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Neighborhood intersections and Hamiltonicity in almost claw-free graphs

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Abstract

Let G be a graph. The partially square graph G^* of G is a graph obtained from G by adding edges uv satisfying the conditions $uv \notin E(G)$, and there is some $w \in N(u) \cap N(v)$, such that $N(w) \subseteq N(u) \cup N(v) \cup \{u, v\}$. Let $t > 1$ be an integer and $Y \subseteq V(G)$, denote $n(Y) = |\{v \in V(G) \mid \min_{y \in Y}\{\mathrm{dist}_G(v, y)\} \leqslant 2\}|$, $I_t(G) = \{Z \mid Z$ is an independent set of $G, |Z| = t\}$. In this paper, we show that a k-connected almost claw-free graph with $k \geqslant 2$ is *hamiltonian* if $\sum_{z \in Z} d(z) \geqslant n(Z) - k$ in G for each $Z \in I_{k+1}(G^*)$, thereby solving a conjecture proposed by Broersma, Ryjáček and Schiermeyer. Zhang's result is also generalized by the new result. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Hamiltonian graphs; Almost claw-free graphs; Claw center; The global insertion

1. Introduction

In this paper, we consider only finite, undirected graphs $G = (V, E)$ of order n without loops or multiple edges. We use the notations and terminology in [4]. The independence number of G and its subgraph induced by $A \subseteq V(G)$ are, respectively, denoted by $\alpha(G)$ and $G[A]$. $G + H$ denotes the union of vertex-disjoint graphs G and H. The join of vertex-disjoint graphs G and H is denoted by $G \vee H$. If A, H are subsets of $V(G)$ or subgraphs of G, we denote by $N_H(A)$ the set of vertices in H which are adjacent to some vertex in A. For simplicity, we adopt $N(A)$ if $H = G$. The open neighborhood, the closed neighborhood and the degree of vertex v are, respectively, denoted by $N(v) = \{u \in V(G) \mid uv \in E(G)\}$, $N[v] = N(v) \cup \{v\}$ and $d(v) = |N(v)|$. $\delta(G)$ denotes the minimum degree of G. A dominating set of G is a subset S of $V(G)$ such that every vertex of G belongs to S or is adjacent to a vertex of S. The domination number, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. To each pair (u, v) of vertices at distance 2, we associate the set $J(u, v) = \{w \in N(u) \cap N(v) \mid N(w) \subseteq N[u] \cup N[v]\}$. Let $Z \subseteq V(G)$, $|Z| = p$, and $t > 1$ be an integer. Put,

$$S_i(Z) = \{v \in V(G) \mid |N(v) \cap Z| = i\}, \quad s_i(Z) = |S_i(Z)| \quad \text{for } i = 0, 1, \ldots, p$$

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and

$$I_t(G)=\{Y \mid Y \text{ is an independent set of } G, |Y|=t\}.$$

Let G be connected and $Z \subseteq V(G)$. Denote

$$N_i(Z)=\left\{v \in V(G) \mid \min_{z \in Z}\{\mathrm{dist}_G(v,z)\}=i\right\} \quad (i=0,1,2,\ldots)$$

and

$$n(Z)=|N_0(Z)\cup N_1(Z)\cup N_2(Z)|=\left|\left\{v \in V(G) \mid \min_{z \in Z}\{\mathrm{dist}_G(v,z)\}\leqslant 2\right\}\right|,$$

where $\mathrm{dist}_G(v,z)$ stands for the distance between v and z in G.

A claw in a graph is an induced subgraph $G[\{u,x,y,z\}]$ isomorphic to $K_{1,3}$ in which the vertex u of degree 3 is called claw-center. A graph is claw-free if it does not contain a claw as an induced subgraph.

Definition 1.1 (*Ryjáček* [10]). A graph G is almost claw-free if there exists an independent set $A \subseteq V(G)$ such that $\alpha(G[N(v)])\leqslant 2$ for every $v \notin A$ and $\gamma(G[N(v)])\leqslant 2 < \alpha(G[N(v)])$ for every $v \in A$.

Definition 1.2 (*Ainouche and Kouider* [3]). The partially square graph G^* of G is a graph satisfying $V(G^*)=V(G)$ and $E(G^*)=E(G)\cup\{uv \mid uv \notin E(G)$, and $J(u,v)\neq\emptyset\}$.

Definition 1.3. The square graph G^2 of G is a graph satisfying $V(G^2)=V(G)$ and $E(G^2)=E(G)\cup\{uv \mid \mathrm{dist}_G(u,v)=2\}$.

2. Properties

Property 2.1 (Ryjáček [10]). Every almost claw-free graph is $K_{1,5}$-free and $K_{1,1,3}$-free.

Property 2.2. Let G be an almost claw-free graph and G^* its partially square graph. Then $s_3(Z)=0$ in G for each $Z \in I_3(G^*)$.

Proof. By contradiction. Suppose that $Z=\{u_1,u_2,u_3\}$ is an independent set of G^*, and $uu_i \in E(G)$ $(i=1,2,3)$. Clearly, $\{u_1,u_2,u_3\}$ is independent in G and u is a claw center. Note that G is almost claw-free. There exists some vertex $w \in N(u)\setminus\{u_1,u_2,u_3\}$ dominating two vertices of $\{u_1,u_2,u_3\}$. We assume w dominates u_1 and u_2. By the definition of G^* and $u_1u_2 \notin E(G^*)$, we have $J(u_1,u_2)=\emptyset$. Then there must exist some vertex $u_4(\notin\{u_1,u_2,u\})$ in $N(w)$ such that $u_4 \notin N[u_1]\cup N[u_2]$. Thus, $G[\{w,u_1,u_2,u_4\}]\cong K_{1,3}$. However, $wu \in E(G)$ and u is a claw center, a contradiction.

Clearly, Property 2.2 is equivalent to saying that G^* is claw-free if G is almost claw-free.

Property 2.3. Let G be an almost claw-free graph and G^* its partially square graph. Then $s_2(Z)\leqslant 2$ in G for each $Z \in I_2(G^*)$. Moreover, if $S_2(Z)=\{u_1,u_2\}$, we have $u_1u_2 \notin E(G)$.

Proof. By contradiction. Suppose that $Z=\{w_1,w_2\}$ is independent in G^* and $\{u_1,u_2,u_3\}\subseteq S_2(Z)$. By the definition of G^* and $w_1w_2\notin E(G^*)$, it is easy to see that u_1,u_2,u_3 are claw centers in G. Then, $\{u_1,u_2,u_3\}$ is independent in G and $G[\{w_1,u_1,u_2,u_3\}]\cong K_{1,3}$, a contradiction. Hence $s_2(Z)\leqslant 2$ in G for each $Z\in I_2(G^*)$. Moreover, if $S_2(Z)=\{u_1,u_2\}$, it is not difficult to get $u_1u_2\notin E(G)$.

3. Hamiltonicity

The following results on claw-free graphs are known.

Theorem 3.1 (Matthews and Sumner [9]). *A 2-connected claw-free graph G is hamiltonian if* $\delta(G)\geqslant\frac{1}{3}(n-2)$.

Theorem 3.2 (Broersma [5], Liu and Tian [8]). *Let G be a 2-connected claw-free graph. If* $\sum_{z\in Z}d(z)\geqslant n-2$ *for each* $Z\in I_3(G)$, *then G is hamiltonian.*

Zhang generalized Theorem 3.2 to k-connected claw-free graphs for any positive integer $k\geqslant 2$ as follows.

