JOURNAL OF ALGEBRA 150, 378–387 (1992)

The Jacobson Radical of Commutative Semigroup Rings

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Communicated by T. E. Hall

Received January 26, 1990

In this paper we consider semiprimitive commutative semigroup rings and related matters. A ring is said to be *semiprimitive* if the Jacobson radical of it is equal to zero. This property is one of the most important in the theory of semigroup rings, and there is a prolific literature pertaining to the field (see [14]).

All semiprimitive rings are contained in another interesting class of rings. Let $\mathscr{E}$ denote the class of rings R such that $J(R) = B(R)$, where J and B are the Jacobson and Baer radicals. Clearly, every semiprimitive ring is in $\mathscr{E}$. This class appears, for example, in the theory of PI-rings and in commutative algebra. (In particular, every finitely generated PI-ring and every Hilbert ring are in $\mathscr{E}$.) Therefore, it is of an independent interest. Meanwhile it is all the more interesting because any characterization of the semigroup rings in $\mathscr{E}$ will immediately give us a description of semiprimitive semigroup rings. Indeed, a ring R is semiprimitive if and only if $R \in \mathscr{E}$ and R is semiprime, i.e., $B(R) = 0$. Semiprime commutative semigroup rings have been described by Parker and Gilmer [12] and, in other terms, by Munn [9]. So it suffices to characterize semigroup rings in $\mathscr{E}$.

Semigroup rings of $\mathscr{E}$ were considered by Karpilovsky [5], Munn [6–9], Okninski [10], and others. In this paper commutative semigroup rings which are in $\mathscr{E}$ will be described completely.

To this end one should know the structure of the Jacobson radical $J(R[S])$. In [2] Jespers described $J(R[S])$ under rather weak assumptions on R. They hold, in particular, for every commutative R. Here we shall give another (quite short) description of $J(R[S])$ which does not require any restriction on R. Besides, it is specially fitted for testing whether an element is in $J(R[S])$, and this is essential for our proofs.

0021-8693/92 $5.00

1. Notation and Preliminaries

For details we refer to [1, 4]. Throughout the paper only commutative semigroups will be considered.

Let p be a prime number. A semigroup S is said to be *separative* (*p-separative*) if for every $s, t \in S$ the equality $s^2 = st = t^2$ ($s^p = t^p$) implies $s = t$. The least separative (p-separative) congruence on S is denoted by $\xi(\xi_p)$. Explicitly

$$\xi = \{(s, t) \mid \exists n: st^n = t^{n+1} \text{ and } s^n t = s^{n+1}\},$$

$$\xi_p = \{(s, t) \mid \exists n: s^{p^n} = t^{p^n}\}.$$

For unification we set $\xi_0 = \xi$.

Let R be a ring, ρ be a congruence on S. Then $I(R, S, \rho)$ denotes the ideal $\{\sum_i r_i(s_i - t_i) \mid r_i \in R, (s_i, t_i) \in \rho\}$ of $R[S]$. Set $\mathscr{R}_n(R) = \{r \in R \mid nr \in \mathscr{R}(R)\}$, where $\mathscr{R}$ is the Baer or the Jacobson radical. Let $\mathbb{P}$ be the set of all prime numbers.

Proposition 1. (Munn [9]). *Let $R[S]$ be a commutative semigroup ring. Then*

$$B(R[S]) = B(R)[S] + I(R, S, \xi) + \sum_{p \in \mathbf{P}} I(B_p(R), S, \xi_p).$$

A semigroup S is said to be *Archimedean* if for any two elements of S, each divides some power of the other.

Proposition 2. (Jespers, Krempa, and Wauters [3]). *Let R be a commutative ring, S be an Archimedean semigroup. If S is periodic, then*

$$J(R[S]) = J(R)[S] + I(R, S, \xi) + \sum_{p \in \mathbf{P}} I(J_p(R), S, \xi_p).$$

Otherwise,

$$J(R[S]) = B(R)[S] + I(R, S, \xi) + \sum_{p \in \mathbf{P}} I(B_p(R), S, \xi_p).$$

Note that in the case of a non-commutative R the results corresponding to Propositions 1 and 2 are proved in [11, 3].

