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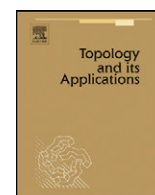
Three geometric applications of quandle homology

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In this paper we describe three geometric applications of quandle homology. We show that it gives obstructions to tangle embeddings, provides the lower bound for the 4-move distance between links, and can be used in determining periodicity of links.

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1. Definitions and preliminary facts

Definition 1. A *quandle*, X, is a set with a binary operation $(a, b) \mapsto a * b$ such that:

(1) For any $a \in X$, $a * a = a$.
(2) For any $a, b \in X$, there is a unique $c \in X$ such that $a = c * b$.
(3) For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$ (right distributivity).

Note that the second condition can be replaced with the following requirement: the operation $*b: Q \to Q$, defined by $*b(x) = x * b$, is a bijection. The inverse map to $*b$ is denoted by $\bar{*}b$.

Definition 2. A *rack* is a set with a binary operation that satisfies conditions (2) and (3) from the definition of quandle.

The following are some of the most commonly used examples of quandles.

– Any group G with conjugation as the quandle operation: $a * b = b^{-1}ab$.
– Let n be a positive integer. For elements $i, j \in \{0, 1, \ldots, n - 1\}$, define $i * j \equiv 2j - i$ (mod n). Then $*$ defines a quandle structure called the *dihedral quandle*, R_n. It can be identified with the set of reflections of a regular n-gon with conjugation as the quandle operation.
– Any $\mathbb{Z}[t, t^{-1}]$-module M is a quandle with $a * b = ta + (1 - t)b$, for $a, b \in M$, called the *Alexander quandle*. Moreover, if n is a positive integer, then $\mathbb{Z}_n[t, t^{-1}]/(h(t))$ is a quandle for a Laurent polynomial $h(t)$.
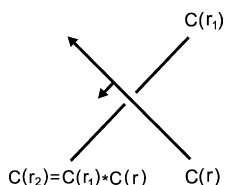
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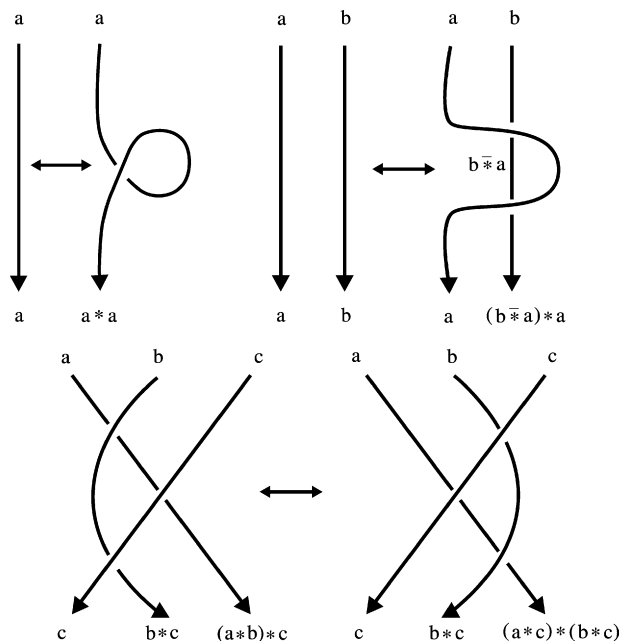**Fig. 1.** The rule of quandle coloring at a crossing.

Fig. 2. Reidemeister moves and quandle axioms.

The last example can be vastly generalized [16]; for any group G and its automorphism $\tau : G \to G$, G becomes a quandle when equipped with the operation $g * h = \tau(gh^{-1})h$. If we consider the anti-automorphism $\tau(g) = g^{-1}$, we obtain another well-known quandle, *Core(G)*, with $g * h = hg^{-1}h$.

Very likely the earliest work on racks is due to J.C. Conway and G.C. Wraith [9,12], who studied the conjugacy operation in a group. The notion of quandle was introduced independently by D. Joyce [16] and S. Matveev [20].

Joyce introduced the fundamental knot quandle, that is a classifying invariant of classical knots up to orientation-reversing homeomorphism of topological pairs [16]. However, just like in the case of fundamental groups, it is very hard to decide whether two given knot quandles are isomorphic. There are several other knot invariants derived from quandles that are easier to work with. For example, one can consider the family of all homomorphisms from the fundamental knot quandle to the given quandle, i.e., the set of all quandle colorings. The cardinality of this set is a knot invariant.