Theorem 3.3 (Zhang [12]). *Let G be a k-connected claw-free graph with* $k\geqslant 2$. *If* $\sum_{z\in Z}d(z)\geqslant n-k$ *for each* $Z\in I_{k+1}(G)$, *then G is hamiltonian.*

Theorem 3.3 was extended by the following Theorem.

Theorem 3.4 (Ainouche and Broersma [1]). *If G is a k-connected claw-free graph* $(k\geqslant 2)$ *with* $\alpha(G^2)\leqslant k$, *then G is hamiltonian.*

Ainouche and Kouider in [3], considered the independence number of partially square graphs and proved the following.

Theorem 3.5 (Ainouche and Kouider [3]). *Let G be a k-connected graph with* $k\geqslant 2$ *and* G^* *its partially square graph. If* $\alpha(G^*)\leqslant k$, *then G is hamiltonian.*

Our objective is to generalize results on claw-free graphs to almost claw-free graphs. Following are some results on hamiltonicity in almost claw-free graphs.

Theorem 3.6 (Broersma et al. [6]). *A 2-connected almost claw-free graph G is hamiltonian if* $\delta(G)\geqslant\frac{1}{3}(n-2)$.

Theorem 3.7 (Broersma et al. [6]). *Let G be a 2-connected almost claw-free graph. If* $\sum_{z\in Z}d(z)\geqslant n$ *for each* $Z\in I_3(G)$, *then G is hamiltonian.*

Broersma et al. in [6] conjectured that $\sum_{z\in Z}d(z)\geqslant n-2$ for each $Z\in I_3(G)$ implies hamiltonicity in 2-connected almost claw-free graphs. This conjecture was verified for

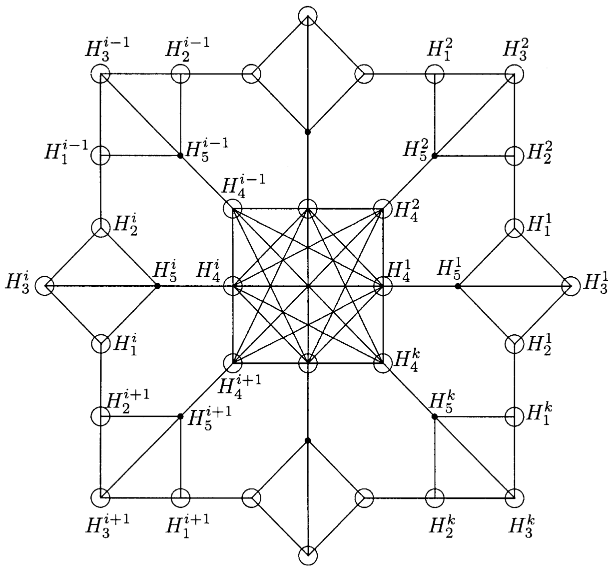

Fig. 1. A k-connected almost claw-free hamiltonian graph.

$n \geqslant 79$ by Li and Tian [7], and proved in [2] for another class containing the class of almost claw-free graphs.

In this paper, we will prove the following result.

Theorem 3.8. *Let G be a k-connected almost claw-free graph with $k \geqslant 2$, and G^* its partially square graph. If $\sum_{z \in Z} d(z) \geqslant n(Z) - k$ in G for each $Z \in I_{k+1}(G^*)$, then G is hamiltonian.*

Clearly, Theorem 3.8 is best possible, it modifies and generalizes Theorems 3.3, 3.6 and 3.7. Of course, it solves the conjecture proposed by Broersma et al. [6].

Now, for $k \geqslant 2$, we construct a graph G_k as follows (see Fig. 1). Let $H_j^i \cong K_k$, $H_5^i \cong K_1$, where $i = 1, 2, \ldots, k$, $j = 1, 2, 3, 4$. Let $H^i = (((H_1^i + H_2^i) \vee H_3^i) + H_4^i) \vee H_5^i$, and $V(H^1)$, $V(H^2), \ldots, V(H^k)$ be pairwise vertex-disjoint. Set

$$V(G_k) = \bigcup_{i=1}^{k} V(H^i),$$

$$E(G_k) = \bigcup_{i=1}^{k} E(H^i) \cup \left(\bigcup_{t=1}^{k-1} E(H_1^t \vee H_2^{t+1}) \cup E(H_1^k \vee H_2^1) \right)$$
$$\cup\ E(H_4^1 \vee H_4^2 \vee H_4^3 \vee \cdots \vee H_4^k).$$

Obviously, G_k is a k-connected almost claw-free hamiltonian graph which is not claw-free, and $\sum_{z \in Z} d(z) = 4k^2 = n(Z) - k$ in G for each $Z \in I_{k+1}(G^*)$. G_k shows that it is meaningful to find the sufficient condition for the hamiltonicity of almost claw-free graphs. On the other hand, for $G = G_k (k \geqslant 2)$, we have $n = 4k^2 + k$, and $\delta(G) < (n-2)/3$, therefore G_k doesn't satisfy the condition of Theorem 3.6.

To prove Theorem 3.8, we will relate in Section 4 the concept of global insertion introduced in [2], and use the global insertion Lemma 4.1 to prove some new lemmas. The proof of Theorem 3.8 is given in Section 5.

4. The global insertion concept

Let G be a k-connected non-hamiltonian graph and C its a maximal cycle of G (that is, there is no cycle C' in G such that $V(C) \subset V(C')$), in the sense of the vertex inclusion, in which an orientation is fixed. For simplicity, we use the same notation to mean a subgraph, its vertex set or its edge set. If $x \in V(C)$, denote by x^+ and x^- the successor and the predecessor of x along the orientation of C, respectively. Set $x^{++} = (x^+)^+$, $x^{--} = (x^-)^-$. If $u, v \in V(C)$, then $C[u,v]$ denotes the consecutive vertices on C from u to v in the chosen direction of C, and $C(u,v] = C[u,v] - \{u\}$, $C[u,v) = C[u,v] - \{v\}$, $C(u,v) = C[u,v] - \{u,v\}$. The same vertices, in the reverse order, are, respectively, denoted by $\bar{C}[v,u]$, $\bar{C}[v,u)$, $\bar{C}(v,u]$ and $\bar{C}(v,u)$. Let H be a component of $G - V(C)$. Assume, $N_C(H) = \{v'_1, v'_2, \ldots, v'_m\}$ with $m \geqslant k \geqslant 2$, and $v'_1, v'_2, \ldots, v'_m$ occur on C in the order of their indices. The subscripts will be taken modulo m. Let $\{x,y\} \subseteq N_C(H)$. We denote by xHy one of the longest (x,y)-paths with all its internal vertices in H.

In [2], a relation $\sim$ on $V(C)$ is defined by the condition $u \sim v$ if there exists a path with endpoints u, v and no internal vertex in C. Such a path is called a connecting path between u and v and is denoted by uRv, where $R := V \setminus V(C)$. Note that if one of u or v is not a vertex v'_i, any connecting path uRv is disjoint from H. If x, y, t, z are distinct vertices of C such that $z \in \{t^+, t^-\}$, $x \sim t$, $y \sim z$, then the paths xRt and yRz are said to be crossing at x, y if either ($z = t^+$ and $t \in C[y^+, x^{--}]$) or ($z = t^-$ and $t \in C[x^{++}, y^-]$).

Definition 4.1 (*Ainouche et al.* [2]). For all $i \in \{1,2,\ldots,m\}$, a vertex $u \in C(v'_i, v'_{i+1})$ is called globally path insertible (GPI for short) if

(i) each vertex in $C(v'_i, u)$ is GPI or $u = (v'_i)^+$;
(ii) there exist $w, w^+ \in C[v'_{i+1}, v'_i]$ and $v \in C(v'_i, u]$ (possibly $v = u$) such that either ($u \sim w, v \sim w^+$) or ($u \sim w^+, v \sim w$).