2. A Description of the Jacobson Radical

A semigroup Γ is called a *semilattice* if it entirely consists of idempotents. A semigroup S is said to be a *semilattice Γ of its subsemigroups S_α* ($\alpha \in \Gamma$) if $S = \bigcup_{\alpha \in \Gamma} S_\alpha$, $S_\alpha \cap S_\beta = \varnothing$ when $\alpha \neq \beta$, and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for any α, β. By

Theorem 4.13 in [1] each semigroup can be uniquely represented as a semilattice of its Archimedean subsemigroups S_α. The semigroups S_α are called the *Archimedean components* of S.

Let R be an arbitrary (not necessary commutative) ring, $x \in R[S]$, $x = \sum_{t \in S} x_t t$. Set $x_\alpha = \sum_{t \in S_\alpha} x_t t$. The semilattice generated in Γ by all α such that $x_\alpha \neq 0$ will be called the *support* of x and denoted by supp (x). (This definition of a support differs from the standard one, cf. [2]. It is the new concept, that will work in our proofs.) Consider the natural partial order $\leqslant$ on Γ defined by $\alpha \leqslant \beta \Leftrightarrow \alpha\beta = \alpha$. Let max (x) denote the set of elements in supp (x) maximal with respect to this order. Clearly the sets supp (x) and max (x) are finite. The following lemma was proved in [16] for the case of a two-element semilattice Γ.

LEMMA 1. *Let R be an arbitrary ring, S be a commutative semigroup with Archimedean components S_α, $\alpha \in \Gamma$. The radical $J(R[S])$ is the largest ideal among ideals I of $R[S]$ such that $x_\mu \in J(R[S_\mu])$ for any $x \in I$, $\mu \in$ max (x).*

Proof. Let M be the set of ideals I of $R[S]$ such that $x_\mu \in J(R[S_\mu])$ for any $x \in I$, $\mu \in$ max (x). By the proof of Theorem 1 in [15], $J(R[S]) \in M$.

On the other hand, take any I in M. We claim that I is quasiregular (and so $I \subseteq J(R[S])$). Suppose the contrary and choose x in I which does not have a right quasi-inverse and |supp (x)| is minimal. Let $\mu \in$ max (x). Then $x_\mu \in J(R[S_\mu])$, and $x_\mu + a + x_\mu a = 0$ for some $a \in J(R[S_\mu])$. Consider the element $y = -x - xa$. Clearly $y \in I$ and $y_\mu = a$. Further, set $z = x + y + xy$. Evidently $z \in I$ and supp $(z) \subseteq$ supp $(x) \setminus \{\mu\}$. By the choice of x there exists u such that $z + u + zu = 0$. Then $x + (y + u + yu) + x(y + u + yu) = 0$. So x is quasi-invertible, giving a contradiction. Thus $I \subseteq J(R[S])$. We have proved that $J(R[S])$ is the largest ideal in M. (This also can be proved as a corollary of Lemma 1.3 in [16].)

Now let us consider a separative semigroup T. By Theorem 4.16 in [1] the Archimedean components T_α of T are cancellative. Denote by Q_α the group of quotients of T_α. Let e_α denote the identity element of Q_α. Set $Q = \bigcup_{\alpha \in \Gamma} Q_\alpha$. The multiplication of T can be easily extended on the whole Q so that $e_\alpha e_\beta = e_{\alpha\beta}$. Let $\mu \in \Gamma$, $x \in R[Q_\mu]$, and Λ be a finite (or empty) subset of $\mu\Gamma$. Then (μ, x, Λ) denotes the product $x \prod_{\lambda \in \Lambda} (e_\mu - e_\lambda)$. If $\Lambda = \varnothing$, then $(\mu, x, \Lambda) = x$. Following [13] we say that (μ, x, Λ) is a *simplest element*, if $xe_\alpha \in J(R[Q_\alpha])$ for any $\alpha \in \mu\Gamma \setminus \Lambda\Gamma$. Note that $(\mu, x, \Lambda)e_\alpha = 0$ for any $\alpha \in \Lambda\Gamma$. The set of the simplest elements of $R[Q]$ is denoted by $Si(R[Q])$. Put $Si(R[T]) = R[T] \cap Si(R[Q])$.

