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The configuration of bitangents of the Klein curve

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Abstract

The configuration of bitangents of a smooth quartic curve in $\boldsymbol{P}^2(\boldsymbol{C})$ has been a classical object of study. In particular for the Klein curve $xy^3+yz^3+zx^3=0$ it is highly symmetric (Baker 1935; Klein 1879). Key concepts are Steiner sets and Aronhold sets (Dickson, 1961). We give a complete description of these sets for the Klein curve and of their orbits under the group of the curve, using the relation between the geometric configuration, the Coxeter graph (Coxeter, 1983) in various appearances and the regular 2-graph on 28 points (Taylor, 1977). Also a model is provided for the self-dual configuration of $21+28$ points and $21+28$ lines associated with the Klein curve.

1. A model for the Klein curve and an encoding for its bitangents

The Klein curve is the unique plane quartic curve with the maximal number 168 of automorphisms. Coxeter [2] mentions a model of the Klein curve due to Ciani. Let c be a complex root of $x^2+x+2=0$. Consider the curve C with equation

$$x^4+y^4+z^4+3\bar{c}(y^2z^2+z^2x^2+x^2y^2)=0,$$

in $\boldsymbol{P}^2(\boldsymbol{C})$. It is defined over $\boldsymbol{Q}[c]=\boldsymbol{Q}[\sqrt{-7}]$ and reduces mod $\sqrt{-7}$ to the double conic $(x^2+y^2+z^2)^2=0$ in $\boldsymbol{P}^2(\boldsymbol{F}_7)$. Coxeter lists the 28 bitangents of C: they reduce mod $\sqrt{-7}$ to the 28 rational secant lines of the conic, which is isomorphic to $\boldsymbol{P}^1(\boldsymbol{F}_7)$, and each tangent point reduces to a rational point of the conic. Hence the bitangents can be encoded by the set Γ_1 of unordered pairs of distinct elements of $\boldsymbol{P}^1(\boldsymbol{F}_7)=\{0,1,2,3,4,5,6,\infty\}$. If the lines in $\boldsymbol{P}^2(\boldsymbol{C})$ are indicated by the coefficients of their equations, the correspondence is

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$0\infty=[1,1,1]$	$01=[1,0,\bar{c}]$	$02=[0,\bar{c},1]$	$03=[1,c^2,-1]$
$1\infty=[-1,c^2,1]$	$12=[-\bar{c},1,0]$	$13=[1,-1,1]$	$14=[0,-\bar{c},1]$
$2\infty=[c^2,1,-1]$	$23=[1,1,c^2]$	$24=[1,0,-\bar{c}]$	$25=[1,-c^2,1]$
$3\infty=[\bar{c},0,1]$	$34=[-c^2,1,1]$	$35=[0,1,-\bar{c}]$	$36=[1,-\bar{c},0]$
$4\infty=[1,-1,c^2]$	$45=[1,1,-1]$	$46=[1,c^2,1]$	$04=[\bar{c},1,0]$
$5\infty=[1,\bar{c},0]$	$56=[-\bar{c},0,1]$	$05=[-1,1,c^2]$	$15=[c^2,1,1]$
$6\infty=[0,1,\bar{c}]$	$06=[c^2,-1,1]$	$16=[1,1,-c^2]$	$26=[-1,1,1]$.

Moreover, the automorphisms of C are reduced to automorphisms of the conic; this establishes an isomorphism $H=\mathrm{Aut}(C)\cong \mathrm{PSL}(2,\boldsymbol{F}_7)$. The set Γ_1 carries a graph structure: two pairs ab and cd are connected iff $abcd$ is a harmonic tetrad in $\boldsymbol{P}^1(\boldsymbol{F}_7)$. This gives an incarnation of the Coxeter graph Γ (cf. [2, 4]). See Fig. 1 with its bold labels, which is the same as Fig. 2 of Coxeter [2].