Definition 3. ([8]) Let X be a fixed quandle. Let K be a given diagram of an oriented classical link, and let R be the set of over-arcs of the diagram. The normals to arcs are given in such a way that the pair (tangent, normal) matches the usual orientation of the plane. A quandle coloring C is a map $C : R \to X$ such that at every crossing, the relation depicted in Fig. 1 holds. More specifically, let r be the over-arc at a crossing, and r_1, r_2 be under-arcs such that the normal of the over-arc points from r_1 to r_2. Then it is required that $C(r_2) = C(r_1) * C(r)$.

The axioms for a quandle correspond to the Reidemeister moves via quandle colorings of knot diagrams. This correspondence is illustrated in Fig. 2.

2. Rack and quandle homology

Rack homology and homotopy theory were first defined and studied in [13], and a modification to quandle homology theory was given in [4] to define knot invariants in a state-sum form (so-called cocycle knot invariants).

Here we recall the definition of rack, degenerate and quandle homology after [8].
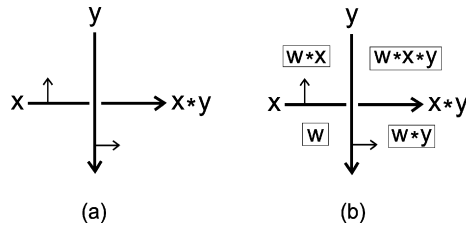
Fig. 3. Coloring and shadow coloring.

Definition 4.

(i) For a given rack X, let $C_n^R(X)$ be the free abelian group generated by n-tuples $(x_1, x_2, \ldots, x_n)$ of elements of X; in other words, $C_n^R(X) = \mathbb{Z}X^n = (\mathbb{Z}X)^{\otimes n}$.

Define a boundary homomorphism $\partial_n : C_n^R(X) \to C_{n-1}^R(X)$ by

$$\partial_n(x_1, x_2, \ldots, x_n) = \sum_{i=2}^{n} (-1)^i \big((x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) - (x_1 * x_i, x_2 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n) \big).$$

$(C_*^R(X), \partial_*)$ is called the rack chain complex of X.

(ii) Assume that X is a quandle. Then there is a subchain complex $C_n^D(X) \subset C_n^R(X)$, generated by n-tuples $(x_1, \ldots, x_n)$ with $x_{i+1} = x_i$ for some i. The subchain complex $(C_n^D(X), \partial_n)$ is called the degenerate chain complex of a quandle X.

(iii) The quotient chain complex $C_n^Q(X) = C_n^R(X)/C_n^D(X)$ is called the quandle chain complex.

(iv) The (co)homology of rack, degenerate, and quandle chain complexes is called rack, degenerate, and quandle (co)homology, respectively.

(v) For an abelian group G, define the chain complex $C_*^Q(X; G) = C_*^Q \otimes G$, with $\partial = \partial_* \otimes id$. The groups of cycles and boundaries are denoted respectively by $ker(\partial_n) = Z_n^Q(X; G) \subset C_n^Q(X; G)$ and $Im(\partial_{n+1}) = B_n^Q(X; G) \subset C_n^Q(X; G)$. The nth quandle homology group of a quandle X with coefficient group G is defined as

$$H_n^Q(X; G) = H_n\big(C_*^Q(X; G)\big) = Z_n^Q(X; G)/B_n^Q(X; G).$$

Rack homology and quandle homology were studied by many authors, for example in [2,4,5,8,11,13,19,21]. Free part of rack (and quandle) homology is known for a large class of racks and quandles [11,21]. However, there are many open problems concerning the torsion part.

In this paper we will show how to use the information about homology of quandles in solving some geometric problems concerning knots and links. The effectiveness of these methods grows together with better understanding of quandle homology.

3. Application to tangle embeddings

First, we will explain, following [15,6,7], the procedure of assigning a cycle in quandle homology to an oriented colored link diagram. 2-cycles correspond to diagrams with the usual quandle coloring, and 3-cycles are assigned to diagrams with shadow colorings.

Definition 5. Let Q be a fixed quandle, D be a link diagram, and $\widetilde{R}$ be the set of arcs and regions separated by the underlying immersed curve of D. A shadow coloring of D is a function $\widetilde{C} : \widetilde{R} \to Q$ satisfying the following two conditions:

(1) The rules of labeling of arcs are as in the ordinary quandle coloring.
(2) Coloring of regions satisfies the condition illustrated in Fig. 3(b), i.e., if R_1 and R_2 are two regions separated by an arc r colored by x, and the normal vector to r points from R_1 to R_2, then the color of R_2 must be equal to $w * x$, where w is the color of R_1.