Proposition 1 shows that $I(R, S, \xi) \subseteq J(R[S])$. Clearly $R[S]/I(R, S, \xi) \cong R[S/\xi]$. Therefore it suffices to describe the Jacobson radical for the semigroup $T = S/\xi$. In this case we state

THEOREM 1. *Let* $x \in R[T]$, $\mu \in \max(x)$, Λ *be the set of maximal elements in the finite set* $\mu\,\mathrm{supp}(x) \setminus \{\mu\}$, $y = (\mu, x_\mu, \Lambda)$. *Then*

(1) $x \in J(R[T]) \Leftrightarrow x \in R[T] \cap J(R[Q])$;

(2) $x \in J(R[Q]) \Leftrightarrow y, x-y \in J(R[Q])$;

(3) $y \in J(R[Q]) \Leftrightarrow y \in Si(R[Q])$.

Assertions (1) and (2) reduce the inclusion $x \in J(R[T])$ to y, $x-y \in J(R[Q])$. Since $|\mathrm{supp}(x-y)| < |\mathrm{supp}(x)|$, applying (2) several times one can reduce $x \in J(R[T])$ to some inclusions of the form $y \in J(R[Q])$, which can be checked with (3). Note that $Si(R[Q])$ is defined in terms of the radicals of the components $R[Q_\alpha]$.

Proof of Theorem 1. (1) Take $x \in R[T] \cap J(R[Q])$, $\mu \in \max(x)$. Since $x \in J(R[Q])$, Lemma 1 yields $x_\mu \in J(R[Q_\mu])$. By Proposition 2 we get $x_\mu \in J(R[T_\mu])$, for Q_μ and T_μ are Archimedean. Then Lemma 1 implies $J(R[T]) \supseteq R[T] \cap J(R[Q])$.

Now take $x \in J(R[T])$. Denote by I the ideal generated by x in $R[Q]$. Choose z in I. Then $z = \sum_i a_i x b_i$, where a_i, $b_i \in R[Q]^1$. Let $\mu \in \max(z)$, $t \in T_\mu$. Evidently $xt \in J(R[T])$. By Lemma 1 and Proposition 2, $(xt)_\mu \in J(R[T_\mu]) \subseteq J(R[Q_\mu])$. Therefore $z_\mu = z_\mu e_\mu = \sum_i (a_i t)_\mu (xt)_\mu (b_i t)_\mu t^{-3} \in J(R[Q_\mu])$. Then Lemma 1 implies $I \subseteq J(R[Q])$, completing the proof of (1).

(2) Let $x \in J(R[Q])$. Take any nonzero element z of the ideal generated in $R[Q]$ by y. Say $z = \sum_i a_i y b_i$, where $a_i, b_i \in R[Q]^1$, and set $u = \sum_i a_i x b_i$. We may assume that each product $a_i x_\mu b_i$ is a homogeneous element, i.e., $a_i x_\mu b_i \in R_{\alpha_i}$ for some $\alpha_i \in \Gamma$ (otherwise we would split a_i or b_i into several summands). Then

$$z_\alpha = \Bigl(\sum_{\alpha_i \geqslant \alpha} a_i y b_i\Bigr)_\alpha = \Bigl[\Bigl(\sum_{\alpha_i \geqslant \alpha} a_i x b_i\Bigr)\Bigl(\prod_{\lambda \in \Lambda} (e_\mu - e_\lambda)\Bigr)\Bigr]_\alpha.$$