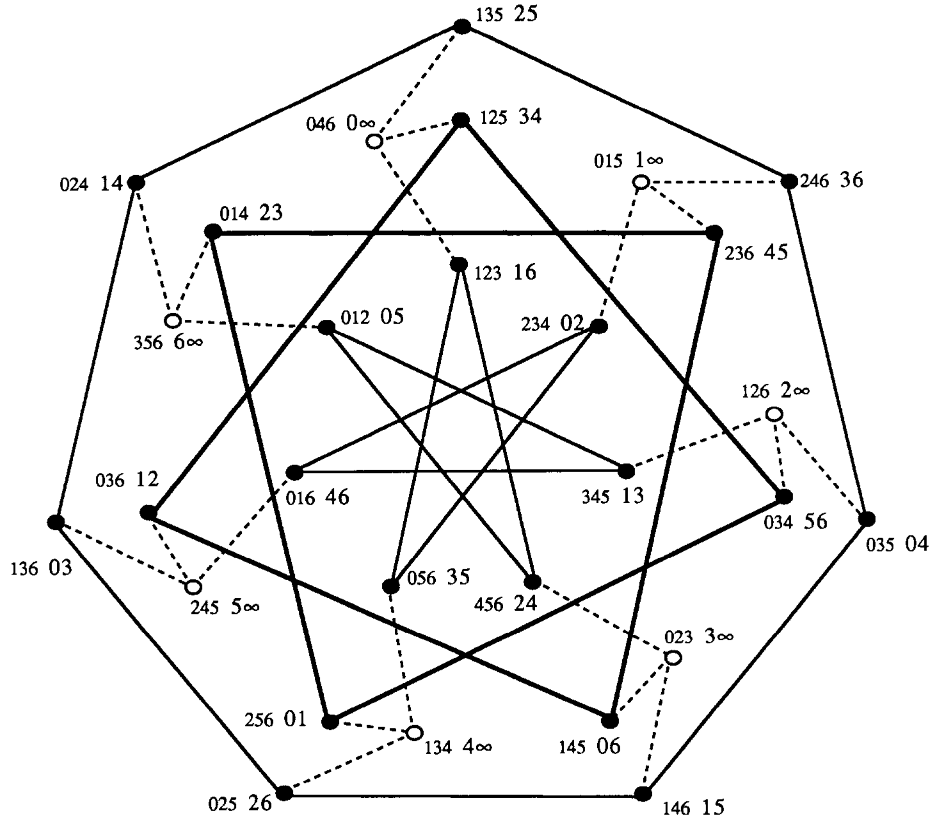

Fig. 1.

We have $G_1 = \mathrm{Aut}(\Gamma_1) \cong \mathrm{PGL}(2, \boldsymbol{F}_7)$, which contains H as a subgroup of index 2. In this incarnation the distances >1 in Γ also have an algebraic significance, see Lemma 1.1.

Lemma 1.1. *Let $ab, cd \in \Gamma_1$. Write ρ for the distance in the Coxeter graph and $(a\,b\,c\,d)$ for the cross-ratio of the points $a, b, c, d \in \boldsymbol{P}^1(\boldsymbol{F}_7)$. Then*

$$(a\,b\,c\,d) = -1 \iff \rho(ab, cd) = 1,$$

$$(a\,b\,c\,d) \in \{2, 4\} \iff \rho(ab, cd) = 2,$$

$$(a\,b\,c\,d) \in \{3, 5\} \iff \rho(ab, cd) = 4,$$

$$\#\{a, b, c, d\} = 3 \iff \rho(ab, cd) = 3.$$

Proof. By inspection from Fig. 1 we find that $\rho(0\infty, 1a) = 3$ for $a \in \{0, \infty\}$, $=1$ for $a = 6$, $=2$ for $a \in \{2, 4\}$ and $=4$ for $a \in \{3, 5\}$. The statement for the remaining pairs follows using G_1. $\square$

2. A second incarnation of the Coxeter graph

Let Γ_2 be the graph of which the points are the triangles of $\boldsymbol{P}^2(\boldsymbol{F}_2)$, two points being adjacent if they are disjoint (as to vertices or, equivalently, as to edges). See the previous figure and its lower case labels. Then $\Gamma_2 \cong \Gamma$ and $\mathrm{Aut}(\Gamma_2)$ is faithfully represented by the group of collineations and correlations of $\boldsymbol{P}^2(\boldsymbol{F}_2)$, see [4].

In Fig. 2 the points are labelled such that the lines are the cyclic shifts mod 7 of 013. The lines are labelled by twice the sum of the labels of their points. In that way concurrent triples of lines have as labels again the cyclic shifts of 013 and the map interchanging point i and line i for all i is a correlation. Call the vertices and sides of a triangle its elements. We then have Lemma 2.1.

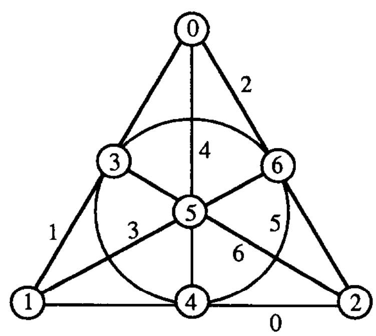

Fig. 2.

Lemma 2.1. *If two triangles have mutual distance* 1, 2, 3, 4, *respectively, in* Γ_2, *then they have in common* 0, 4, 2, 3 *elements, respectively.*

Proof. We denote a triangle by the triple of its vertices. Modulo PSL(3, 2) we can take the first triangle as 012, one at distance 1 from it as 356 and one at distance 2 from it as 024. One at distance 3 from it can be taken as either 135 or 136, one at distance 4 from it as either 025 or 246 (see Fig. 3). The lemma now follows by inspection. □

Note that the correlation mentioned above fixes 012, 356 and 024. It interchanges 135 with 136 and 025 with 246.

With a triangle △ (012, say) there is a unique point not on any of its sides (5) and a unique line not through any of its vertices (346). We call these the *central point* and the *central line* of △. The triangle △ has distance 4 to precisely 6 other triangles. Three of these (015, 025, 125) are the triangles having the same central line as △. Such triangles have also mutual distance 4, and mutually share a side and two vertices. The four vertices involved (0, 1, 2 and 5) are those of a 4-gon. Dual statements hold for the other three triangles at distance 4 from △ (036, 134, 246). The correlations that leave invariant 012 interchange the two triples of triangles, which are pairwise joined by edges of Γ_2 (see Fig. 3). Evidently in Γ the relation "at distance 4 from" is the disjoint union of two relations, each defining a graph consisting of 7 4-cliques. Two points x and y are connected in Γ if and only if there are points z_1 and z_2 such that $\langle x, z_1\rangle$ and $\langle y, z_2\rangle$ are in one of these relations and $\langle x, z_2\rangle$ and $\langle y, z_1\rangle$ in the other. Note that by Γ only the pair of these relations is defined, not each of them separately.