Note that despite the fact that near the crossing there is more than one way to go from one region to another, the third quandle axiom (the right distributivity) guarantees unique colors near a crossing.

Let D be a link diagram colored with elements of a finite quandle X. Each positive crossing represents a pair $(x, y) \in C_2^Q(X)$, where x is the color of an under-arc away from which points the normal of the over-arc labeled y (see Fig. 3(a)). In the case of negative crossing, we write $-(x, y)$. The sum of such 2-chains taken over all crossings of the diagram forms a 2-cycle (see [6] for details). Thus, it represents an element in $H_2^Q(X)$.

In the case of shadow coloring, each positive crossing corresponds to the triple $(w, x, y) \in C_3^Q(X)$, where w is the color of so-called source region. It is the region near the crossing such that both normal vectors to the arcs colored by x and y
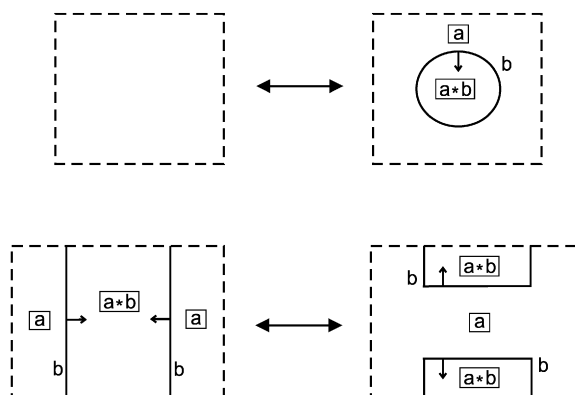
Fig. 4. Moves on shadow-colored diagrams that do not change homology class.

point away from this region (see Fig. 3(b), where colors assigned to the regions are depicted as letters enclosed within squares). A negative crossing represents the triple $-(w, x, y)$. The sum of such signed triples taken over all crossings of D gives an element of $H_3^Q(X)$ [6].

Carter, Kamada, and Saito gave a list of moves on colored or shadow-colored link diagram that do not change the homology class represented by this diagram [7,8]. Their list includes Reidemeister moves and two moves illustrated in Fig. 4. We are going to use these moves in our construction of obstructions to tangle embeddings. The first move is creating or deleting a trivial component with appropriate shadow coloring. The second move allows to change the connections between arcs of the diagram, if these arcs have the same color and opposite orientation.

A 2n-tangle consists of n disjoint arcs in the 3-ball. We ask the following question, that was first considered by D.A. Krebes [18]. For a given knot K, and a tangle T, can we embed T into K? In other words, is there a diagram of T that extends to a diagram of K? This problem is important due to its applications in the study of DNA. A number of knot invariants have been used to find criteria for tangle embeddings (see for example [28,29]).

Let us recall the definition of a special type of colorings of tangles that will be essential for defining homological obstructions to tangle embeddings.

Definition 6. Let D_T be a tangle diagram, and Q be a quandle. A *boundary-monochromatic coloring* of D_T is a map from the set of arcs of D_T to quandle Q satisfying the usual conditions for quandle colorings of knot diagrams, and an additional requirement that all boundary points receive the same color.

If a tangle T embeds into a knot K, then each boundary-monochromatic coloring of D_T can be extended trivially to the whole diagram of K, i.e., all arcs outside D_T receive the color of the boundary points of D_T. Thus, the existence of nontrivial boundary-monochromatic colorings of D_T gives the first basic obstruction to tangle embeddings, for T can possibly embed only into knots admitting at least the same number of nontrivial colorings (see also [18]).

Definition 7. A boundary-monochromatic shadow coloring of a tangle diagram D is obtained from the ordinary boundary-monochromatic coloring of D by choosing a color of any region of D and extending this coloring to other regions according to the rules of Definition 5. Notice that such extension is unique.

Lemma 8. *Every boundary-monochromatic coloring of an oriented diagram D of a tangle T with elements of a fixed quandle X represents an element in $H_2^Q(X)$. Every boundary-monochromatic shadow coloring of D represents an element in $H_3^Q(X)$.*

Proof. The fact that all boundary points of D have the same color allows us to take any closure of a diagram D and obtain a colored link diagram that represents an element in $Z_2^Q(X)$ (or in $Z_3^Q(X)$ in the case of shadow coloring). Any two such closures can be transformed one into another by a sequence of homology moves illustrated in Fig. 4. Therefore, D (as well as T) represents an element in quandle homology, i.e., element represented by any of its closures. □

Now we can define obstructions to tangle embeddings using quandle homology.