Take any $\alpha \in \max(z)$. Evidently $\alpha \in \mu\Gamma$, since $\mathrm{supp}(y) \subseteq \mu\Gamma$. If $\alpha \in \Lambda\Gamma$, then the support of the sum $s = \sum_{\alpha_i \geqslant \alpha} a_i x b_i$ is contained in Λ because of the maximality of α. Hence $s \prod_{\lambda \in \Lambda} (e_\mu - e_\lambda) = 0$ yielding $z_\alpha = 0$, a contradiction. Thus α is not in $\Lambda\Gamma$. Clearly $z_\beta = \sum_{\alpha_i = \beta} a_i x_\mu b_i = u_\beta$ for any $\beta \in \mu\Gamma \setminus \Lambda\Gamma$, and so $\alpha \in \max(u)$. Besides $z_\alpha = u_\alpha \in J(R[Q_\alpha])$, since $x \in J(R[Q])$. By Lemma 1, $y \in J(R[Q])$, and so does $x-y$. The converse is trivial.

(3) Let $y \in Si(R[Q])$. Take any element z of the ideal generated by y in $R[Q]$, say $z = \sum_i a_i y b_i \neq 0$, where $a_i, b_i \in R[Q]^1$. Let $\alpha \in \max(z)$. If $\alpha \in \Lambda\Gamma$ then $ye_\alpha = 0$, and so $z_\alpha = (ze_\alpha)_\alpha = 0$. Therefore $\alpha \in \mu\Gamma \setminus \Lambda\Gamma$. Evidently y may be written as $y = x + y'$ where $\mathrm{supp}(y') \subseteq \Lambda\Gamma$. Then $\mathrm{supp}(\sum_i a_i y' b_i) \subseteq \Lambda\Gamma$ and so $ye_\alpha = xe_\alpha$. Since y is simplest, $ye_\alpha = xe_\alpha \in J(R[Q_\alpha])$. So $z_\alpha = \sum_i (a_i e_\alpha)(ye_\alpha)(b_i e_\alpha) \in J(R[Q_\alpha])$, implying $y \in J(R[Q])$

by Lemma 1. Conversely, let $y \in J(R[Q])$, $\alpha \in \mu\Gamma \setminus \Lambda\Gamma$. Then $xe_\alpha = (ye_\alpha)_\alpha \in J(R[Q_\alpha])$, since $\alpha \in \max(ye_\alpha)$.

COROLLARY [13]. *$J(R[Q])$ is the additive group generated by $Si(R[Q])$.*

Proof. Take any $z \in J(R[Q])$ and set $n = |\text{supp}(z)|$. If $n = 1$, then Lemma 1 shows that $z \in Si(R[Q])$. If $n > 1$, then Theorem 1 and induction on n give the result.

3. Main Result and Corollaries

We need a few definitions. Let G be a finite subgroup of a semigroup T, I be an ideal generated in T by a finite (or empty) set of idempotents which does not contain G. Put down all subgroups $H_1, ..., H_n$ of G such that $H_i = \{h \in G \mid ht_i = et_i\}$ for a non-periodic element $t_i \in GTI$, where e is the identity of G. Numerate the elements of $G = \{g_1, ..., g_m\}$. The *matrix of the conjugacy relation* of G by H_i is the $(m \times m)$-matrix $D_i = [d_{jk}]$ such that

$$d_{jk} = \begin{cases} 1 & \text{when } g_j \in H_i g_k, \\ 0 & \text{otherwise.} \end{cases}$$

Set $D_I(G) = [D_1 \mid D_2 \mid \cdots \mid D_n]$. If $n = 0$ (i.e., G has no subgroup with the property mentioned or, equivalently, there is not any non-periodic element in $GT \setminus I$), then set $D_I(G) = [0]$.