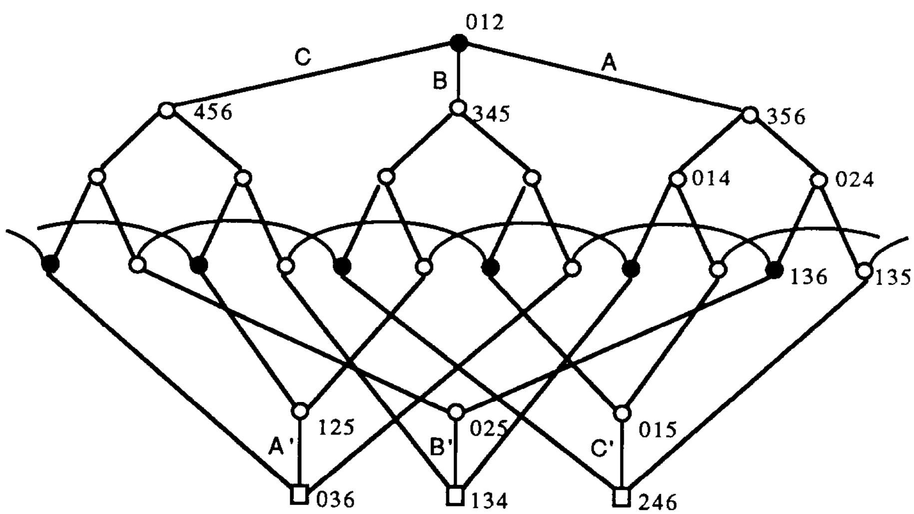

Fig. 3.

We note in passing that Γ_2 has six 7-cycles through each vertex (see 012 and the arcs in Fig. 3), so 24 in total. In the figure one easily recognizes the stabilizer S_3 of 012 in PSL(3, 2): the reflection in the vertical line gives an automorphism of Γ_2 and there is also an automorphism of order 3 that cyclically shifts the 12 vertices on a horizontal line over 4 positions; both do not interchange the two triples of vertices at distance 4 from 012. This S_3 acts transitively on the 6 arcs in the figure. Thus we have a regular action of the subgroup of order 168 of Aut(Γ_2) on the set of pairs consisting of a vertex and a 7-cycle through it, and the stabilizer of a 7-cycle must be the cyclic group of order 7. In Aut(Γ_2) it is the dihedral group of order 14, as can be seen in Fig. 1.

3. The reflection points

We investigate the relation between geometric properties of the configuration of bitangents and graph-theoretical properties of Γ_2. The bijection from the set of bitangents onto the set of triangles establishes an isomorphism between H and the subgroup PSL(3, 2) of Aut(Γ_2). The centres of the 21 involutions of H each are incident with 4 bitangents and each bitangent contains 3 of these centres (see [1] or the following). The 21 involutions of PSL(3, 2) $\cong H$ correspond to the 21 centre/axis pairs. Each involution leaves invariant 4 triangles, having their central point on the axis and their central line through the centre. For instance for centre 4 and axis 045 the invariant triangles are 012, 356, 125 and 036. In Γ_2 they span two 'antipodal' edges (see Fig. 4 and the pairs AA', BB', CC' in Fig. 3). Their mutual distances are 1, 1, 4, 4, 4, 4. Since there are 21 of these edge-pairs we have a 1–1 correspondence, and 4 bitangents

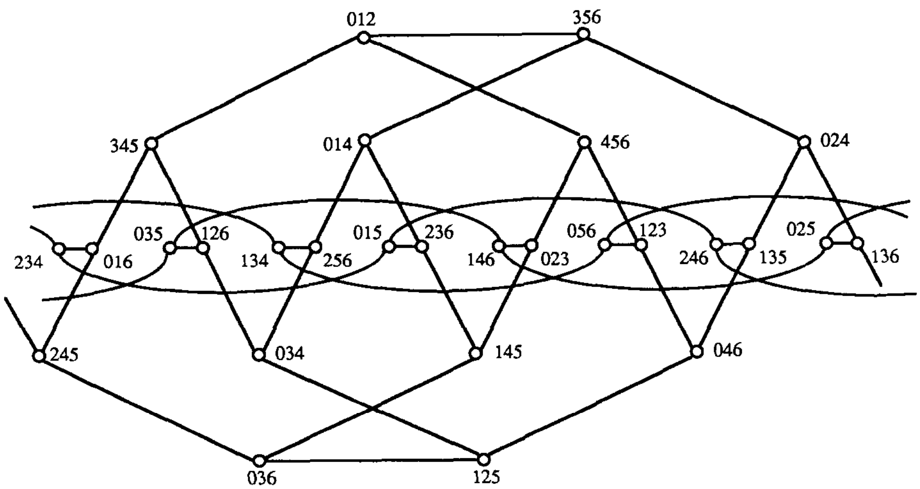

Fig. 4.

through a reflection point must correspond to 4 triangles that are the endpoints of two antipodal edges. It thus may be expected that those 4 bitangents admit a 'natural' partition into two pairs (see also [2, Section 4, last paragraph]).

Now in PSL(3, 2) there are 8 collineations that leave invariant the centre 4 and the axis 045 (namely the involutions with axis 045 and centre 0, 4 or 5, those with centre 4 and axis 124 or 346, and the two of order 4 that interchange 0 and 5 and cyclically permute 1, 3, 2, 6). They form a dihedral group. Its action on the above set of triangles has a kernel of order 2 (generated by the initial involution); the image is the Klein group on the triangles. The two elements of order 4 have the same image; it interchanges the 4 triangles in pairs, fixing the antipodal edges. We have Lemma 3.1.