Note that $u \sim v$ if $uv \in E(G)$. By replacing the connecting path with the edge in (ii), Wu et al. independently introduced the T-insertion concept in [11]. Clearly, T-insertion concept is a special case of the global edge insertion.

Let u be a GPI vertex in $C(v'_i, v'_{i+1})$ ($i \in \{1,2,\ldots,m\}$). By the technique in [2] we can insert the vertices of $C(v'_i, u]$ into the path $C[v'_{i+1}, v'_i]$. Consider that $u_1 = u, v_{u_1}$

and the insertion edge $w_{u_1}w_{u_1}^+$; $u_2 = v_{u_1}^-, v_{u_2}$ and the insertion edge $w_{u_2}w_{u_2}^+$; $\cdots$; $u_s = v_{u_{s-1}}^-, v_{u_s} = (v_i')^+$ and the insertion edge $w_{u_s}w_{u_s}^+$, where for $j \in \{1,2,\ldots,s\}, v_{u_j}$ is the first vertex in $C(v_i', u_j]$ such that (ii) holds, and based on this, $w_{u_j}w_{u_j}^+$ is the first edge in $C[v_{i+1}', v_i']$ such that (ii) holds. By the choice above, $w_{u_1}w_{u_1}^+, w_{u_2}w_{u_2}^+, \ldots, w_{u_s}w_{u_s}^+$ are different from each other. Similarly, the paths connecting the vertices w_{u_j} and $w_{u_j}^+$ to the vertices u_j and v_{u_j} are all pairwise internally disjoint by the choice of the v_{u_j}'s and the maximality of C. Thus, in the path $C[v_{i+1}', v_i']$, replace the edge $w_{u_j}w_{u_j}^+$ ($j \in \{1,2,\ldots,s\}$) by the path $C[v_{u_j}, u_j]$ or $\bar{C}[u_j, v_{u_j}]$, for the resulting (v_{i+1}', v_i')-path P_u, we have $V(P_u) = C[v_{i+1}', u]$. $C(v_i', u]$ is therefore inserted into the path $C[v_{i+1}', v_i']$. Denote $E(u) = \{w_{u_1}w_{u_1}^+, w_{u_2}w_{u_2}^+, \ldots, w_{u_s}w_{u_s}^+\}$, and call it the inserted edge set of $C(v_i', u]$.

Lemma 4.1 (Ainouche et al. [2]). *For all $i \in \{1,2,\ldots,m\}$, let x_i' be the first vertex on $C(v_i', v_{i+1}')$ along C which is not globally path insertible. Then*

(a) *For each i, x_i' exists.*
Set $X' = \{x_0', x_1', \ldots, x_m'\}$ with $x_0' \in V(H)$.
(b) *For $1 \leqslant i \neq j \leqslant m$ and for any $u_i \in C(v_i', x_i']$ and $u_j \in C(v_j', x_j']$, $u_i \nsim u_j$ and there are no crossing paths at u_i, u_j.*
(c) *X is independent.*
(d) *For $0 \leqslant i \neq j \leqslant m$, $J(x_i', x_j') = \emptyset$. In particular any common neighbor of at least two vertices of X' must be a claw-center.*

In the rest of this paper, we pick up $\{v_1, v_2, \ldots, v_k\} \subseteq \{v_1', v_2', \ldots, v_m'\}$. The subscripts of $(v_i)'s$ will be taken modulo k. For each $i \in \{1,2,\ldots,k\}$, let x_i be the first non-GPI in $C(v_i, v_{i+1})$. Set $X = \{x_0, x_1, \ldots, x_k\}$, where $x_0 \in V(H)$. Denote $J_X = \bigcup_{i=1}^{k} C[x_i, v_{i+1}]$, $K_X = V(G) \backslash J_X$. For $X = \{x_0, x_1, \ldots, x_k\}$, let $C[z_1, z_2) \subseteq C[x_t, v_{t+1}]$ ($t \in \{1,2,\ldots,k\}$). If $C(z_1, z_2) \cap S_0(X) = \emptyset$, and $z_1 \in N_2(X) \cup X$, $z_2 \in S_0(X) \cup \{v_{t+1}^+\}$, then $C[z_1, z_2)$ is called a CX-segment. A CX-segment $C[z_1, z_2)$ is called simple if $C(z_1, z_2) \subseteq S_1(X)$. By Lemmas 4.1(b)–(d) and the maximality of C, the following Lemma holds.

Lemma 4.2 (Wu et al. [11]). (a) *If u is a GPI, then $u^+ \notin N_C(H)$.*

(b) *$X \in I_{k+1}(G^*)$, $K_X \subseteq S_0(X) \cup S_1(X)$, $K_X \cap N_0(X) = \{x_0\}$.*

(c) *If $u \in N_C(H) \backslash \{v_1, v_2, \ldots, v_k\}$, $y \in \bigcup_{j=1}^{k} C(v_j, x_j]$, then $u^+ y \notin E(G)$.*

(d) *Let, $C[z_1, z_2)(\subseteq C[x_t, v_{t+1}], t \in \{1,2,\ldots,k\})$ be a CX-segment, and $M_i = N(x_i) \cap C(z_1, z_2)$ ($i \in \{0,1,\ldots,k\}$). Then $M_t, M_{t-1}, \ldots, M_1, M_k, M_{k-1}, \ldots, M_{t+1}, M_0$ (some of them may be empty) form consecutive subpaths of $C(z_1, z_2)$ which can have only their end-vertices in common, and $|M_i| \leqslant 1$, $i \in \{0,1,2,\ldots,k\} \backslash \{t\}$.*

Lemma 4.3.

$$\sum_{i=0}^{k} |N(x_i) \cap K_X| \leqslant |K_X| - 1 - \sum_{l>2} |N_l(X) \cap K_X|.$$

Proof. By Lemma 4.2(b), we have $K_X \subseteq S_0(X) \cup S_1(X)$, and $K_X \cap N_0(X) = \{x_0\}$. Then

$$\begin{aligned}\sum_{i=0}^{k} |N(x_i) \cap K_X| &= |S_1(X) \cap K_X| = |N_1(X) \cap K_X| \\ &= |K_X| - |N_0(X) \cap K_X| - \sum_{l>2} |N_l(X) \cap K_X| - |N_2(X) \cap K_X| \\ &\leqslant |K_X| - 1 - \sum_{l>2} |N_l(X) \cap K_X|.\end{aligned}$$

Lemma 4.4. *If $C[z_1,z_2)$ is a simple CX-segment, then* $\sum_{i=0}^{k} |N(x_i) \cap C[z_1,z_2)| = |C[z_1,z_2)| - 1$.

Proof. By the definition of a simple CX-segment, it is easy to see that

$$\sum_{i=0}^{k} |N(x_i) \cap C[z_1,z_2)| = |S_1(X) \cap C[z_1,z_2)| = |C(z_1,z_2)| = |C[z_1,z_2)| - 1.$$

Now, we assume that G is almost claw-free, and A is the set of all claw centers. By the definition of an almost claw-free graph, A is independent. By Lemma 4.1(d), we have the following:

Lemma 4.5. *For any $\{i,j\} \subseteq \{0,1,\ldots,k\}$, we have $N(x_i) \cap N(x_j) \subseteq A$. Therefore, for any $i \in \{1,2,\ldots,k\}$, $N(x_i) \cap N_C(H) \subseteq A$.*

Lemma 4.6. *For any $t \in \{3,4,\ldots,k+1\}$, we have $S_t(X) = \emptyset$. Therefore, for any $\{i,j\} \subseteq \{1,2,\ldots,k\}$, $N(x_i) \cap N(x_j) \cap N_C(H) = \emptyset$.*

Proof. By Lemma 4.2(b), $X \in I_{k+1}(G^*)$. Then the result directly follows from Property 2.2.