Theorem 9. *If a tangle T embeds into a link L then for every boundary-monochromatic (shadow) coloring α of a diagram D of T there exists a (shadow) coloring β of any diagram $\overline{D}$ of L, that represents the same homology class as the one represented by α.*

Proof. If T embeds in L, then there exists a diagram $\widetilde{D}$ of L such that D is a part of it. Any boundary-monochromatic (shadow) coloring of D extends trivially to a coloring β of $\widetilde{D}$. Then, using homology moves from Fig. 4, one can destroy all
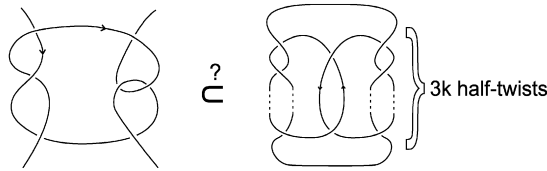
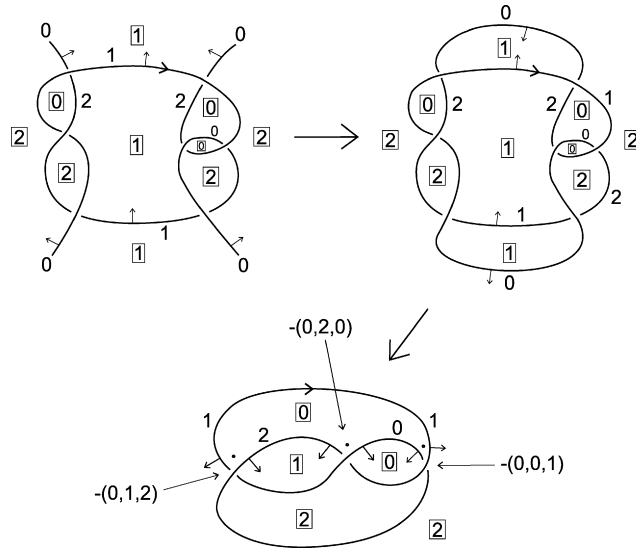**Fig. 5.** An example of a problem of embedding a given tangle into a link.

Fig. 6. A shadow-colored tangle that represents a generator of $H_3^Q(R_3)$.

crossings in $\widetilde{D}$ that are outside of D, and remove trivial components that may appear during this process. As a result one obtains one of the closures of D. Homology class does not depend on the closure. Therefore, the cycle represented by α equals the cycle represented by β in $H_2^Q(X)$ (or $H_3^Q(X)$ in the case of shadow colorings). Finally, any coloring of $\widetilde{D}$ gives a coloring of any other diagram $\overline{D}$ of L by a sequence of Reidemeister moves (they do not change the homology class). $\quad\Box$

Example 10. Fig. 5 illustrates an example of a tangle T, and a family of links that have $3k$, $k \in \mathbb{N}$, half-twists on each side. Let L denote any member of this family. We can use the third quandle homology of the dihedral quandle R_3 to show that T does not embed into L. Fig. 6 shows an example of a boundary-monochromatic shadow coloring of a diagram of T with elements of R_3. The numerator closure of this tangle is a shadow-colored trefoil knot. This coloring represents a chain $-(0, 1, 2) - (0, 0, 1) - (0, 2, 0)$ that gives a generator of $H_3^Q(R_3)$ [24]. On the other hand, every coloring of the link L represents 0 in $H_3^Q(R_3)$ (see Fig. 7). We can see it as follows. Quandle R_3 is the simplest nontrivial example of a Burnside kei [23], and is invariant under 3-moves and, more generally, under $3k$-moves. That is why we can label the top arcs and the corresponding bottom arcs of the diagram of L with the same elements a, b, c. Such labeling forces relations $b * c = b$ and $c * b = c$ that imply the equality $b = c$ in R_3. Because of this equality it is possible to perform a homology move on the diagram of L (see Fig. 5), that transforms it into unlink representing 0 in homology. From Theorem 9 follows that T cannot be embedded into L.

Another quandle-based approach to the tangle embedding problem, using quandle cocycle invariants, was proposed in [1].