Lemma 3.1. *Every bitangent b is intersected in each of its 3 reflection points by 3 other bitangents, one of which is distinguished by the fact that it is interchanged with b by the two elements of order 4 in* Aut(*C*) *that fix the reflection point. Defining these distinguished bitangents as the neighbours of b we get a Coxeter graph on the set of bitangents.*

We mention in passing that the 21 involutions of PSL(2, 7) are realized on the conic in $\boldsymbol{P}^2(\boldsymbol{F}_7)$ as the involutions that have their centers in an internal point (a point lying on no tangent), since involutions with their centre in an external point have fixed points and cannot belong to PSL(2, 7).

4. Klein groups and 8-cycles

In PSL(3, 2) two involutions commute if and only if they share the centre or the axis, so a Klein subgroup contains, apart from the identity, the 3 involutions with a given centre or the 3 involutions with a given axis. Thus, as already noted by Klein [5], there are 14 Klein groups forming 2 conjugacy classes. A dihedral group of order 8 contains two Klein groups; they share an involution. So one of them contains the involutions with a certain centre, the other those with a certain axis through that centre. Now we have only $7\times 3=21$ such pairs of Klein groups. In the previous section we have seen that each centre/axis pair yields a dihedral group. Apparently these are the only dihedral groups of order 8. The involution that gave rise to such a group is the one shared by its Klein groups. It corresponds to a pair of antipodal edges in Γ_2, and the points at distance 3 from all 4 endpoints of these edges span an 8-cycle, as we can observe in Fig. 4 (the arcs). It has the dihedral group as its stabilizer in H. Since there are $\frac{1}{8}(28\times 6)=21$ 8-cycles we have again a bijection. (It can be verified that the antipodal pairs corresponding to the other involutions of the two Klein groups each consist of two of the small horizontal edges in Fig. 4).

Now from Fig. 4 we take the cycle corresponding to the centre/axis pair 4/045: 234–015–246–035–146–025–134–056–234. The set $\{234, 246, 146, 134\}$ is an orbit under the Klein group belonging to the centre 4, the set $\{015, 035, 025, 056\}$ is an orbit under the Klein group belonging to the axis 045 and (consequently) both are orbits under the dihedral group. Note that of the pairs $\{234, 146\}$, $\{246, 134\}$, $\{015, 025\}$ and $\{035, 056\}$, opposite in the 8-cycle, the first two consist of triangles sharing a vertex and two sides, whereas for the second two this is a side and two vertices, cf. Lemma 2.1.

In the sequel we shall use the following lemma.

Lemma 4.1. *Let K be a Klein subgroup of* $\mathrm{PGL}(3, C)$ *and P and Q points in $\boldsymbol{P}^2(C)$. Then there is a conic through P and Q that is invariant under K. It is unique if and only if P and Q are in different K-orbits.*

Proof. By suitable choice of coordinates we may suppose that K is generated by the involutions

$$x:y:z \mapsto -x:y:z \quad \text{and} \quad x:y:z \mapsto x:-y:z.$$

The conics invariant under K are those with equations of the form $ax^2+by^2+cz^2=0$. Such a conic contains $P=p_1:p_2:p_3$ and $Q=q_1:q_2:q_3$ if and only if

$$ap_1^2+bp_2^2+cp_3^2=0=aq_1^2+bq_2^2+cq_3^2.$$

There is a solution $a:b:c$, and it is unique if and only if $p_1^2:p_2^2:p_3^2 \neq q_1^2:q_2^2:q_3^2$, that is, if and only if $p_1:p_2:p_3 \neq \pm q_1:\pm q_2:\pm q_3$. $\square$

5. Another incarnation of the Coxeter graph

The stabilizer of a triangle in PSL(3, 2) is a subgroup isomorphic to S_3, and it is easily verified (e.g., embedding PSL(3, 2) in S_7) that every subgroup of type S_3 is the stabilizer of a triangle. All these subgroups are conjugate. However, as follows from Section 3, stabilizers of disjoint triangles (e.g., 012 and 356) are conjugate under a collineation of order 4 (e.g., (05)(1326) of which the square belongs to both stabilizers. Conversely, a collineation of order 4 is of type $(a)(bc)(defg)$ with abc, adf and aeg lines, and one verifies that a triangle $\triangle$ fixed by its square is mapped by it onto a triangle disjoint from $\triangle$. Thus we have a Coxeter graph of which the points are the subgroups of type S_3 in PSL(3, 2).

6. Some geometry

We denote the reflection point in $\boldsymbol{P}^2(\boldsymbol{C})$ that corresponds to the centre/axis pair 4/045 in $\boldsymbol{P}^2(\boldsymbol{F}_2)$ by 05. Note that the bitangents through it correspond to the triangles having 0 or 5 as their central point and the other point as a vertex. Let a be the bitangent corresponding to the triangle 012 ($a=012$ for short). Then $A=05$, $A_1=15$, $A_2=25$ are the reflection points on a. The other bitangents through A with the reflection points on them are

$b = 356$ with $B_1 = 03$ and $B_2 = 06$,

$c = 125$ with $C_1 = 01$ and $C_2 = 02$,

$d = 036$ with $D_1 = 35$ and $D_2 = 56$.