Lemma 4.7. *For any $i \in \{1,2,\ldots,k\}$, $N(x_0) \cap N(x_i) \subseteq \{v_i\}$.*

Proof. By contradiction. Suppose that there exists some vertex $w \in N(x_0) \cap N(x_i) \backslash \{v_i\}$. Then we have $w \in A$ by Lemma 4.5, and $w \in C(x_i, v_i^-)$ by Lemma 4.2(a), (b) and the definition of x_i. Consider $\{w^-, w^+, x_0\} \cup \{x_i\} (\subseteq N(w))$. It is clear that x_0 and x_i have no common neighbor in $N(w)$. Suppose first that x_0 and w^+ have a common neighbor v in $N(w)$. By the maximality of C, it is easy to see that $v \in V(C)$, $v \notin \{w^+, w^-\}$ and $v^+x_0, v^-x_0 \notin E(G)$. By $w \in A$, we have $v \notin A$, then $v^+v^- \in E(G)$. Thus, the cycle $C_1 = C[w^+, v^-]C[v^+, w)wHvw^+$ in G contains C, a contradiction. Hence x_0 and w^+ have no common neighbor in $N(w)$. By symmetry, x_0 and w^- have no common neighbor in $N(w)$.

Since G is almost claw-free and $w \in A$, we have $\gamma(G[N(w)]) \leqslant 2$. Then there exists some vertex $u \in N(w)$ dominating $\{w^+, w^-\} \cup \{x_i\}$. Clearly, $u \in V(C)$. By Lemmas 4.1(b), 4.2(c), we have $u \in C(x_i, v_i] \backslash \{w\}$. By $w \in A$ and $uw \in E(G), u \notin A$. We will consider three cases.

Case 1: $u = v_i$.

Note that $u \in N(x_i) \cap N_C(H)$. By Lemma 4.5, we have $u \in A$, a contradiction.

Case 2: $u \in C(x_i, w)$.

In fact, $u^+ x_i \notin E(G)$ (Otherwise, suppose that $u^+ x_i \in E(G)$. Note that since $\{w, u\} \subseteq N(x_i)$, we have $\{uu^+, ww^+\} \cap E(x_i^-) = \emptyset$, that is all the inserted edges of vertices in $C(v_i, x_i)$ are not in $\{uu^+, ww^+\}$. Then the vertices of $C(v_i, x_i)$ can be inserted into the cycle $C[u^+, w)\, wHv_i\, \bar{C}(v_i, w^+]\bar{C}[u, x_i]u^+$. Denote by C_2 the resulting cycle. Clearly, $V(C) \subset V(C_2)$, a contradiction). Thus $u \neq w^-$. Moreover, $u^+ w^+ \in E(G)$ (Otherwise $u^+ w^+ \notin E(G)$. By Lemmas 4.1(b) and 4.2(c), $x_i w^+ \notin E(G)$. Then $G[\{u, u^+, w^+, x_i\}] \cong K_{1,3}$, a contradiction). It is easy to see that $\{ww^+, ww^-, uu^+\} \cap E(x_i^-) = \emptyset$ by the definition of x_i and $wx_i, ux_i \in E(G)$. Then the vertices of $C(v_i, x_i)$ can be inserted into the cycle $C[x_i, u]\bar{C}[w^-, u^+]C[w^+, v_i)v_i Hwx_i$. Let C_3 denote the resulting cycle, we have $V(C) \subset V(C_3)$, a contradiction.

Case 3: $u \in C(w, v_i)$.

In fact, $u^+ w^+ \notin E(G)$ (If not, we assume $u^+ w^+ \in E(G)$. Note that $\{u, w\} \subseteq N(x_i)$, we have $\{ww^+, uu^+\} \cap E(x_i^-) = \emptyset$. Then the vertices of $C(v_i, x_i)$ can be inserted into the cycle $C[w^+, u]C[x_i, w)\, wHv_i\, \bar{C}(v_i, u^+]w^+$. Denote by C_4 the resulting cycle, we have $V(C) \subset V(C_4)$, a contradiction). Moreover, $x_i u^+ \notin E(G)$ (Otherwise, it is easy to see that the vertices of $C(v_i, x_i)$ can be inserted into the cycle $C[x_i, w^-]\bar{C}[u, w)\, wHv_i$ $\bar{C}[v_i, u^+]x_i$. Let C_5 denote the resulting cycle. Clearly, $V(C) \subset V(C_5)$, a contradiction). By Lemmas 4.1(b) and 4.2(c), we have $x_i w^+ \notin E(G)$. Thus, $G[\{u, u^+, w^+, x_i\}] \cong K_{1,3}$, a contradiction.

Lemma 4.8. *Let,* $C[z_1, z_2)\, (\subseteq C[x_t, v_{t+1}], t \in \{1, 2, \ldots, k\})$ *be a CX-segment. Then the following statements hold.*

(1) *For any* $i \in \{3, 4, \ldots, k\}$, $|C[z_1, z_2) \cap S_i(X)| = 0$.

(2) *If* $u \in S_2(X) \cap C(z_1, z_2)$, $y \in C(u, z_2)$, *then* $yu \in E(G)$. *Therefore,* $y \notin A$ *and* $|C(z_1, z_2) \cap S_2(X)| \leqslant 1$.

(3) $\sum_{i=0}^{k} |N(x_i) \cap C[z_1, z_2)| \leqslant |C[z_1, z_2)|$.

Proof. (1) It is easy to see that (1) holds by Lemma 4.6.

(2) By contradiction. Suppose that y is the first vertex in $C(u, z_2)$ nonadjacent to u. Then $yu \notin E(G)$, $y \neq u^+$ and $y^- u \in E(G)$. Note that $u \in S_2(X) \cap C(z_1, z_2)$. By the definition of CX-segment, we set $y^- \in N(x_j)(j \neq t)$. It is clear that $x_j \notin N(u) \cup N(y)$ by Lemma 4.2(d). Thus $G[\{y^-, y, x_j, u\}] \cong K_{1,3}$, $y^- \in A$. By Lemma 4.5, $u \in A$. This contradicts $y^- u \in E(G)$. Hence $yu \in E(G)$.

(3) By (1) and (2), we have

$$\begin{aligned}\sum_{i=0}^{k} |N(x_i) \cap C[z_1, z_2)| &= |S_1(X) \cap C(z_1, z_2)| + 2|S_2(X) \cap C(z_1, z_2)| \\ &= |C(z_1, z_2)| + |S_2(X) \cap C(z_1, z_2)| \leqslant |C[z_1, z_2)|.\end{aligned}$$

Lemma 4.8 holds.

For $t \in \{1,2,\dots,k\}$, $C[x_t, v_{t+1}] \setminus \bigcup_{l>2} N_l(X)$ can be divided into some disjoint CX-segments. Assume that all these CX-segments are $C[z_{11}, z_{12})$, $C[z_{21}, z_{22}), \dots,$ $C[z_{m1}, z_{m2})$, occurring consecutively along the direction of the path $C[x_t, v_{t+1}]$. It is possible to have $z_{j2} = z_{j+1,1}$ for some j.

Lemma 4.9. *If $v_{t+1} \notin S_2(X)$, then $C[z_{m1}, z_{m2})$ is simple.*

Proof. Suppose, to the contrary, that $C[z_{m1}, z_{m2})$ is not simple. By Lemma 4.8(2), $|C(z_{m1}, z_{m2}) \cap S_2(X)| = 1$. Let $w \in C(z_{m1}, z_{m2}) \cap S_2(X)$. Then $w \neq v_{t+1}$. By Lemma 4.5, $w \in A$. To get the contradiction, we first show two claims.