For a ring R denote by $\pi(R)$ the set of all q such that q is prime or zero and $J(R)/B(R)$ has a nonzero element with an additive period q. (Here an element with an additive period 0 is a non-periodic element.) We say that G is *q-complete* in T, if q divides $|G|$ or q does not divide the determinant of an $(m \times m)$-submatrix of $D_I(G)$ (for any I).

THEOREM 2. *Let $R[S]$ be a commutative semigroup ring, ξ the least separative congruence on S, and $T = S/\xi$. The Jacobson radical $J(R[S])$ is nil if and only if for any $q \in \pi(R)$ every finite subgroup G of T is q-complete in T.*

Theorem 2 and Proposition 1 give us a description of semiprimitive commutative semigroup rings.

COROLLARY 1. *A commutative semigroup ring $R[S]$ is semiprimitive if and only if R is semiprime, S is separative, and p-separative for every prime $p \in \pi(R)$, each finite subgroup G in S is q-complete in S for any $q \in \pi(R)$.*

Note that when R is a field a description of semiprimitive $R[S]$ was given in [6].

Now we show that all the previous results on commutative semigroup rings of the class $\mathscr{E}$ are in fact partial cases of Theorem 2. The previous results are listed in Corollaries 2–5.

COROLLARY 2 [9]. *If $J(R)$ is nil, then $J(R[S])$ is nil.*

This follows from Theorem 2 because $J(R)=B(R)$ if and only if $\pi(R)=\varnothing$; that is, there are no q in $\pi(R)$.

COROLLARY 3 [7]. *If S has no idempotent elements, then $J(R[S])$ is nil.*

This is clear because if S has no idempotents, then T does not have any subgroup.

COROLLARY 4 [9]. *Let S be a periodic semigroup. Then $J(R[S])$ is nil if and only if $J(R)$ is nil.*

Indeed, a periodic S does not have a non-periodic element. Therefore all $D_l[G]$ are equal to $[0]$, and so every finite subgroup is not q-complete in S for each q. So $J(R[S])$ is nil if and only if $\pi(R)=0$, which is equivalent to $J(R)$ is nil.

COROLLARY 5 [7]. *Let S be a semilattice of cancellative and non-periodic $S_\alpha, \alpha\in\Gamma$. Then $J(R[S])$ is nil.*

Indeed, let us take a finite subgroup G in S. There is α such that $G\subseteq S_\alpha$. Fix a non-periodic element t in S_α. Then $H=\{h\in G \mid ht=et\}=\{e\}$, for S is cancellative. Hence the matrix of the conjugacy relation of G by H is the identity matrix. It's determinant is equal to 1, and q does not divide 1. Therefore G is q-complete in S for every q, not only for $q\in\pi(R)$.

4. PROOF OF THE MAIN THEOREM

LEMMA 2. *Let $F=R/B(R)$, $T=S/\xi$. The radical $J(R[S])$ is nil if and only if $J(F[T])$ is nil.*

Proof. This easily follows from Proposition 1 and the isomorphisms $R[S]/I(R,S,\xi)\cong R[T]$, $R[T]/B(R)[T]\cong F[T]$.

Recall that $T=\bigcup_{\alpha\in\Gamma}T_\alpha$, Q_α denotes the group of quotients of T_α, e_α is the identity of Q_α, and $Q=\bigcup_{\alpha\in\Gamma}Q_\alpha$. Say that a subgroup G of T is q-incomplete in T if G is not q-complete in T. Note that $\pi(R)=\pi(F)$. In view of Lemma 2, Theorem 2 is equivalent to the following

LEMMA 3. *$J(F[T])$ has a non-nilpotent element if and only if T has a q-incomplete finite subgroup for some $q \in \pi(F)$.*