We now have 9 reflection points. There is a bitangent through two of them if and only if the corresponding pairs are not disjoint and their union is not a collinear triple. We thus find 8 more bitangents

$B_1D_1 = 056$ and $B_2D_2 = 035$, intersecting in 36,

$B_1C_2 = 234$ and $B_2C_1 = 146$, intersecting in 04,

$A_1D_1 = 134$ and $A_2D_2 = 246$, intersecting in 45,

$A_1C_1 = 025$ and $A_2C_2 = 015$, intersecting in 12.

Evidently the two special pairs a, b and c, d are characterized by the fact that no two reflection points on the bitangents of such a pair are on a third bitangent. The bitangent 012 is intersected in the reflection points 05, 15, 25, respectively, by the bitangents 036, 134, 246, respectively. These 4 bitangents have mutual distance 4 (see Lemma 2.1) and a common central point 5. Likewise it is intersected by 125, 025, 015, giving a quadruple with common central line 346. The two triples of bitangents form a line-perspective pair of triangles with vertices 45, 56, 35 and 12, 02, 01. Finally we have the bitangents 345, 356, 456, the neighbors of 012 in the Coxeter graph. They also form a triangle that is line-perspective with the other two. Using coordinates as in Section 1, and the isomorphism between Γ_1 and Γ_2 as given in [4] and in Fig. 1, it can be calculated that all three triangles are perspective from one point $-1:1:\bar{c}^2$. This point, the 'Hermitian pole' of the bitangent $[-1, 1, c^2]$ we started with, is the pole of that bitangent with respect to the 2 conics belonging (as explained in the next section) to the first two triples of bitangents (see [2], Section 10).

We have seen in Section 4 that every reflection is in the centre of a unique dihedral subgroup of H and commutes with precisely 4 other reflections. So its centre is on the axes of these 4 reflections and its axis passes through their centres (which implies that a reflection line is not a bitangent). These are all incidences between reflection points and reflection lines: the bitangents through a point p contain 8 other reflection points, so there is only room for 4 reflection lines through p with precisely 3 more reflection points on each. See [2], Sections 6 and 7, for a different approach.

We shall now denote a reflection point by the pair $\langle p, l\rangle$ of the centre and axis of the corresponding involution of the Fano plane and the axis of the same reflection by $\langle l, p\rangle$. Then we have: $\langle p, l\rangle$ is on $\langle m, q\rangle$ iff $p = q$ and $l \neq m$ or $p \neq q$ and $l = m$. If we represent a bitangent by the pair $\langle m, q\rangle$ of the central line m and the central point q of the corresponding triangle, then a reflection point $\langle p, l\rangle$ is on the bitangent $\langle m, q\rangle$ iff $p \in m$ and $q \in l$.

In Ciani's coordinatization every reflection point is the Hermitian pole of the corresponding reflection line (see [2], Section 6). So the Hermitian poles of the bitangents also play a (dual) role: each is incident with 3 reflection lines. As Coxeter ([2], Section 7) remarks, these poles and the reflection points are all intersection points of reflection lines since $21\times\binom{4}{2}+28\binom{3}{2}=\binom{21}{2}$. Moreover there are no incidences between the 28 bitangents and their Hermitian poles [2]. Calling $\langle q,m\rangle$ the pole of the bitangent $\langle m,q\rangle$, we thus can fully describe the configuration of $28+21$ lines and $28+21$ points as follows.

Theorem 6.1. *Let P be the set of points and L that of lines of the Fano plane. There is a labelling of the 21 reflection points of the Klein curve and the 28 other intersection points of reflection lines by the elements of $P\times L$ and a labelling of the 21 reflection lines and the 28 bitangents by the elements of $L\times P$ in such a way that*

(i) *$\langle p,l\rangle\in P\times L$ represents a reflection point iff $p\in l$, $\langle l,p\rangle\in L\times P$ represents a reflection line iff $l\ni p$,*

(ii) *the reflection with centre $\langle p,l\rangle$ has axis $\langle l,p\rangle$,*

(iii) *the reflection point $\langle p,l\rangle$ is on the reflection line $\langle m,q\rangle$ iff $p=q$ and $l\neq m$, or $p\neq q$ and $l=m$,*

(iv) *the reflection point $\langle p,l\rangle$ is on the bitangent $\langle m,q\rangle$ iff $p\in m$ and $q\in l$,*

(v) *the reflection line $\langle l,p\rangle$ passes through the point $\langle q,m\rangle$ with $q\notin m$ iff $p\in m$ and $q\in l$.*

There are no other incidences between the 49 points and the 49 lines.

Remark. In Fig. 2 a point p and a line l carry a label from F_7. Denote these labels by p' and q', respectively. Then p is on l if and only if $p'+l'\in\{1,2,4\}$. If we replace the labels $\langle p,l\rangle$ in the theorem by $\langle p',l'\rangle$, and the labels $\langle l,p\rangle$ by $\langle\!\langle l',p'\rangle\!\rangle$ we get, for instance: $\langle p',l'\rangle$ represents a reflection point and $\langle\!\langle m',q'\rangle\!\rangle$ a bitangent through it if and only if $p'+l', p'+m', q'+l'\in\{1,2,4\}$ and $m'+q'\notin\{1,2,4\}$.