4. The structure of $H_2^Q(R_4)$

In order to provide examples for the next two applications, we will now analyze the second homology group of the dihedral quandle R_4.

Let us recall that the dihedral quandle R_4 is a set $\{0, 1, 2, 3\}$ with operation $i * j \equiv 2j - i \pmod{4}$. It consists of two orbits (with respect to the action of R_4 on itself by the right multiplication): $\{0, 2\}$ and $\{1, 3\}$.

To simplify our notation, and make it more general, we will write the elements of this quandle as $\{a, b, a*b, b*a\}$, where a and b are representatives of different orbits. Note that the elements of R_4 (when written as longer products involving a
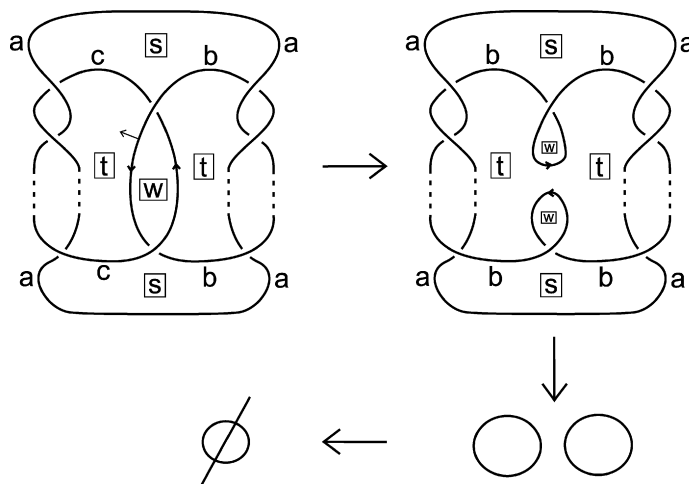
Fig. 7. A nontrivial knot, whose every coloring represents 0 in $H_3^Q(R_3)$.

and b) can be determined by looking at the first letter in the word, and the parity of the letter from the second orbit that appears in the rest of the word. For example, if s denotes the number of b's in the word, then

$$a * b * \cdots = \begin{cases} a, & \text{if } s \text{ is even,} \\ a * b, & \text{if } s \text{ is odd.} \end{cases}$$

It is known (see for example [19]) that $H_2^Q(R_4) = Z^2 \oplus (Z_2)^2$. We will show that the free part is generated by

$$f_1 = (a, b) + (a * b, b),$$
$$f_2 = (b, a) + (b * a, a),$$

and that the generators of the torsion part are

$$t_1 = (a, a * b),$$
$$t_2 = (b, b * a).$$

The first part of the statement follows from evaluating the cocycles

$$\chi_{(a,b)} + \chi_{(a,b*a)},$$
$$\chi_{(b,a)} + \chi_{(b,a*b)},$$

on f_1 and f_2. Here, $\chi_{(x,y)}$ denotes the characteristic function of (x, y), and the above cocycles were proven to be generators of $H_Q^2(R_4, \mathbb{Z}) = \mathbb{Z}^2$ in [4]. To prove the second part, we first notice that t_1 and t_2 are either torsion elements or 0, since we have

$$\partial\big(-(a * b, b, a * b) - (a * b, b * a, a * b)\big) = 2t_1,$$
$$\partial\big(-(b * a, a, b * a) - (b * a, a * b, b * a)\big) = 2t_2.$$

To prove nontriviality, we use the following cocycles c_1 and $c_2 \in H_Q^2(R_4, \mathbb{Z}_2)$:

$$c_1 = \chi_{(a,b)} + \chi_{(a*b,b)} + \chi_{(a,a*b)} + \chi_{(a*b,a)},$$
$$c_2 = \chi_{(b,a)} + \chi_{(b*a,a)} + \chi_{(b,b*a)} + \chi_{(b*a,b)}.$$

Since $c_1(t_1) = 1$ and $c_2(t_2) = 1$, t_1, t_2 and c_1, c_2 must be nontrivial. We also note that c_1 and c_2 evaluate trivially on f_1 and f_2.

5. The lower bound for the 4-move distance between links

Definition 11. An n-move is a replacement of n half-twists by two parallel strings or vice versa in a link diagram (see Fig. 8).