Proof. First we prove the "only if" part. Choose in $J(F[T])$ a non-nilpotent element x with minimal $|\operatorname{supp}(x)|$. Let $\mu \in \max(x)$. Then $\operatorname{supp}(x^n) = \operatorname{supp}(x)$ for each n, and so x_μ is not nilpotent. Further, the element $y = x_\mu x$ is not nilpotent, for $y_\mu = x_\mu^2$. Hence $\operatorname{supp}(y) = \operatorname{supp}(x)$, that is $\mu \operatorname{supp}(x) = \operatorname{supp}(x)$. Therefore $\max(x) = \{\mu\}$. Let Λ be the set of maximal elements of $\operatorname{supp}(x) \setminus \{\mu\}$, $y = (\mu, x_\mu, \Lambda)$. By Theorem 1, $y \in Si(F[Q])$. We are to prove that $y \in F[T]$.

To this end we first prove that $e_\lambda \in T$ for every $\lambda \in \Lambda$. Suppose the contrary. Then T_λ does not have any idempotent, and so all elements in T_λ are non-periodic. Denote by P (and N) the set of periodic (non-periodic) elements of Q_λ. Then $T_\lambda \subseteq N$. The definition of a simplest element implies $y_\mu \in J(F[Q_\mu])$. Hence $J(F[Q_\mu])$ is not nil. This and Propositions 1, 2 show that Q_μ is a periodic group. Therefore $y_\lambda = e_\lambda y_\mu \in F[P]$. On the other hand, $x_\lambda \in F[T_\lambda] \subseteq F[N]$, implying $x_\lambda \neq y_\lambda$. Consider $z = x - y$. Clearly $\lambda \in \max(z)$. Since $x, y \in J(F[Q])$, Lemma 1 shows that $z_\lambda \in J(F[Q_\lambda])$. By Proposition 2, $J(F[Q_\lambda]) = \sum_{p \in P} I(B_p(F), Q_\lambda, \xi_p)$, since Q_λ is not periodic. Evidently, ξ_p can not join a periodic element with a non-periodic one. Therefore $y_\lambda \in F[P]$, $x_\lambda \in F[N]$, and $x_\lambda - y_\lambda \in J(F[Q_\lambda])$ yield $x_\lambda, y_\lambda \in J(F[Q_\lambda])$. By Propositions 1, 2 $J(F[Q_\lambda])$ is nil, and so x_λ is nilpotent. Hence $w = x - x_\lambda$ is in $J(F[T])$. Meanwhile w is not nilpotent, for $w_\mu = x_\mu$. However, $|\operatorname{supp}(w)| < |\operatorname{supp}(x)|$ contradicting the choice of x. We have shown that $e_\lambda \in T_\lambda$ for any $\lambda \in \Lambda$.

Now take any $\gamma \in \operatorname{supp}(y) \setminus \{\mu\}$. There are $\lambda_1, ..., \lambda_m$ such that $\gamma = \lambda_1 \cdots \lambda_m$. Further $y_\gamma = k x_\mu e_{\lambda_1} \cdots e_{\lambda_m}$ for an integer k. Since $x_\mu \in F[T]$ and all $e_{\lambda_i} \in F[T]$ we get $y_\gamma \in F[T]$. Therefore $y \in F[T]$.

Propositions 1 and 2 show that $J(F[T_\mu])$ is nil modulo $J(F)[T_\mu]$. Hence $y_\mu^m \in J(F)[T_\mu]$. Since $y^{m-1} y = (\mu, x_\mu^m, \Lambda)$ we may for simplicity of notation assume that $y_\mu \in J(F)[T_\mu]$. Further, $y^m = (\mu, x_\mu^m, \Lambda)$ because $(\prod_{\lambda \in \Lambda} (e_\mu - e_\lambda))$ is an idempotent. Denote by $p(y^m)$ the additive period of y^m. Obviously $p(y^m)$ divides $p(y^{m+1})$. If there is a periodic element among $y, y^2, y^3, ...$ then we choose m such that $p(y^m)$ is the smallest possible period. For simplicity of notation assume that $m = 1$. Then $p(y) = p(y^2) = ...$ If all $y, y^2, ...$ are non-periodic then $0 = p(y) = p(y^2) = ...$ Thus we may assume that from the very beginning all the elements $y_\mu, y_\mu^2, ...$ are of same additive period. Denote it by d. Let $F_d = \{f \in F \mid df = 0\}$. Since F_d is an ideal of F, we get $y \in J(F_d[T])$. To simplify the notation, assume that $F = F_d$. If $d = 0$, then we denote by I the set of periodic elements of F and put $q = 0$. If $d \neq 0$, then d can be written as $d = qr$ for a prime number q, and we set $I = F_r$. Let $K = F/I$ and y denote also the image of y in $K[T]$. Then in both the cases $q \in \pi(K)$, for