7. Steiner sets of bitangents of a smooth quartic curve

We give some classical results concerning the configuration of bitangents of a smooth quartic curve X in $P^2(C)$. A set of four distinct bitangents $l_1,\ldots,l_4$ of X whose eight tangent points are on a conic we call a coherent tetrad. Each X has 315 coherent tetrads. These define a relation between unordered pairs of distinct bitangents of X by: $lm\sim pq\leftrightarrow lmpq$ is a coherent tetrad. Remarkably this is an equivalence relation on the set of 378 unordered pairs of distinct bitangents. An equivalence class for this relation is called a Steiner set of pairs of bitangents; each Steiner set contains 6 pairs and there are 63 Steiner sets. The bitangents occurring in one Steiner set are all different. See [3], Section 185. We will determine the Steiner sets for the curve C of Section 1 and the H- and G_1-orbits in the set of coherent tetrads of C and in the set of Steiner sets. Observe that the data of the Steiner sets and of the coherent tetrads are equivalent: a tetrad is coherent if and only if it consists of two pairs from the same Steiner set. A triple is called coherent if it is formed from a pair of bitangents and a bitangent of another pair in the Steiner set containing the first pair. These triples are called syzygetic in [3], and are the edges of the regular 2-graph on 28 points, see Section 6 in [6] (from which we took the term 'coherent'; we return to this in the remarks in Section 9).

8. The coherent tetrads of bitangents of C.

Coxeter has observed that the following form coherent tetrads of bitangents of C:

(1) Four bitangents of C passing through one point. These tetrads are in 1–1 correspondence with the centres of involutions in H. A typical one is $\{0\infty, 16, 23, 45\}$, consisting of the pairs 0∞–16 and 23–45 which are connected in Γ; the remaining distances in the tetrad are 4. We shall say that these tetrads have a *distance pattern* $(1, 1, 4, 4, 4, 4)$; in this we write the distances of disjoint pairs of the tetrad next to each other.

(2) 2×7 tetrads in which all distances are equal to 4 ([2] Table 2 and Fig. 7). These 14 tetrads form one orbit under G_1, but two orbits under the action of H. Each unordered pair of distance 4 in Γ occurs in exactly one of these tetrads, and H has two orbits on these pairs; see Section 2.

(3) 28 tetrads consisting each of a vertex of Γ and its 3 neighbours ([2], p. 135). The distance pattern is $(1, 2, 1, 2, 1, 2)$.

To these we can add the following.

(4) Each of the 21 8-cycles in Γ gives rise to 2 tetrads with distance pattern $(2, 2, 2, 2, 4, 4)$, see Section 4. A tetrad with this distance pattern clearly is on a unique 8-cycle, so there are 42 of these tetrads. Take such a tetrad. It is the orbit of a Klein group K in H. Let p and q be the tangent points of one of its bitangents. Take a conic as in Lemma 4.1. It contains the images under K of p and q, which are the tangent points of the other bitangents of the tetrad. So the tetrad is coherent.

The tetrads of type (1) to (4) count for 105 tetrads. In none of these a pair with distance 3 occurs. Let us consider 3 more types of tetrads in which four pairs have distance 3 (i.e. contain a common symbol), to be constructed as follows. Take two disjoint pairs ab and cd. These give rise to the tetrad $\{ac, bd, ad, bc\}$. We distinguish three cases, according to whether $(a\,b\,c\,d) \in \{2, 4\}, \{3, 5\}$ or $\{6\}$.

(5) 84 tetrads with distance pattern $(1, 2, 3, 3, 3, 3)$; e.g., $\{01, 02, \infty 1, \infty 2\}$. Such a tetrad must consist of the endpoints of an edge m of Γ and two points each opposite to m in a 7-cycle, and at mutual distance 2. From Fig. 3 we see that for an edge m there are 4 points at distance 3 from each of its endpoints, forming 2 pairs with distance 2, corresponding to the pairs of 7-cycles that share precisely m. Observe that there are indeed 42×2 of such pairs of 7-cycles. We have one H-orbit, since the stabilizer of $\{\{0, \infty\}, \{1, 2\}\}$ in PSL(2, 7) has order 2. (Note: there also exist tetrads with distance pattern $(1, 3, 2, 3, 3, 3)$).

(6) 84 tetrads with distance pattern $(4,4,3,3,3,3)$; e.g., $\{01,03,\infty 1,\infty 3\}$. From Fig. 3 one can derive that for two points at distance 4 (there are 2 cases!) there are 4 points having distance 3 to both, forming 2 pairs with distance 4 (the other distances are 3). So there are indeed $\frac{1}{4}(28\times 6\times 2)=84$ tetrads with these distances. Note that the 4 points mentioned form again a tetrad of this kind; we thus have a set of 6 points consisting of 3 pairs with distance 4 while all other distances are 3. Since the stabilizer of $\{\{0,\infty\},\{1,3\}\}$ in PSL(2,7) has order 4, there are two H-orbits. There is only one G_1-orbit.

(7) 42 tetrads with distance pattern $(2,2,3,3,3,3)$; e.g., $\{01,06,\infty 1,\infty 6\}$. From Fig. 3 we see that for points x and y at distance 3 there are only 2 points z at distance 3 from x that have distance 2 from y. For only one choice of z there is a point at distance 2 from x and at distance 3 from y and z. The four points are the neighbours of an edge. So we have indeed 42 tetrads with this distance pattern, and clearly they are in one H-orbit.

It remains to show that tetrads of type (5), (6) or (7) are coherent. Note that we then have found a total of 315 coherent tetrads and thus have them all. Its suffices to give the proof for one tetrad out of every orbit.