Of particular interest in knot theory are 4-moves. One of the reasons is the following old conjecture [17,25,26].

n half-twists
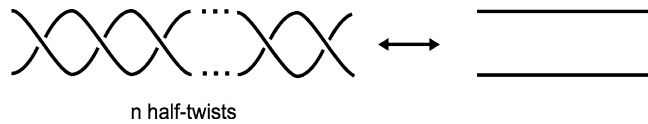
Fig. 8. n-move.
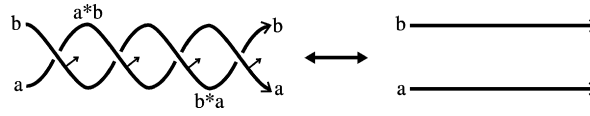
Fig. 9. A colored 4-move represents a cycle in $H_2^Q(R_4)$.

Conjecture 12. *(Nakanishi, 1979.) Every knot is* 4*-move equivalent to the trivial knot. In other words, every knot can be transformed into a trivial knot using* 4*-moves and Reidemeister moves.*

Not every link is 4-move equivalent to a trivial link; in particular, the linking matrix modulo 2 is preserved by 4-moves. Furthermore, Nakanishi demonstrated that the Borromean rings cannot be reduced to the trivial link of three components [22,26]. Kawauchi expressed the question for links as follows:

Problem 13. ([17])

 (i) Is it true that if two links are link-homotopic then they are 4-move equivalent?
(ii) In particular, is it true that every 2-component link is 4-move equivalent to the trivial link of two components or to the Hopf link?

A 3-component counterexample to this problem was provided in [10]. The second part of the question remains open, and is actively investigated.

In this paper we consider the following problem.

Problem 14. If two links are 4-move equivalent, what is the minimal number of 4-moves needed to transform one into the other?

Therefore, it is natural to make the following definition.

Definition 15. The n-move distance, $d_n(L_1, L_2)$, between two links L_1 and L_2, is the minimal number of n-moves realizing the n-move equivalence, or ∞ if L_1 and L_2 are not n-move equivalent.

For example, the 4-move distance between the trivial link of two components and the Hopf link is ∞, as indicated by their linking matrices modulo 2.

We will now explain how the quandle homology of R_4 can be often used to obtain the lower bound for $d_4(L_1, L_2)$.

Lemma 16. *Any R_{2k}-coloring of the two oriented strings with $2k$ half-twists represents a cycle in $H_2^Q(R_{2k})$.*

Proof. It is known (see for example [25,26,23]) that the dihedral quandle R_n is an invariant under n-moves. In other words, for any R_n-coloring of the two strings with n half-twists, the colors of the two initial arcs are the same as colors of the corresponding final arcs (see Fig. 9 for an illustration of this fact in the case of 4-move). Colorings with dihedral quandles do not depend on the orientation of the link. However, if we want to analyze quandle homology, the orientation has to be taken into account. In the case of an even number of half-twists, for any orientation (parallel or anti-parallel) of the twisted strings, it is possible to join the upper left arc with the upper right arc, and the lower left arc with the lower right arc, without introducing any additional crossings. Thus, we obtain a properly colored and oriented, uniquely determined link that represents an element in $H_2^Q(R_n)$. □

Remark 17. In the case when n is odd and the orientation of the strings is anti-parallel, the chain determined by the n colored crossings considered above is often not a cycle.

Lemma 18. *Let c be a cycle representing some R_4-coloring of the two oriented strings with 4 half-twists. If both strings have colors from a single orbit, then either c is homologically trivial or $c = \pm(f_1 + f_2 + t_1 + t_2)$, where f_1, f_2, t_1, t_2 are as in the previous section.*
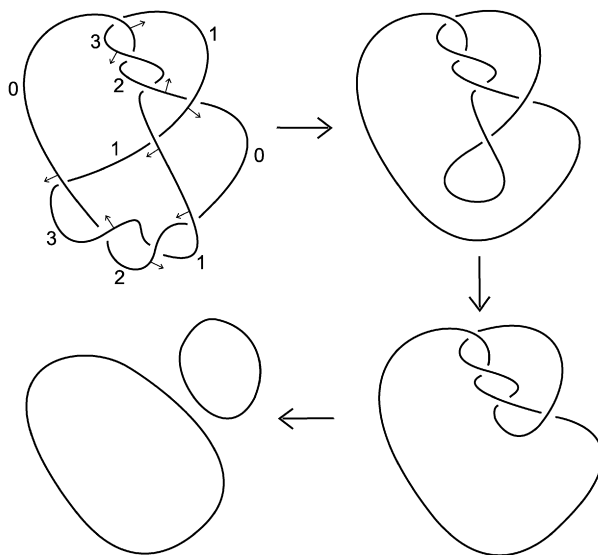
Fig. 10. A link reduced to the trivial link by two 4-moves.