$y_\mu \in J(K)[T_\mu]$. Evidently y is a non-nilpotent simplest element of $K[T]$, and K is a ring of characteristic q.

Clearly y_μ is of the form $y_\mu = \sum_{i=1}^{k} a_i s_i$, where $0 \neq a_i \in K$, $s_i \in T_\mu$. Denote by G or $G(y)$ the subsemigroup generated in T by $s_1, ..., s_k$. Since T_μ is periodic, G is a finite group. We may assume that from the very beginning y is chosen so that the cardinality of G is minimal. Now we shall prove that G is q-incomplete in T.

First we show that q does not divide $|G|$. Suppose the contrary and represent G as a direct product $H \times E$, where H is the largest q-subgroup of G. Then $|E| < |G|$. Write s_i as $s_i = (h_i, b_i)$, where $h_i \in H$, $b_i \in E$. Set $z = \sum_{i=1}^{k} a_i(h_i, b_i) - a_i(e_\mu, b_i)$. The elements (h_i, b_i) and (e_μ, b_i) are in the relation ξ_q with each other, since H is a q-group. By Proposition 1, $z \in B(K[T])$. Put $c = y_\mu - z$, $d = (\mu, c, \Lambda)$. Evidently $d - y = (\mu, z, \Lambda) \in B(K[T])$, and so $d \in Si(K[T])$ by Theorem 1. Further, d is not nilpotent and $G(d) \subseteq E \subseteq G(y)$, a contradiction with the minimality of $G(y)$. Thus q does not divide $|G|$.

Let I be the ideal generated in T by all e_λ, $\lambda \in \Lambda$. Put down all subgroups $H_1, ..., H_n$ of G such that $H_i = \{h \in G \mid ht_i = e_\mu t_i\}$ for a non-periodic element t_i of $GT \backslash I$. Denote by D_i the matrix of the relation of G by H_i and set $D_I(G) = [D_1 \mid \cdots \mid D_n]$. We are to prove that q divides every $(m \times m)$-minor of $D_I(G)$.

Since char $K = q$, it suffices to prove the equality $(a_1, ..., a_m) D_I(G) = 0$, where $y_\mu = \sum_{i=1}^{m} a_i g_i$, $G = \{g_1, ..., g_m\}$. This is equivalent to equalities $(a_1, ..., a_m) D_i = 0$, $i = 1, ..., n$. Let $(a_1, ..., a_m) D_i = (b_1, ..., b_m)$. We claim that $b_j = 0$.