Now $\{0\infty,16,23,45\}$ is of type (1) and $\{0\infty,45,13,26\}$ is of type (2), so the pairs $\{0\infty,45\}$, $\{16,23\}$ and $\{13,26\}$ are in the same Steiner set. Therefore, $\{16,13,26,23\}$ is coherent, and of type (6) since $(1\,2\,3\,6)=5$. Multiplication by 5, which is in $\mathrm{PGL}(2,7)\setminus\mathrm{PSL}(2,7)$, transforms the above tetrad of type (2) into $\{0\infty,46,15,23\}$. Since the pair $\{0\infty,23\}$ is also in the tetrad of type (1) above, $\{15,16,45,46\}$ is coherent. It is also of type (6) since $(1\,4\,5\,6)=5$, but in the other H-orbit since no cyclic shift transforms it into the first one.

Again we use $\{0\infty,16,23,45\}$ of type (1), now with $\{0\infty,16,25,34\}$ of type (3), to find $\{23,25,43,45\}$. It is of type (5), since $(2\,4\,3\,5)=2$.

Finally $\{0\infty,16,34,25\}$ and $\{0\infty,16,24,35\}$ of type (3) yield $\{24,25,34,35\}$. It is of type (7), since $(2\,3\,4\,5)=6$.

9. The types of Steiner sets of C

We have seen that all tetrads of type $\{ac,bc,ad,bd\}$ are coherent, of type (5), (6) or (7). It follows that for any a,b the 6 pairs $\{ax,bx\}$ with $x\neq a,b$ form a Steiner set. Clearly there are 28 of these sets, all equivalent under H. Note that all pairs have distance 3. The 12 points of Γ that are involved are those at distance 3 from the point ab.

In the 105 tetrads of type (1), (2), (3) or (4) distance 3 does not occur. Therefore, they are of type $\{ab,cd,ef,gh\}$ with $a,\ldots,h$ all different. Since $105=\frac{1}{4!}\binom{8}{2}\binom{6}{2}\binom{4}{2}$ all tetrads involving the eight elements $0,\ldots,6,\infty$ are coherent. Thus for every partition $\{a,b,c,d\}\cup\{e,f,g,h\}$ of the set of these elements we find a Steiner set

$$\{\{ab,cd\},\{ac,bd\},\{ad,bc\},\{ef,gh\},\{eg,fh\},\{eh,fg\}.$$

If a, b, c, d or e, f, g, h is, in some order, a harmonic quadruple, we can map it (in that order) onto $0, \infty, 1, 6$ or $0, \infty, 6, 1$ by an element of PSL(2, 7). So under H the Steiner set is equivalent to

$$\{\{0\infty, 16\}, \{01, 6\infty\}, \{06, 6\infty\}, \{23, 45\}, \{24, 35\}, \{25, 34\}\},$$

and the orbit has length 21, since there are 42 harmonic quadruples. Note that the distances in the pairs are $1, 2, 2, 1, 2, 2$. The twelve points of Γ involved are the 4 endpoints of two antipodal edges and their 8 neighbors.

In the remaining cases we can, mod PSL(2, 7), take $a=0$, $b=\infty$. We can take $c=1$ if c or d is a square, and since $(0\ \infty\ 1\ 6)=(0\ 1\ \infty\ 4)=(1\ \infty\ 0\ 2)=-1$ we can then take $d=3$ or 5. If both c and d are nonsquares we can take $c=3$, $d=5$ or 6. But $(03\ \infty\ 5)=(3\ \infty\ 06)=-1$. We are left with the cases where $\{a, b, c, d\}=\{0, \infty, 1, 3\}$ or $=\{0, \infty, 1, 5\}$. The corresponding Steiner sets are

$$\{\{0\infty, 13\}, \{01, 3\infty\}, \{03, 1\infty\}, \{24, 56\}, \{25, 46\}, \{26, 45\}\},$$

$$\{\{0\infty, 15\}, \{01, 5\infty\}, \{05, 1\infty\}, \{24, 36\}, \{23, 46\}, \{26, 34\}\}.$$

Note that all distances in the pairs are 4. Since $(0\infty 13)=5=(0\infty 51)$ there is equivalence under G_1. Now by the map $z \mapsto z/(z+4)$ from PSL(2, 7) we see that in the first set the first 3 pairs as well as the second 3 pairs are equivalent, and by $z \mapsto (2+2z)/(1+4z)$ that all 6 pairs are equivalent. Likewise for the second set, by $z \mapsto z/(z+2)$ and $z \mapsto (2+4z)/(1+z)$. So the sets could only be equivalent under H if $\{0\infty, 13\}$ and $\{0\infty, 15\}$ are equivalent. Now $0, \infty, 1, 3$ in this order becomes $\infty, 0, 3, 1$ by $z \mapsto 3/z$, and fixing 0 we can cyclically permute $\infty, 1, 3$ by the above $z \mapsto z/(z+4)$. So equivalence would imply that there is a map in PSL(2, 7) that maps $0, \infty, 1, 5$ in this order onto $0, \infty, 1, 3$ or onto $0, \infty, 3, 1$. Both are impossible. Finally, the stabilizer of $\{\{0, \infty, 1, 3\}, \{2, 4, 5, 6\}\}$ in PSL(2, 7) has order 24, so the orbit of the first Steiner set has order 7, and so has that of the second one, there being 63 Steiner sets. We summarize.