Claim 1. $v_{t+1} \in C[z_{m1}, z_{m2})$, *and* $v_{t+1}x_0, wv_{t+1}^+ \in E(G)$.

Suppose that $v_{t+1} \notin C[z_{m1}, z_{m2})$. By the definition of CX-segment and $|C(z_{m1}, z_{m2}) \cap S_2(X)| = 1$, we have $z_{m2} \in N_2(X)$. Then $C[z_{m2}, v_{t+1}]$ still has some other CX-segments. This contradicts the assumption.

By $v_{t+1} \notin S_2(X)$, $w \in C(z_{m1}, z_{m2}^-) \cap S_2(X)$. By Lemma 4.8(2), $wv_{t+1} \in E(G)$ and $v_{t+1} \notin A$. Then $v_{t+1}x_0 \in E(G)$(Otherwise, suppose that $v_{t+1}x_0 \notin E(G)$. Since $v_{t+1} \in S_1(X)$, we set $v_{t+1}x_j \in E(G)$, where $j \in \{1,2,\dots,k\}$. By Lemma 4.5, $v_{t+1} \in A$, a contradiction). Moreover, $wv_{t+1}^+ \in E(G)$. (If not, we assume $wv_{t+1}^+ \notin E(G)$. By $v_{t+1} \in N(x_0)$ and Lemma 4.2(d), $x_0 w \notin E(G)$. By the maximality of C, $v_{t+1}^+ x_0 \notin E(G)$. Then $G[\{v_{t+1}, x_0, w, v_{t+1}^+\}] \cong K_{1,3}$, a contradiction.)

Claim 2. *Let $w \in N(x_i) \cap S_2(X) \cap C[z_{m1}, z_{m2})$. Then the following statements hold.*

(1) $\{w^+, w^-, v_{t+1}^+\} \cup \{x_i\} \subseteq N(w)$ $(i \neq 0, t+1)$.
(2) *Let $u \in N_C(w)$. If u dominates $\{x_i\} \cup \{w^+, w^-\}$, then $u \in C(v_{t+1}^+, v_i] \cup C(x_i, w)$; If u dominates $\{x_i, v_{t+1}^+\}$, then $u \in C(x_i, w)$.*
(3) *There is no vertex in $N(w)$ dominating $\{x_i\} \cup \{w^+, w^-\}$.*
(4) *There is no vertex in $N(w)$ dominating $\{x_i, v_{t+1}^+, w^+\}$.*
(5) *There is no vertex in $N(w)$ dominating $\{x_i\} \cup \{v_{t+1}^+, w^-\}$.*

Now, we show these statements one by one.

(1) By Claim 1, wv_{t+1}^+, $v_{t+1}x_0 \in E(G)$. By $wv_{t+1}^+ \in E(G)$, we have $\{w^+, w^-, v_{t+1}^+\} \cup \{x_i\} \subseteq N(w)$. By $v_{t+1}x_0 \in E(G)$, Lemma 4.2(d) and $w \neq v_{t+1}$, we have $x_i \neq x_0$. If $x_i = x_{t+1}$, then $w = v_{t+1}^-$ by Lemma 4.2(d). This contradicts the definition of x_{t+1}. Hence $x_i \neq x_{t+1}$. (1) holds.

(2) In fact, $C[z_{m1}, z_{m2}) = C[z_{m1}, v_{t+1}]$ by Claim 1. Clearly, $u \neq w$. By Lemma 4.2(d), $w \in N(x_i) \cap S_2(X)$ and $u \in N(x_i)$, we have $u \notin C(w, v_{t+1}]$.

If u dominates $\{x_i\} \cup \{w^+, w^-\}$, then we have $u \neq v_{t+1}^+$ by Lemma 4.1(b) and $i \neq t+1$. Moreover, we have $u \notin C(v_i, x_i]$ by Lemma 4.2(d), $w \in S_2(X)$ and $\{w^+, w^-\} \subseteq N(u)$. Thus, $u \in C(v_{t+1}^+, v_i] \cup C(x_i, w)$.

If u dominates $\{x_i, v_{t+1}^+\}$, then we have $u \notin \{v_{t+1}^+\} \cup C(v_i, x_i]$ by Lemma 4.1(b) and $i \neq t+1$. Moreover, $u \notin C(v_{t+1}^+, v_i]$ (Suppose that $u \in C(v_{t+1}^+, v_i]$. By Lemma 4.1(b), we have $u^+v_{t+1}^+, v_{t+1}^+x_i \notin E(G)$. By Lemma 4.5 and $u \notin A$, we have $u \neq v_i$. By the definition of x_i, $u^+x_i \notin E(G)$. Then, $G[\{u, u^+, x_i, v_{t+1}^+\} \cong K_{1,3}$, a contradiction.) Thus, $u \in C(x_i, w)$.

(3) Suppose, to the contrary, that there exists a vertex $u \in N(w)$ dominating $\{x_i\} \cup \{w^+, w^-\}$. By the maximality of C, $u \in V(C)$. By (2), $u \in C(v_{t+1}^+, v_i] \cup C(x_i, w)$. By $w \in A$, we have $u \notin A$.

If $u \in C(v_{t+1}^+, v_i]$, then we have $u^-x_i \notin E(G)$ as x_i is not GPI. By Lemma 4.1(b) and $wv_{t+1}^+ \in E(G)$, $x_iw^+ \notin E(G)$. Thus, $u^-w^+ \in E(G)$. (Otherwise, $G[\{u, u^-, x_i, w^+\} \cong K_{1,3}$, a contradiction.) Note that $\{u, w\} \subseteq N(x_i)$, we have $\{uu^-, ww^+, v_{t+1}^+v_{t+1}\} \cap E(x_i^-) = \emptyset$ by Lemma 4.1(b) and the definition of x_i. The vertices of $C(v_i, x_i)$ can be inserted into the cycle $C[x_i, w]C[v_{t+1}^+, u^-]\ C[w^+, v_{t+1})\ v_{t+1}Hv_i\bar{C}\ (v_i, u]x_i$. Denote by C_1 the resulting cycle, we have $V(C) \subset V(C_1)$, a contradiction.

If $u \in C(x_i, w)$, then $u^+x_i \notin E(G)$ (If not, we assume $u^+x_i \in E(G)$. By Claim 1, $wv_{t+1}^+ \in E(G)$. Note that $\{uu^+, ww^+, v_{t+1}v_{t+1}^+\} \cap E(x_i^-) = \emptyset$. The vertices of $C(v_i, x_i)$ can be inserted into the cycle $C[x_i, u]\ C[w^+, v_{t+1})\ v_{t+1}Hv_i\ \bar{C}(v_i, v_{t+1}^+]\ \bar{C}[w, u^+]x_i$. Let C_2 denote the resulting cycle. Clearly, $V(C) \subset V(C_2)$, a contradiction). Thus, $u \neq w^-$. By $wv_{t+1}^+ \in E(G)$ and Lemma 4.1(b), we have $x_iw^+ \notin E(G)$. Then $w^+u^+ \in E(G)$ (Otherwise, $G[\{u, x_i, w^+, u^+\}] \cong K_{1,3}$, a contradiction). Note that $\{uu^+, ww^+, ww^-, v_{t+1}v_{t+1}^+\} \cap E(x_i^-) = \emptyset$ and $\{x_i, v_{t+1}^+\} \subseteq N(w)$, the vertices of $C(v_i, x_i)$ can be inserted into the cycle $C[x_i, u]\ \bar{C}[w^-, u^+]\ C[w^+, v_{t+1})\ v_{t+1}Hv_i\ \bar{C}(v_i, v_{t+1}^+]wx_i$. Denote by C_3 the resulting cycle, we have $V(C) \subset V(C_3)$, a contradiction.