Proof. First, for a given quandle X, we define the map

$$*_a : C_n^Q(X) \to C_n^Q(X)$$

determined by $*_a(w) = w * a$, for any $w \in X^n$, or more precisely,

$$*_a(x_1, \ldots, x_n) = (x_1, \ldots, x_n) * a = (x_1 * a, \ldots, x_n * a).$$

We are going to use the following fact from [24]: if z is a cycle, then $*_a(z)$ is also a cycle, homologous to z, for any $a \in X$.

It follows that $*_b(t_1) = *_b(a, a*b) = (a*b, a)$ is homologous to t_1, and $*_a(t_2) = *_a(b, b*a) = (b*a, b)$ is homologous to t_2. Therefore, any pair of different elements of R_4 from the same orbit represents a torsion in $H_2^Q(R_4)$. It follows that if the strings are nontrivially colored by elements from the same orbit, then such coloring represents a cycle homologous to $\pm 4t_1$ or $\pm 4t_2$. It can be checked by inspection that each coloring that uses the elements from different orbit gives a cycle c that decomposes into two smaller cycles: $c = \pm(c_1 + c_2)$, where $c_1 = (a, b) + (a*b, b*a)$ and $c_2 = (b, a) + (b*a, a*b)$. Sometimes cycles $c_1' = (a*b, b) + (a, b*a)$ or $c_2' = (b*a, a) + (b, a*b)$ appear (as in Fig. 9), but $c_1 = *_b(c_1')$ and $c_2 = *_a(c_2')$, so there is no difference in homology. Finally, we check that

$$\partial\big((a, b, a*b) - (a, b, b*a)\big) = c_1 - (f_1 - t_1),$$
$$\partial\big((b, a, b*a) - (b, a, a*b)\big) = c_2 - (f_2 - t_2),$$

and, since homologically $t_1 = -t_1$ and $t_2 = -t_2$, the proof is finished. $\square$

Corollary 19. *Let L be an oriented link diagram colored with elements of the dihedral quandle R_4, and let c be a cycle in $H_2^Q(R_4)$ represented by this coloring. Then, for any 4-move performed on the diagram, c either remains unchanged or is replaced by $c \pm (f_1 + f_2 + t_1 + t_2)$.*

Corollary 20. *The 4-move distance, $d_4(L_1, L_2)$, between links L_1 and L_2, such that at least one of them admits nontrivial R_4-colorings, can be analyzed by comparing the multiplicities of $f_1 + f_2$ appearing in the cycles represented by the colorings.*

Example 21. We consider the coloring of the link illustrated in Fig. 10. It represents a cycle $c \in H_2^Q(R_4)$ of the form:

$$(a, b) + (a*b, b*a) + (b, a) + (b*a, a*b) + (a, b*a) + (a*b, b) + (b*a, a) + (b, a*b).$$

Using a similar technique as in the proof of Lemma 18, we can conclude that

$$c = 2(f_1 + f_2 + t_1 + t_2) = 2(f_1 + f_2).$$

Thus, at least two 4-moves are necessary to reduce it to the trivial link with two components (whose colorings represent 0 in homology). It cannot be reduced to the Hopf link, since the Hopf link admits only colorings using elements of one orbit, and this property is preserved by 4-moves. As shown in Fig. 10, two 4-moves suffice to make the reduction.
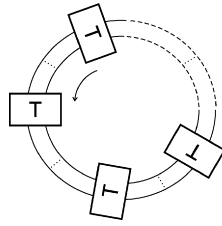
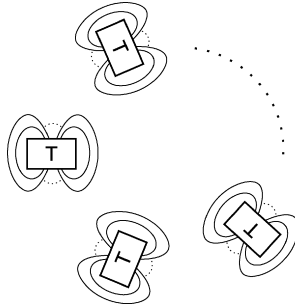**Fig. 11.** Invariant diagram of a periodic link.

Fig. 12. Homological decomposition of a colored invariant diagram.

We note that any link with a coloring representing a cycle that is not a multiple of $f_1 + f_2 + t_1 + t_2$ would be a counterexample to the second part of Kawauchi's question, because the colorings of the Hopf link and the colorings of the trivial link do not give any nontrivial classes in $H_2^Q(R_4)$. No such link has been found so far. However, since every cycle from the second homology can be represented by a colored virtual link [8], above technique provides virtual counterexamples to the question. One such example is a virtual link with a coloring representing $f_1 + f_2 + t_1$.

