condition that $x \in X$, $y \in S$, and

[1] The subject is supported by NSFC and NSFH.

0021-9045/01 $35.00

$|y| \leqslant |x|$ imply $y \in X$ and $\|y\| \leqslant \|x\|$. The norm on a Banach ideal space is called order-continuous if in addition it satisfies the condition that $x_n \downarrow 0$ implies $\|x_n\| \to 0$. It is well known that $L^p (1 \leqslant p \leqslant \infty)$, Orlicz spaces, Lorentz spaces, Orlicz–Lorentz spaces, and Musielak–Orlicz spaces are all Banach ideal spaces. Criterions of order-continuity, strict convexity, smoothness, and reflexivity of these spaces can be found in the literature.

For a Banach ideal space $X(G, \Sigma, \mu)$ (which is simply denoted by $X(\Sigma)$), let Σ' be a σ-sublattice of the σ algebra Σ and $X(\Sigma') = \{x \in X(\Sigma) : x \text{ is } \Sigma' \text{ measurable}\}$. If $\pi(\cdot \mid X(\Sigma'))$ exists, it is called a prediction operator. Prediction operators have wide applications in probability, Bayes estimation theory, prediction theory, and many other fields. Various authors have studied these in the past 30 years [2–12]. For example, Dykstra has studied the case L^2 in [4]. Landers and Rogge have studied the case L^p in [6] and then obtained a result of Orlicz space L_M for the modular in [5]. Duan and Chen gave a necessary condition (which is not sufficient) and a sufficient condition (which is not necessary) for prediction operators in L_M in [11]. In 1995, Wang *et al.* [12] obtained a necessary and sufficient condition for an operator in L_M to be a prediction operator. The purpose of this paper is to generalize the main result in [12] to Banach ideal spaces.

THEOREM. *Let $X(\Sigma)$ be a reflexive, strictly convex, and smooth Banach ideal space. Then the operator $T: X(\Sigma) \to X(\Sigma)$ is a prediction operator (i.e., there exists σ-sublattice $\Sigma' \subset \Sigma$ such that $T(\cdot) = \pi(\cdot \mid X(\Sigma'))$ if and only if it satisfies the following conditions:*

(i) $T(Tx) = Tx (x \in X(\Sigma))$;

(ii) $r \in TX$ *if* $r \in R$;

(iii) $x, y \in TX$ *and* $\alpha, \beta \in R^+$ *imply* $\alpha x + \beta y \in TX$;

(iv) $\{x_n\} \subset TX$ *and* $\|x_n - x\| \to 0 (n \to \infty)$ *imply* $x \in TX$;

(v) $x \in TX$ *and* $r \in R$ *imply* $x \vee r, x \wedge r \in TX$;

(vi) $\chi_A \in TX$ *and* $\alpha, \beta \in R^+$ *and* $r \in R$ *imply* $\|x - Tx\| \leqslant \|x - \alpha Tx - \beta \chi_A + r\|$,

where R is the set of all real numbers, R^+ is the set of non-negative numbers, and χ_A is the characteristic function of the set A.

Proof. Necessity. Suppose Σ' is a σ-sublattice of Σ and $T(\cdot) = \pi(\cdot \mid X(\Sigma'))$. Obviously, $TX = X(\Sigma')$, so (i) is clear.

Note that $r \in R$ implies $r \in X(\Sigma')$ since $\phi, G \in \Sigma'$, so (ii) is true.

It is easy to find that $X(\Sigma')$ is a closed convex cone in $X(\Sigma)$ since Σ' is a σ-sublattice. So (iii) and (iv) are true.

For brevity, we use the notation $\{x>a\}$ for the inverse image of the set (a, ∞) under the mapping $x: G \to R$. Since

$$\{x \vee r > a\} = \begin{cases} G & (a < r) \\ \{x > a\} & (a \geqslant r) \end{cases}$$

and

$$\{x \wedge r > a\} = \begin{cases} \{x > a\} & (a < r) \\ \phi & (a \geqslant r), \end{cases}$$

we see that $x \vee r$ and $x \wedge r$ are Σ'-measurable, and hence $x \vee r$, $x \wedge r \in X(\Sigma') = TX$. This establishes (v).

Since $Tx \in TX, \chi_A \in TX, r \in TX$, and TX is a convex cone, we have $\alpha Tx + \beta \chi_A - r \in TX$. Therefore $\|x - Tx\| \leqslant \|x - (\alpha Tx + \beta \chi_A - r)\|$, and so (vi) holds.