(4) Suppose, to the contrary, that there exists a vertex $u \in N(w)$ dominating $\{x_i, v_{t+1}^+, w^+\}$. By the maximality of C, $u \in V(C)$. By (2), $u \in C(x_i, w)$. Then we have x_iu^+, $x_iv_{t+1}^+ \notin E(G)$ by Lemma 4.1(b). Thus, $u^+v_{t+1}^+ \in E(G)$. (Otherwise, $G[\{u, v_{t+1}^+, x_i, u^+\}] \cong K_{1,3}$, a contradiction.) Note that $\{uu^+, ww^+, v_{t+1}v_{t+1}^+\} \cap E(x_i^-) = \emptyset$. By inserting the vertices of $C(v_i, x_i)$ into the cycle $C[x_i, u]\ C[w^+, v_{t+1})\ v_{t+1}Hv_i\ \bar{C}(v_i, v_{t+1}^+]\ C[u^+, w]x_i$, we get a cycle which contains C, a contradiction.

(5) Suppose, to the contrary, that there is a vertex $u \in N(w)$ dominating $\{x_i\} \cup \{v_{t+1}^+, w^-\}$. By the maximality of C, $u \in V(C)$. By Lemma 4.1(b), $u \neq w^-$. Then we have $u \in C(x_i, w^-)$ by (2). By Lemma 4.1(b), $v_{t+1}^+x_i$, $x_iu^+ \notin E(G)$. Thus, $v_{t+1}^+u^+ \in E(G)$. (Otherwise, $G[\{u, u^+, x_i, v_{t+1}^+\}] \cong K_{1,3}$, a contradiction.) Note that $\{uu^+, ww^-, v_{t+1}v_{t+1}^+\} \cap E(x_i^-) = \emptyset$, the vertices of $C(v_i, x_i)$ can be inserted into the cycle $C[x_i, u]\ \bar{C}[w^-, u^+]\ C[v_{t+1}^+, v_i)\ v_iHv_{t+1}\ \bar{C}(v_{t+1}, w]x_i$. Denote by C_4 the resulting cycle. Clearly, $V(C) \subset V(C_4)$, a contradiction.

The proof of Claim 2 is over.

Now we prove Lemma 4.9. By Claim 2(1) and $w \in S_2(X)$, we have $\{w^+, w^-, v_{t+1}^+\} \cup \{x_i, x_j\} \subseteq N(w)$, where $\{i, j\} \subseteq \{1, 2, \ldots, k\} \setminus \{t+1\}$. By Lemma 4.5 and since A is independent, it is easy to see that there is no vertex in $N(w)$ dominating $\{x_i, x_j\}$. Then $\gamma(G[N(w)]) > 2$ by Claims 2(3)–(5), a contradiction. Hence Lemma 4.9 holds.

For given C and its $\{v_1, v_2, \ldots, v_k\}$, set $\mathscr{C} = \{C' | C'$ is a cycle in G, $V(C') = V(C)$, and $v_1, v_2, \ldots, v_k$ occur on C' in the order of the indices $\}$. $\mathscr{C}$ is the set of some cycles in G. Clearly, C' is a maximal cycle in G for each $C' \in \mathscr{C}$.

Lemma 4.10. *Let $C \in \mathscr{C}$ be a cycle having maximum number of globally path insertible vertices in $\mathscr{C}$. If $v_t \in S_2(X)$, then $m \geqslant 2$, and there exists some $j \in \{1, 2, \ldots, m-1\}$, such that $C[z_{j1}, z_{j2})$ is simple.*

Proof. In fact, $v_t \in S_2(X) \cap N_C(H)$. Then we have $v_t \in N(x_0) \cap N(x_t)$ by Lemmas 4.6, 4.7, and $v_t \in A$ by Lemma 4.5. Therefore, we have $u \notin A$ if $u \in N(v_t)$. By the definition of x_t, $x_t v_t^-, v_t^+ v_t^- \notin E(G)$. By the maximality of C, $x_0 v_t^+, x_0 v_t^- \notin E(G)$. Considering $N(v_t)$, we have $\{v_t^+, v_t^-, x_0\} \cup \{x_t\} \subseteq N(v_t)$. (It is possible to have $x_t = v_t^+$.) It is clear that x_0 and x_t have no common neighbor in $N(v_t)$. Suppose, to the contrary, that Lemma 4.10 does not hold.

Claim 1. *There is no vertex in $N(v_t)$ dominating $\{x_0, v_t^+\}$.*

Suppose that there exists a vertex $u \in N(v_t)$ dominating $\{x_0, v_t^+\}$. By the maximality of C, we have $u \in V(C)$, $u \neq v_t^+$ and $u^+, u^- \notin N(x_0)$. Note that $u \in N_C(x_0)$ and $u \notin A$, we have $u^+ u^- \in E(G)$. Then the cycle $C[v_t^+, u^-]C[u^+, v_t)v_t H u v_t^+$ in G contains C, a contradiction.

Claim 2. *There is no vertex in $N(v_t)$ dominating $\{x_0, v_t^-\}$.*

Suppose that there exists a vertex $u \in N(v_t)$ dominating $\{x_0, v_t^-\}$. By the maximality of C, we have $u \in V(C)$, $u \neq v_t^-$ and $u^-, u^+ \notin N(x_0)$. Then $u^+ u^- \in E(G)$. Thus, the cycle $C[v_t, u^-]C[u^+, v_t^-]u H v_t$ in G contains C, a contradiction. Claim 2 holds.

Since G is almost claw-free, we have $\gamma(G[N(v_t)]) \leqslant 2$. Then there exists a vertex $u \in N(v_t)$ dominating $\{x_t\} \cup \{v_t^+, v_t^-\}$ by Claims 1, 2. It is easy to see that $u \in V(C)$. By $v_t^- x_t, v_t^- v_t^+ \notin E(G)$, we have $u \notin \{v_t^+, v_t^-\} \cup \{x_t\}$. By $u \notin A$ and Lemma 4.5, $u \notin N_C(H)$.

Claim 3. $u \in C(x_t, v_{t+1})$, $u^+ u^- \notin E(G)$.

Suppose that $u \notin C(x_t, v_{t+1})$. Since $v_t x_t, u v_t^- \in E(G)$, we have $u \notin C(v_t, x_t]$ by the definition of x_t. By $u \notin N_C(H)$, we have $u \in C(v_{t+1}, v_t)$. Thus, we may let $u \in C(v_p, v_{p+1})$ $(\subseteq C(v_{t+1}, v_t))$. By $u x_t \in E(G)$ and Lemma 4.1(b), $u \in C(x_p, v_{p+1})$. By $x_t u \in E(G)$ and the definition of x_t, we have $u^+ x_t, u^- x_t \notin E(G)$ and $uu^+, uu^- \notin E(x_t^-)$. Consider $G[\{u, u^+, u^-, x_t\}]$. It is clear that $u^+ u^- \in E(G)$. For each $i \in \{t+1, t+2, \ldots, p-1\}$, we have $uu^+, uu^- \notin E(x_i^-)$. (Otherwise, there exists some $j \in \{t+1, t+2, \ldots, p-1\}$, such that $\{uu^+, uu^-\} \cap E(x_j^-) \neq \emptyset$. Then there is a vertex $v \in C(v_j, x_j)$ satisfying $uv \in E(G)$. By Lemma 4.1(b), $x_t v, u^+ v \notin E(G)$. Thus, $G[\{u, u^+, v, x_t\}] \cong K_{1,3}$, a contradiction.) By symmetry, $u^+ u, uu^- \notin E(x_i^-)$ for each $i \in \{p+1, p+2, \ldots, t-1\}$. Hence there is a cycle $C_1 = C[v_t^+, u^-]C[u^+, v_t]u v_t^+$ satisfying $V(C) = V(C_1)$, however C_1 has more globally path insertible vertices than C. This contradicts the choice of C. So $u \in C(x_t, v_{t+1})$.

