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A-model and generalized Chern–Simons theory

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Abstract

The relation between open topological strings and Chern–Simons theory was discovered by Witten. He proved that A-model on T^*M where M is a three-dimensional manifold is equivalent to Chern–Simons theory on M and that A-model on arbitrary Calabi–Yau 3-fold is related to Chern–Simons theory with instanton corrections. In present Letter we discuss multidimensional generalization of these results.

1. Introduction

In present Letter we analyze the relation between multidimensional A-model of open topological strings and generalized Chern–Simons theory. Such a relation was discovered by Witten [1] in three-dimensional case; we generalize his results. Our approach is based on rigorous mathematical results of [2–6]; in three-dimensional case it gives mathematical justification of some of Witten's statements.

In modern language Witten considers A-model in presence of a stack of N coinciding D-branes wrapping a Lagrangian submanifold M. In the neighborhood of Lagrangian submanifold a symplectic manifold V looks like T^*M. In the case $V = T^*M$, $\dim M = 3$ Witten shows that A-model is equivalent to Chern–Simons theory on M. He considers also the case when V is a Calabi–Yau 3-fold and shows that in this case Chern–Simons action functional on M acquires instanton corrections.

We remark that one can analyze instanton corrections to Chern–Simons functional combining results by Fukaya [4] and Cattaneo–Froehlich–Pedrini [2] and that this approach works also in multidimensional case.

To study the origin of Chern–Simons functional and its generalizations one can replace the stack of N coinciding D-branes by N Lagrangian submanifolds depending on ε and tending to the same limit as $\varepsilon \to 0$. This situation was studied by Fukaya–Oh [5] and Kontsevich–Soibelman [6]; we will show that the appearance of Chern–Simons functional follows from their results.

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2. Generalized Chern–Simons theory

Multidimensional generalization of Chern–Simons theory can be constructed in the following way. We consider differential forms on d-dimensional compact manifold M taking values in Lie algebra $\mathcal{G}$. One assumes that $\mathcal{G}$ is equipped with invariant inner product. We will restrict ourselves to the only case we need: $\mathcal{G} = gl(N)$; then invariant inner product can be defined as $\langle a, b\rangle = \operatorname{Tr} ab$ where Tr denotes the trace in vector representation of $\mathcal{G} = gl(n)$. The graded vector space $\Omega^*(M) \otimes \mathcal{G}$ of such forms will be denoted by $\mathcal{E}$. The bilinear form $\langle C, C'\rangle = \int_M \operatorname{Tr} C \cdot C'$ specifies an odd symplectic structure on $\mathcal{E}$ if $\dim M$ is odd and even symplectic structure if $\dim M$ is even.

The generalized Chern–Simons functional $CS(C)$ is defined by the standard formula

$$\mathcal{S}(C) = CS(C) = \frac{1}{2}\int_M \operatorname{Tr} C\, dC + \frac{1}{3}\int_M \operatorname{Tr} C[C, C] \tag{1}$$

where $C \in \mathcal{E} = \Omega^*(M) \otimes gl(n)$ and d stands for the de Rham differential. We can replace d in (1) by the differential d_A corresponding to flat connection A; corresponding functional will be denoted by $\mathcal{S}_A$. Notice that the functional $\mathcal{S}_A$ for arbitrary flat connection in trivial vector bundle can be obtained from the functional (1) with the standard de Rham differential by means of shift of variables. It is easy to see that for any solution A of equation $dA + \frac{1}{2}[A, A] = 0$ we have

$$\mathcal{S}(C + A) = \mathcal{S}(A) + \frac{1}{2}\int_M \operatorname{Tr}\bigl(C\, dC + C[A, C]\bigr) + \frac{1}{3}\int_M \operatorname{Tr} C[C, C]. \tag{2}$$

If A is a 1-form such a solution determines a flat connection and (2) coincides (up to a constant) with the corresponding action functional. This remark permits us to reduce the study of Chern–Simons functional with flat connection to the study of functional (1).

In the case when $\dim M$ is odd $\mathcal{E}$ is an odd symplectic space hence we can define an odd Poisson bracket on the space of functionals on $\mathcal{E}$ (on the space of preobservables); the functional $\mathcal{S}_A$ obeys the BV classical master equation $\{\mathcal{S}_A, \mathcal{S}_A\} = 0$ and therefore can be considered as an action functional of classical mechanical system in BV-formalism. Corresponding equations of motion have the form

$$d_A C + \frac{1}{2}[C, C] = 0.$$

The functional $\mathcal{S}$ determines an odd differential δ on the algebra of preobservables by the formula $\delta(\mathcal{O}) = \{\mathcal{S}, \mathcal{O}\}$; homology of δ are identified with classical observables.

In the case of even-dimensional manifold M the functional (1) has an interpretation in terms of BFV-formalism. The Poisson bracket on the space of functionals on $\mathcal{E}$ (on the space of preobservables) is even; the operator δ can be interpreted as BRST operator and its homology as classical observables.