Sufficiency. Let $\Sigma' = \{A \in \Sigma : \chi_A \in TX\}$. Divide the proof into three steps as follows:

1. We prove that Σ' is a σ-sublattice of Σ.

By (ii) and (i), $\chi_\Phi = 0 = T0 = T\chi_\Phi$, $\chi_G = 1 = T1 = T\chi_G$. So$\chi_\Phi, \chi_G \in TX$, that is, $\Phi, G \in \Sigma'$. If $A, B \in \Sigma'$, then $\chi_A, \chi_B \in TX$. Observe that $\chi_{A \cup B}(t) = (\chi_A(t) + \chi_B(t)) \wedge 1$ and $\chi_{A \cap B}(t) = (\chi_A(t) + \chi_B(t) - 1) \vee 0$. Thus, by (iii) and (v), $\chi_{A \cup B}, \chi_{A \cap B} \in TX$. So $A \cup B, A \cap B \in \Sigma'$.

Let $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n \in \Sigma'(n = 1, 2, \ldots)$ and $A_1 \subset A_2 \subset A_3 \subset \cdots$. By Theorem 10 (Ogasawara) in [1, Chap. 10, Sect. 4] we find that $X(\Sigma)$ is order-continuous since $X(\Sigma)$ is reflexive. Therefore $\|\chi_A - \chi_{A_n}\| \to 0$ as $n \to \infty$ because $\mu(A/A_n) \to 0$ as $n \to \infty$. Using (iv), $\chi_A \in TX$ and it follows $A \in \Sigma'$. If $A = \bigcap_{n=1}^{\infty} A_n$ where $A_n \in \Sigma'(n = 1, 2, \ldots)$ and $A_1 \supset A_2 \supset A_3 \supset \cdots$. By the above same arguments we can deduce that $A \in \Sigma'$. Thus we have proved that Σ' is a σ-sublattice of Σ.

2. We prove that $X(\Sigma') = TX$.

Suppose $x \in TX$. For any $a \in R$ we write $y_n(t) = ((3/2)^n (x(t) - a) \vee 0) \wedge 1$. By (iii) and (v) we see that $y_n \in TX (n = 1, 2, \ldots)$. Observe that $x(t) \leqslant a$ implies $y_n(t) = 0 = \chi_{\{x-a\}}(t)$ and $x(t) > a$ implies $y_n(t) \to 1 = \chi_{\{x>a\}}(t)$ $(n \to \infty)$. We can deduce $\chi_{\{x>a\}} - y_n \downarrow 0$ $(n \to \infty)$ and so $\|\chi_{\{x>a\}} - y_n\| \to 0$ $(n \to \infty)$ by the order-continuity of $X(\Sigma)$. It follows that $\chi_{\{x>a\}} \in TX$ from (iv). That is, $\{x > a\} \in \Sigma'$ and then $x \in X(\Sigma')$. Hence $TX \subset X(\Sigma')$.

Conversely, suppose $x \in X(\Sigma')$. Then $\{x > a\} \in \Sigma'$ and so $\chi_{\{x>a\}} \in TX$ for any $a \in R$.

Notice that $x \vee (-m) \downarrow x$ as $m \to \infty$ and thus $\|x \vee (-m) - x\| \to 0$ as $m \to \infty$. We need only prove $x \vee (-m) \in TX (m = 1, 2, \ldots)$ to show that $x \in TX$ by (iv). However, $x \vee (-m) = (x \vee (-m) + m) - m$ and $-m \in TX$, so from (iii) we see that we only need to prove $x \vee (-m) + m \in TX$. Moreover, $x \vee (-m) + m \geqslant 0$ and $x \vee (-m) + m \in X(\Sigma')$, since $X(\Sigma')$ is a convex cone. In what follows, we need only prove that $0 \leqslant x \in X(\Sigma')$ implies $x \in TX$.

By Lemma 3 in [1, Chap. 4, Sect. 3],

$$\left\| x - \sum_{k=0}^{n2^n} (k/2^n)\, \chi_{\{(k+1)/2^n \geqslant x \geqslant k/2^n\}} \right\| \to 0 \qquad \text{as} \quad n \to \infty.$$

However,

$$\sum_{k=0}^{n2^n} (k/2^n)\, \chi_{\{(k+1)/2^n \geqslant x \geqslant k/2^n\}} = (1/2^n) \sum_{k=0}^{n2^n} \chi_{\{x > k/2^n\}} \in TX$$

since $\chi_{\{x > k/2^n\}} \in TX (n = 1, 2, ...; k = 1, 2, ..., n2^n)$ and by using (iii). Consequently $x \in TX$ by (iv). Hence $X(\Sigma') \subset TX$.

3. We prove that $T(\cdot) = \pi(\cdot \mid X(\Sigma'))$.

It is well known that $\pi(\cdot \mid X(\Sigma'))$ has meaning since $X(\Sigma)$ is reflexive and strictly convex.

If $x \in TX$, then $Tx = x = \pi(x \mid X(\Sigma'))$.