Suppose $u^+ u^- \in E(G)$. First we will prove $uu^-, uu^+ \notin E(x_i^-)$ for each $i \in \{1, 2, \ldots, k\} \backslash \{t\}$. We proceed by contradiction. Suppose that there exists some $i_0 \in \{1, 2, \ldots, k\} \backslash \{t\}$, such that $\{uu^-, uu^+\} \cap E(x_{i_0}^-) \neq \emptyset$. Let g be the first vertex in

$C(v_{i_0},x_{i_0})$ adjacent to u. Then $uu^+\notin E(g^-)$. If $v_tv_t^-\notin E(g^-)$, then we have $gx_t,u^+x_t\notin E(G)$ by Lemma 4.1(b). Thus $gu^+\in E(G)$ as $u\notin A$. Hence the vertices of $C(v_{i_0},g)$ can be inserted into the cycle $C[u^+,v_{i_0})\ v_{i_0}Hv_t\ C(v_t,u]\ \bar{C}[v_t^-,g]u^+$. Denote by C_2 the resulting cycle. Clearly, $V(C)\subset V(C_2)$, a contradiction. If $v_tv_t^-\in E(g^-)$, we let h the first vertex in $C(v_{i_0},g)$ adjacent to v_t. Then $v_tv_t^-\notin E(h^-)$. By Lemma 4.1(b) and the choices of h,g, we have $uu^+,uu^-,v_tv_t^+\notin E(h^-)$. Thus we can insert all vertices of $C(v_{i_0},h)$ into the cycle $C[u^+,v_{i_0})\ v_{i_0}Hv_t\ C[h,v_t^-]u\ C[v_t^+,u^-]u^+$. Denote by C_3 the resulting cycle. Then we have $V(C)\subset V(C_3)$, a contradiction.

Thus, we have $uu^+,uu^-\notin E(x_i^-)$ for each $i\in\{1,2,\ldots,k\}\backslash\{t\}$. Considering the cycle $C_4=C[v_t^+,u^-]\ C[u^+,v_t]\ uv_t^+$ in G, we have $V(C)=V(C_4)$, however C_4 has more globally path insertible vertices than C. This contradicts the choice of C too. So $u^+u^-\notin E(G)$.

Claim 4. $u\in S_1(X)$, $u^+\notin N_C(H)$. *Therefore* $u\neq v_{t+1}^-$.

In fact, note that $ux_t\in E(G)$, we have $u\in S_1(X)$ by Lemma 4.5 and $u\notin A$. Moreover $u^+\notin N_C(H)$, otherwise the cycle $\bar{C}[u,v_t)v_tHu^+C(u^+,v_t^-]u$ in G contains C, a contradiction.

Claim 5. $u^+\in N(x_t)$.

Suppose, to the contrary, that $u^+\notin N(x_t)$. By Claim 4, we have $u^+,u^{++}\in C(x_t,v_{t+1}]$ and $u^+\notin N(x_0)$. We distinguish three cases.

Case 1: $u^+\in S_0(X)$.

Let a be the first vertex in $\bar{C}[u,x_t]$ nonadjacent to x_t. Then $C[a,u^+)$ is simple by Lemma 4.2(d) and Claim 4. Since $u^+\in N_2(X)$ and $u^+\in C[x_t,v_{t+1}]$, there must exist some other CX-segments in $C[u^+,v_{t+1}]$. This contradicts the assumption.

Case 2: $u^+\in S_l(X)$ $(l\geqslant 2)$.

Set $u^+\in N(x_i)\cap N(x_j)$. Clearly, $i,j\neq 0,t$. By Lemma 4.1(b), $E(x_i^-)\cap E(x_j^-)=\emptyset$. Then we have either $v_tv_t^-\notin E(x_i^-)$ or $v_tv_t^-\notin E(x_j^-)$. Without loss of generality, we may assume that $v_tv_t^-\notin E(x_i^-)$. By $x_iu^+\in E(G)$ and the definition of x_i, $uu^+\notin E(x_i^-)$. Thus, the vertices of $C(v_i,x_i)$ can be inserted into the cycle $C[v_t,u]\bar{C}[v_t^-,x_i]C[u^+,v_i)\ v_iHv_t$. Denote by C_1 the resulting cycle, we have $V(C)\subset V(C_1)$, a contradiction.

Case 3: $u^+\in S_1(X)$.

Suppose $u^+\in N(x_i)$, where $i\neq 0,t$. Then $v_tv_t^-\in E(x_i^-)$. (Otherwise, the cycle C_1 which is mentioned in Case 2 contains C, a contradiction). Consider $G[\{u,u^+,v_t^-,x_t\}]$. Since $x_tv_t^-,x_tu^+\notin E(G)$ and $u\notin A$, we have $u^+v_t^-\in E(G)$. For $u^{++}(\in C[x_t,v_{t+1}])$, we assume that there exists some $j\in\{0,1,2,\ldots,k\}$ such that $u^{++}\in N(x_j)$. By Lemma 4.2(d), we have either $x_j\in C(v_{t+1},v_i)$ or $j=0$. If $j=0$, then the cycle $C[u^{++},v_t^-]\ \bar{C}[u^+,v_t)\ v_tHu^{++}$ contains C, a contradiction. If $j\neq 0$, then we have $v_tv_t^-\notin E(x_j^-)$ since $v_tv_t^-\in E(x_i^-)$. Thus, the vertices of $C(v_j,x_j)$ can be inserted into the cycle $C[v_t,u^+]\ \bar{C}[v_t^-,x_j]\ C[u^{++},v_j]\ v_jHv_t$. Denote by C_2 the resulting cycle. It is clear that $V(C)\subset V(C_2)$, a contradiction. So $u^{++}\in S_0(X)$. Let a be the first vertex in $\bar{C}[u,x_t]$

nonadjacent to x_t. Then $C[a, u^{++})$ is simple by Lemma 4.2(d). Since $u^{++} \in N_2(X)$ and $u^{++} \in C[x_t, v_{t+1}]$, there exist some other CX-segments in $C[x_t, v_{t+1}]$. This contradicts the assumption too.

Claim 6. *For any* $i \in \{1, 2, \dots, k\}$, $v_t v_t^- \notin E(x_i^-)$.

If $i \in \{t, t-1\}$, it is clear that Claim 6 holds. Then we should prove $v_t v_t^- \notin E(x_i^-)$ for any $i \in \{1, 2, \dots, k\} \backslash \{t, t-1\}$.