Theorem 9.1. *Under H the 63 Steiner sets fall apart in: an orbit of length 28 with distance pattern $3, 3, 3, 3, 3, 3$, an orbit of length 21 with distance pattern $1, 1, 2, 2, 2, 2$ and two orbits of length 7 each with distance pattern $4, 4, 4, 4, 4, 4$.*

Remarks. Without reference to $P^1(F_7)$ Dickson [3] mentions a labelling of the bitangents by the 28 pairs from $\{1,2,3,4,5,6,7,8\}$, due to Hesse and Cayley, in such a way that the Steiner sets are as above, with $0, 1, \ldots, 6, \infty$ replaced by $1, 2, .., 7, 8$. In his section 186 he shows, using combinatorial properties of the Steiner sets, that to prove that the Steiner sets are as given above, it is sufficient to prove for five particularly chosen sets that they are Steiner sets. That same labelling is used in [6] to show that the triples called 'syzygetic' in [3] are the edges of a regular 2-graph. Of course it helped in finding the cases (5), (6) and (7) in Section 8.

10. Aronhold sets

An Aronhold set is a set of seven bitangents with the property that no three of them form a coherent triple (geometrically they are of interest since such a set determines the quartic curve). They are described in [3], Section 186, in terms of the labelling mentioned in the remarks in Section 9, as follows. Coherent triples have the form $\{xy, yz, zw\}$ or $\{uv, wx, yz\}$, with u, v, w, x, y, z all different (We have seen this in the beginning of Section 9.) It follows that an Aronhold set has one of the forms $\{\{ax\} \mid x \neq a\}$, or $\{ab, ac, ad, ae, fg, gh, hf\}$, and that indeed 7 is the largest cardinality a set with our property can have.

Now the 8 sets of the first type clearly form one orbit under H. In Γ they are the sets of 7 points at mutual distance 3 (see the black points in Fig. 3). As to the second type: mod H we can suppose that $\{f, g, h\} = \{0, \infty, 1\}$, which is left invariant by the maps id., $z \mapsto 1/(1-z)$ and $z \mapsto 1 - 1/z$. So the sets in which $a = 2$, 4 or 6 are equivalent, with a stabilizer of order 1; the orbit has length 168. The set with $a = 3$ and that with $a = 5$ are not equivalent, and each has an orbit of length 56. Then Theorem 10.1 follows.

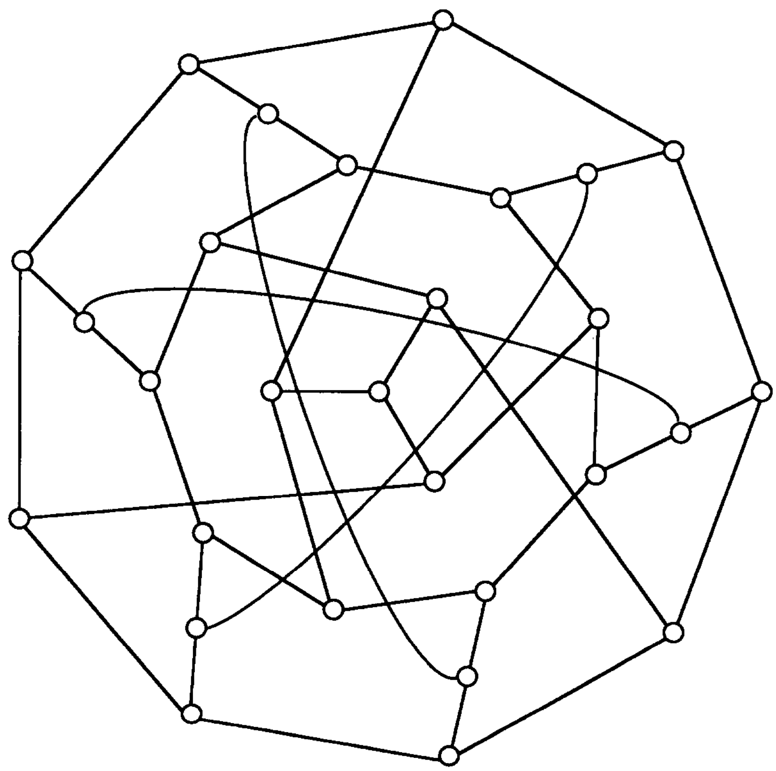

Fig. 5.

Theorem 10.1. *The* 288 *Aronhold sets fall apart into* 4 *orbits under* H, *of lengths* $8, 168, 56$ *and* 56.

11. Intrinsic connection with the Fano plane

As we observed above, the tetrads of type (2) lie in two H-orbits. Reversing the process in Section 1 we regain $\boldsymbol{P}^2(\boldsymbol{F}_2)$ in the following way. Call the tetrads in one H-orbit 'points' and the other ones 'lines'. These are indicated in Fig. 7 of [2] as ○ and ● respectively. Moreover this Fig. 7 contains the connections between tetrads which have nonempty intersection. Let us define an incidence relation between 'points' and 'lines' as disjointness of the corresponding tetrads. Then the resulting geometry P is isomorphic to $\boldsymbol{P}^2(\boldsymbol{F}_2)$. Moreover there is a bijection between the set of vertices of Γ and the set of pairs (p, l) with p a point, l a line of P and $p \notin l$. Observe that the complement of $l \cup \{p\}$ in P is just a set of three noncollinear points. This gives a natural bijection between Γ and the set of triangles in P, such that two adjacent vertices in Γ are mapped to disjoint triangles! The group G_1 appears as the union of the collineations and the correlations of P.

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