Now suppose $x \notin TX$. Since $X(\Sigma)$ is smooth, $x - Tx$ has a unique support functional $f_{x-Tx} \in X(\Sigma)^*$ such that $\|f_{x-Tx}\|_* = 1$ and $f_{x-Tx}(x - Tx) = \|x - Tx\|$. Also $f_{x-Tx}(Tx) = \lim_{\lambda \to 0} (\|x - Tx + \lambda Tx\| - \|x - Tx\|)/\lambda$ exists in the Gateaux sense. Nevertheless, $\|x - Tx + \lambda Tx\| = \|x - (1 - \lambda)\, Tx\| \geqslant \|x - Tx\|$ for $|\lambda|$ sufficiently small by (vi). This implies that $\lim_{\lambda \to 0^+} (\|x - Tx + \lambda Tx\| - \|x - Tx\|)/\lambda \geqslant 0$ and $\lim_{\lambda \to 0^-} (\|x - Tx + \lambda Tx\| - \|x - Tx\|)/\lambda \leqslant 0$. Hence

$$f_{x-Tx}(Tx) = 0. \tag{1}$$

By the same arguments we can deduce that

$$f_{x-Tx}(r) = 0 \qquad (r \in R). \tag{2}$$

For any $\chi_A \in TX$, again by (vi) we have

$$f_{x-Tx}(-\chi_A) = \lim_{\lambda \to 0^+} (\|x - Tx + \lambda Tx\| - \|x - Tx\|)/\lambda \geqslant 0$$

and so $f_{x-Tx}(\chi_A) \leqslant 0$.

For any $0 \leqslant u \in TX$, we have $\|u - (1/2^n) \sum_{k=0}^{n2^n} \chi_{\{u > k/2^n\}}\| \to 0$ as $n \to \infty$. Since $\{u > k/2^n\} \in \Sigma'$, $\chi_{\{u > k/2^n\}} \in TX$, we get $f_{x-Tx}(\chi_{\{u > k/2^n\}}) \leqslant 0$ $(n = 1, 2, ...; k = 1, 2, ..., n2^n)$. Furthermore, $f_{x-Tx}((1/2^n) \sum_{k=0}^{n2^n} \chi_{\{u > k/2^n\}}) \leqslant 0$, and so $f_{x-Tx}(u) \leqslant 0$.

In the general case, since $0 \leqslant u \vee (-m) + m \in TX (m = 1, 2, \ldots)$, we have $f_{x-Tx}(u \vee (-m) + m) \leqslant 0 (m = 1, 2, \ldots)$. Combining with Eq. (2), $f_{x-Tx}(u \vee (-m)) \leqslant 0 (m = 1, 2, \ldots)$. But $\|u - (u \vee (-m))\| \to 0 (m \to \infty)$. So

$$f_{x-Tx}(u) \leqslant 0 (u \in TX = X(\Sigma')). \tag{3}$$

By Eqs. (3) and (1), for any $u \in X(\Sigma')$ we have

$$\begin{aligned} \|x - Tx\| &= f_{x-Tx}(x - Tx) = f_{x-Tx}(x) \leqslant f_{x-Tx}(x - u) \\ &\leqslant \|f_{x-Tx}\|_* \, \|x - u\| = \|x - u\|. \end{aligned}$$

This means $Tx = \pi(x \mid X(\Sigma'))$.

ACKNOWLEDGMENTS

The authors are very grateful to the referees for their many valuable suggestions and to the editor Professor W. Light for his warm help.

REFERENCES

1. L. V. Kantorovich and G. P. Akilov, "Functional Analysis," Pergamon, New York, 1992.
2. R. B. Darst, D. A. Legg, and D. W. Townsend, Prediction in Orlicz spaces, *Manuscripta Math.* **35** (1989), 91–103.
3. D. Landers and L. Rogge, Best approximants in L_ϕ spaces, *Z. Wahrsch. Verw. Gebiete* **51** (1980), 215–237.
4. R. L. Dykstra, A characterization of a conditional expectation with respect to a σ-lattice, *Ann. Math. Statist.* **41** (1970), 689–701.
5. D. Landers and L. Rogge, A characterization of best ϕ approximants, *Trans. Amer. Math. Soc.* **287** (1981), 259–264.
6. D. Landers and L. Rogge, Characterization of p-prediction, *Proc. Amer. Math. Soc.* **76** (1979), 307–309.
7. R. E. Barlow, U. J. Bartholomew, and J. M. Bremner, "Statistical Inference under Order Restrictions," Wiley, New York, 1972.
8. Y. Wang and S. Chen, The best approximation operator in Orlicz spaces, *Pure Appl. Math.* **1** (1986), 44–51.
9. T. Ando and I. Amemiya, Almost everywhere convergence for prediction sequence in L_p, *Z. Wahrsch. Verw. Gebiete* **4** (1965), 113–120.
10. H. D. Brunk, Uniform inequality for conditional p-means given σ-lattices, *Ann. Probab.* **3** (1975), 1025–1030.
11. Y. Duan and S. Chen, On best approximation operator in Orlicz spaces, *J. Math. Anal. Appl.* **178** (1993), 1–8.
12. T. Wang, D. Ji, and Y. Li, Prediction operatior in Orlicz spaces, *Chinese Sci. Bull.* **40** (1995), 1592–1595.