Suppose, to the contrary, that there exists some $i_0 \in \{1, 2, \dots, k\} \backslash \{t, t-1\}$, such that $v_t v_t^- \in E(x_{i_0}^-)$. Let f be first vertex in $C(v_{i_0}, x_{i_0})$ adjacent to v_t. Then we have $v_t v_t^- \notin E(f^-)$. Considering $E(f^-)$ and $E(x_t^-)$, we have $E(f^-) \cap E(x_t^-) = \emptyset$ by Lemma 4.1(b). By Claim 5, $x_t u^+ \in E(G)$. Thus, $uu^+ \notin E(f^-)$ by Lemma 4.1(b). Note that $\{uu^+, v_t v_t^-\} \cap E(x_t^-) = \emptyset$, the vertices in $C(v_{i_0}, f) \cup C(v_t, x_t)$ can be inserted into the cycle $C[x_t, u]\ \bar{C}[v_t^-, f]\ v_t H v_{i_0}\ \bar{C}(v_{i_0}, u^+]x_t$. Denote by C_1 the resulting cycle. It is obvious that $V(C) \subset V(C_1)$, a contradiction.

Claim 7. $v_{t+1}^- \notin N(x_t)$.

Suppose, to the contrary, that $x_t v_{t+1}^- \in E(G)$. By Claim 3, $u^+ u^- \notin E(G)$. Considering $G[\{u, u^+, u^-, v_t\}]$, we have either $u^+ v_t \in E(G)$ or $u^- v_t \in E(G)$ as $uv_t \in E(G)$. If $u^+ v_t \in E(G)$, then the vertices of $C(v_t, x_t)$ can be inserted into the cycle $C[x_t, u]\ \bar{C}[v_t^-, v_{t+1})\ v_{t+1} H v_t\ C[u^+, v_{t+1}^-]x_t$ as $u \neq v_{t+1}^-$ by Claim 4. Thus, we get the cycle that contains C, a contradiction. If $u^- v_t \in E(G)$, note that $u \neq x_t$, we can extend C by inserting the vertices of $C(v_t, x_t)$ into the cycle $C[x_t, u^-]\ v_t H v_{t+1}\ C(v_{t+1}, v_t^-]\ C[u, v_{t+1}^-]x_t$, a contradiction. Hence $v_{t+1}^- \notin N(x_t)$.

Claim 8. *Let*, $C[u, w) \subseteq N(x_t)$. *Then we have* $w \notin N(x_i)$ *for each* $i \in \{0, 1, \dots, k\} \backslash \{t\}$.

In fact, by Claim 7, $w \in C[u^+, v_{t+1}^-]$. First we will prove that there exists a (v_t^-, v_t)-path P containing the vertices of $C[x_t, w)$. By Claim 3, $u^+ u^- \notin E(G)$. Then we have either $v_t u^+ \in E(G)$ or $v_t u^- \in E(G)$ since $u \notin A$. If $v_t u^- \in E(G)$, then $P = v_t^- C[u, w^-]\ C[x_t, u^-]v_t$ is a (v_t^-, v_t)-path which contains all vertices of $C[x_t, w)$. If $v_t u^+ \in E(G)$, then we put $P = v_t^- \bar{C}[u, x_t] \bar{C}[w^-, u^+]v_t$ when $w \neq u^+$, and put $P = v_t^- \bar{C}[u, x_t]v_t$ when $w = u^+$. Thus P contains the vertices of $C[x_t, w)$.

Suppose, to the contrary, that there exists some $i \in \{0, 1, \dots, k\} \backslash \{t\}$ such that $w \in N(x_i)$. If $i = 0$, then the vertices of $C(v_t, x_t)$ can be inserted into the cycle $P[v_t^-, v_t) v_t H w C(w, v_t^-)$. Denote by C_1 the resulting cycle. Then we have $V(C) \subset V(C_1)$, a contradiction. If $i \in \{1, 2, \dots, k\} \backslash \{t\}$, by inserting the vertices of $C(v_t, x_t) \cup C(v_i, x_i)$ into the cycle $P[v_t^-, v_t) v_t H v_i \bar{C}(x_i, w] C[x_i, v_t^-)$, we get a cycle which contains C, a contradiction. Claim 8 holds.

By Claim 7, let w_0^+ be the first vertex in $C[u, v_{t+1})$ nonadjacent to x_t. Then $C[u, w_0] \subseteq N(x_t)$. By Claim 8, $w_0 \in S_1(X)$, $w_0^+ \in S_0(X)$. Let a be the first vertex in $\bar{C}[u, x_t]$ nonadjacent to x_t. Then $C[a, w_0^+)$ is simple by Lemma 4.2(d). By Claim 7, we have $w_0^+ \in C[x_t, v_{t+1})$. Note that $w_0^+ \in N_2(X)$, there exist some other CX-segments in $C[w_0^+, v_{t+1}]$. It contradicts the assumption. The proof of Lemma 4.10 is over.

Lemma 4.11. $\sum_{i=0}^{k} d(x_i) \leqslant n(X) - k - 1$.

Proof. Suppose that $|\{v_1, v_2, \ldots, v_k\} \cap S_2(X)| = p$. By Lemmas 4.9 and 4.10, there are at least $(k-p)+p=k$ simple CX-segments in $J_X \backslash \bigcup_{l>2} N_l(X)$. Thus, by Lemmas 4.4 and 4.8(3), we have

$$\begin{aligned}
\sum_{i=0}^{k} |J_X \cap N(x_i)| &= \sum_{t=1}^{k} \sum_{i=0}^{k} |C[x_t, v_{t+1}] \cap N(x_i)| \\
&= \sum_{t=1}^{k} \sum_{j=1}^{m} \sum_{i=0}^{k} |C[z_{j1}, z_{j2}) \cap N(x_i)| \leqslant \sum_{t=1}^{k} \sum_{j=1}^{m} |C[z_{j1}, z_{j2})| - k \\
&= \sum_{t=1}^{k} \left(|C[x_t, v_{t+1}]| - \sum_{l>2} |N_l(X) \cap C[x_t, v_{t+1}]| \right) - k \\
&= |J_X| - \sum_{l>2} |N_l(X) \cap J_X| - k.
\end{aligned}$$

Note that $V(G) = J_X \cup K_X$. By Lemma 4.3, we have

$$\begin{aligned}
\sum_{i=0}^{k} d(x_i) &= \sum_{i=0}^{k} |J_X \cap N(x_i)| + \sum_{i=0}^{k} |K_X \cap N(x_i)| \\
&\leqslant |J_X| - k - \sum_{l>2} |N_l(X) \cap J_X| + |K_X| - 1 - \sum_{l>2} |N_l(X) \cap K_X| \\
&= n - k - 1 - \sum_{l>2} |N_l(X)| = n(X) - k - 1.
\end{aligned}$$

5. Proof of Theorem 3.8

Proof of Theorem 3.8. Suppose, to the contrary, that G is non-hamiltonian. Let C be a longest cycle in G. Then C is clearly a maximal cycle of G, and $G \backslash V(C)$ has at least a component H. Since G is a k-connected graph with $k \geqslant 2$, we may suppose $\{v_1, v_2, \ldots, v_k\} \subseteq N_C(H)$, and $v_1, v_2, \ldots, v_k$ occur on C in the order of their indices. For the cycle C above and its $\{v_1, v_2, \ldots, v_k\}$, set $\mathscr{C} = \{C' \mid C'$ is a cycle in G, $V(C') = V(C)$, and $v_1, v_2, \ldots, v_k$ occur on C' in the order of the indices$\}$. Let $C \in \mathscr{C}$ be a cycle having maximum number of globally path insertible vertices. By Lemma 4.1(a), for each $i \in \{1, 2, \ldots, k\}$, denote by x_i the first non-GPI vertex in $C(v_i, v_{i+1})$. Pick up an $x_0 \in V(H)$ and let $X = \{x_0, x_1, x_2, \ldots, x_k\}$. By Lemma 4.11, $\sum_{i=0}^{k} d(x_i) \leqslant n(X) - k - 1$. On the other hand, by Lemma 4.2(b), $X \in I_{k+1}(G^*)$, a contradiction.

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