We also remark that the above method can be generalized to quandles R_{2k} and the $2k$-move distance. More generally, it should work with certain rational moves (see [10] for a definition) and the rational move distance.

6. Application to periodicity of links

Definition 22. Let p be a prime number. A link L in S^3 is called p-periodic if there is a Z_p-action on S^3, with a circle as a fixed point set, which maps L onto itself, and such that L is disjoint from the fixed point set. Furthermore, if L is an oriented link, one assumes that each generator of Z_p preserves the orientation of L or changes it to the opposite one.

By the positive solution to the Smith Conjecture, if a link is p-periodic, then L has a diagram $\widetilde{D}$ such that the rotation by an angle $\frac{2\pi}{p}$ about a point away from the diagram leaves $\widetilde{D}$ invariant. There exists an n-tangle T such that L is the closure of T^p, i.e., a tangle obtained by gluing p copies of T in a natural way, as illustrated in Fig. 11 (see also [3,14,27]).

In this section we will show that sometimes we can use quandle homology to prove that a link L is not p-periodic for some prime p.

Theorem 23. *Let p be a prime number, and D be a diagram of a p-periodic link L. If a coloring (shadow coloring) of D with elements of some fixed quandle X represents a homology class c in $H_2^Q(X)$ (or $H_3^Q(X)$ in the case of shadow coloring), then either there exist $p - 1$ different colorings of D that represent the same element in homology as c, or $c = p\widetilde{c}$, for some $\widetilde{c}$ in $H_2^Q(X)$ (or in $H_3^Q(X)$).*

Proof. First, let us note that there are two types of colorings of the invariant diagram $\widetilde{D}$ of a p-periodic link L. One possibility is that all tangles T that are building blocks of $\widetilde{D}$ receive exactly the same coloring. Otherwise, there is some asymmetry in the coloring, and it is possible to obtain from it $p - 1$ different colorings by rotating the given coloring by a multiple of an angle $\frac{2\pi}{p}$. Note that in this process the position of the link diagram $\widetilde{D}$ is not changed, only coloring is rotated. This distinction becomes more clear if we translate each such coloring (via Reidemeister moves) into a coloring of some other, less symmetric diagram of L. If p is not prime, then we might obtain a smaller number of colorings than $p - 1$, because some of them may be identical. Let C be any (shadow) coloring of a diagram D. If C is of the first type, then using the homology move that changes connections between strings with the same color (see Fig. 4), we can decompose colored diagram D into p identical smaller diagrams $\widehat{D}$ (as in Fig. 12). In this case, the element in quandle homology that is represented by the coloring C is equal to $p\widehat{C}$, where $\widehat{C}$ is element of homology corresponding to $\widehat{D}$. If the coloring is of the second type, then each of the $p - 1$ colorings obtained by rotating the original coloring represents the same element in homology. $\square$
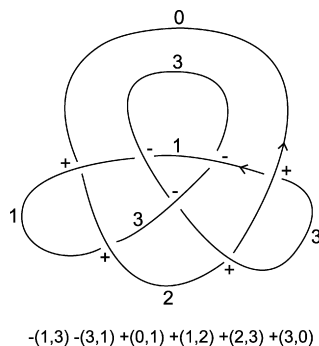
-(1,3) -(3,1) +(0,1) +(1,2) +(2,3) +(3,0)

Fig. 13. A quandle coloring of the link 7_5^2 and the nontrivial cycle it represents.

Example 24. We can use the second quandle homology of the quandle R_4 to show that 2 is the only possible period for the link 7_5^2 (see Fig. 13). This link has 16 colorings using the quandle R_4. Eight of them are either trivial or represent $4t$, where t is an element from the $\mathbb{Z}_2$-torsion. The remaining eight colorings give cycles of the form:

$$(a, b) + (a * b, b * a) + (b, a * b) + (b * a, a) - (b, b * a) - (b * a, b).$$

As in the previous section, we can recognize them as homologous to $f_1 + f_2 + t_1 + t_2$. The possibility of such element being equal to p times some other element, for p prime, is excluded. If the link 7_5^2 were p-periodic, then the aforementioned 8 colorings would have to be partitioned into p-element subsets. Therefore, the only candidate for the prime period of the link 7_5^2 is 2.

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