The definition of D_i shows that $b_j = \sum_{g_k \in H_i g_j} a_k$. Take α in Γ such that $t_i \in T_\alpha$. Since $t_i \in GT \backslash I$, we get $\alpha \in \mu\Gamma \backslash \Lambda\Gamma$, implying $y_\mu e_\alpha \in J(K[T_\alpha])$. In view of the fact that T_α is not periodic, Proposition 2 yields $y_\mu e_\alpha \in I(K, T_\alpha, \xi_q)$. Further, $y_\mu e_\alpha \in K[Ge_\alpha]$ and q does not divide the order of the group Ge_α. Therefore $I(K, Ge_\alpha, \xi_q) = 0$, implying $y_\mu e_\alpha = 0$. Hence $y_\mu t_i = 0$, and so $\sum_{k=1}^{m} a_i g_i t_i = 0$. Therefore $\sum_{k: g_k t_i = g_j t_i} a_k = 0$. The equality $g_k t_i = g_j t_i$ is equivalent to $g_j^{-1} g_k \in H_i$ by the definition of H_i. Hence $b_j = \sum_{g_k \in H_i g_j} a_s = \sum_{g_k t_i = g_j t_i} a_k = 0$, yielding $(a_1, ..., a_m) D_I(G) = 0$. Thus G is q-incomplete in T as required.

Now we will prove the "if" part. Let $q \in \pi(F)$ and T contains a q-incomplete subgroup G. It is well known that a cancellative Archimedean semigroup is a group if it contains an idempotent. Therefore T_μ is a group.

Suppose that T_μ has a non-periodic element t and consider the group $H = \{h \in G \mid ht = et\}$. Clearly $H = \{e\}$. Then the matrix D of the relation of G by H is the identity matrix. Therefore q does not divide det $(D) = 1$, and D lies in the matrix $D_0(G)$. The contradiction with q-incompleteness of G shows that T_μ is a periodic group.

Let $G = \{g_1, ..., g_m\}$. Since G is q-incomplete, q does not divide m and there is an ideal I of T generated by idempotents $e_1, \ldots, e_k$ and such that q divides the determinant of every $(m \times m)$-matrix of $D_I(G)$. Then $e_i \in T_{\lambda_i}$ for some $\lambda_i \in \Gamma$. We may assume that $\lambda_i \leqslant \mu$, because otherwise one could substitude ee_i for e_i and $\lambda_i \mu$ for λ_i without changing the set of non-periodic elements in $GT \backslash I$. Write down all the groups $H_1, \ldots, H_n$ such that $H_i = \{h \in G \mid ht_i = e_\mu t_i\}$ for non-periodic $t_i \in GT \backslash I$. Denote by D_i the matrix of the conjugacy relation of G by H_i and set $D_I(G) = [D_1 \mid \cdots \mid D_n]$. Then q divides the determinant of each $(m \times m)$-submatrix of $D_I(G)$. Therefore the q-element field $GF(q)$ (or the field of rational numbers, if $q = 0$) contains elements $u_1, \ldots, u_m$ such that $(u_1, \ldots, u_m)\, D_I(G) = 0$, $(u_1, \ldots, u_m) \neq 0$. Since $\pi(R) = \pi(F)$, by the choice of q and F there exists a nonzero $r \in F$ such that $qr = 0$. Set $x = u_1 r g_1 + \cdots + u_m r g_m$. Since q does not divide G and $r \notin B(F) = 0$, Proposition 2 shows that x is not nilpotent. Put $\Lambda = \{\lambda_1, \ldots, \lambda_k\}$, $y = (\mu, x, \Lambda)$. We claim that $y \in Si(F[T])$, i.e., $xe_\lambda \in J(F[Q_\lambda])$ for any $\lambda \in \mu\Gamma \backslash \Lambda\Gamma$.

Indeed, if T_λ is periodic then the claim follows from Proposition 2 and $r \in J(F)$. Now consider the case where T_λ has a non-periodic element t. Then $t \notin I$ implying $\{h \in G \mid ht = e_\mu t\} = H_i$ for some i. Write $xt = u_1 r g_1 t + \cdots + u_m r g_m t$. Here $g_j t$ coincides with $g_k t$ if and only if g_j and g_k lie in the same class of the conjugacy relation of G by H_i. This and $(u_1, \ldots, u_m) D_i = 0$ yield $xt = 0$. Therefore $xe = xtt^{-1} = 0$, and so $y \in Si(F[T])$. By Theorem 1, $J(F[T])$ contains y, which was proved to be non-nilpotent. This proves the result.

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