The generalized Chern–Simons action functional (1) was considered in [7,8] in the framework of BV sigma-model. In the definition of BV sigma-model we consider the space $\mathcal{E}$ of maps of ΠTM, where M is a d-dimensional manifold into (odd or even) symplectic Q-manifold X. (One says that a supermanifold equipped with an odd vector field obeying $\{Q, Q\} = 0$ is a Q-manifold. De Rham differential specifies the structure of Q-manifold on ΠTM.) The space of maps of Q-manifold into a Q-manifold also can be regarded as a Q-manifold. From the other side using the volume element on ΠTM and symplectic structure on X we can define odd or even symplectic structure on $\mathcal{E}$. These facts permit us to consider BV or BFV theory where fields are identified with functionals on $\mathcal{E}$.

Numerous topological theories can be obtained as particular cases of BV sigma-model. It was shown in [7] that A-model and B-model can be constructed this way.

To obtain generalized Chern–Simons theory from BV-sigma model we should take $X = \Pi\mathcal{G}$ in this construction. (If $\mathcal{G}$ is a Lie algebra we can consider $\Pi\mathcal{G}$ as a Q-manifold where Q is a vector field $\frac{1}{2} f^{\gamma}_{\alpha\beta} c^{\alpha} c^{\beta} \frac{\partial}{\partial c^{\gamma}}$. We use the notation c^{α} for coordinates in $\Pi\mathcal{G}$ corresponding to the basis e_{α} in $\mathcal{G}$; structure constants of $\mathcal{G}$ corresponding to this basis are denoted by $f^{\alpha}_{\beta\gamma}$. An invariant inner product on $\mathcal{G}$ specifies a symplectic structure on $\Pi\mathcal{G}$.)

In [9] Kontsevich constructed a multidimensional generalization of perturbation series for standard Chern–Simons. It was shown in [8] that the perturbation theory for generalized Chern–Simons theory coincides with Kontsevich generalization. It is important to emphasize that usual correlation functions of multidimensional Chern–Simons theory are trivial, however, one can define non-trivial cohomology classes of some space that play the role of generalized correlation functions. (In [9] this space was related to the classifying space of diffeomorphism group of M, in [8] it was interpreted as moduli space of gauge conditions in the corresponding BV sigma-model.)

Notice that one can construct Chern–Simons functional for every differential associative $\mathbf{Z}_2$-graded algebra $\mathcal{A}$ equipped with invariant inner product $\langle\,,\rangle$. (We assume that the algebra is unital; then the invariant inner product can be written in terms of trace: $\langle a,b\rangle = \operatorname{tr} ab$.) For every N we define the associative algebra $\mathcal{A}_N$ as tensor product $\mathcal{A}\otimes \mathrm{Mat}_N$ where Mat_N stands for the matrix algebra. We define Chern–Simons functional for $A\in\mathcal{A}_N$ by the formula

$$\begin{aligned} CS(A) &= \frac{1}{2}\operatorname{tr} A\,dA + \frac{2}{3}\operatorname{tr} A^3 \\ &= \frac{1}{2}\operatorname{tr} A\,dA + \frac{1}{3}\operatorname{tr} A[A,A]. \end{aligned}$$

(Notice that we need really only the super Lie algebra structure defined by the super commutator in the associative algebra $\mathcal{A}_N$.)

The functional CS coincides with (1) in the case when $\mathcal{A}$ is the algebra $\Omega(M)$ of differential forms on manifold M equipped with a trace $\operatorname{tr} C = \int_M C$.

The construction of CS functional can be generalized to the case when $\mathcal{A}$ is an A_∞-algebra equipped with invariant inner product. Recall that the structure of A_∞-algebra $\mathcal{A}$ on a $\mathbf{Z}_2$-graded space is specified by means of a sequence ${}^{(k)}m$ of operations; in a coordinate system the operation ${}^{(k)}m$ is specified by a tensor ${}^{(k)}m^a_{a_1,\ldots,a_k}$ having one upper index and k lower indices. Having an inner product we can lower the upper index; invariance of inner product means that the tensor ${}^{(k)}\mu_{a_0,a_1,\ldots,a_k} = g_{a_0 a} m^a_{a_1,\ldots,a_k}$ is cyclically symmetric (in graded sense). The Chern–Simons functional can be defined on $\mathcal{A}\otimes\mathrm{Mat}_N$ by means of tensors ${}^{(k)}\mu$; see [10] for details. Notice that two quasiisomorphic A_∞-algebras are physically equivalent, i.e., corresponding Chern–Simons functionals lead to the same physical results. (In this statement we consider action functionals at the level of classical theory.)

A differential associative algebra can be considered as an A_∞-algebra where only operations ${}^{(1)}m$ and ${}^{(2)}m$ do not vanish; in this case both definitions of Chern–Simons functional coincide.

3. Observables of Chern–Simons theory

If Chern–Simons theory is constructed by means of associative graded differential algebra $\mathcal{A}$ with inner product it is easy to check that classical observables of this theory correspond to cyclic cohomology of $\mathcal{A}$. This fact is equivalent to the statement that infinitesimal deformations of $\mathcal{A}$ into A_∞-algebra with inner product are labeled by cyclic cohomology $HC(\mathcal{A})$ of $\mathcal{A}$ [11]. (Recall, that classical observables are related to infinitesimal deformations of the theory.) Algebra $\mathcal{A}$ determines Chern–Simons theory for all N the observables we were talking about were defined for every N.

As we mentioned the generalized Chern–Simons theory corresponds to the algebra of differential forms $\Omega^*(M)$ with de Rham differential. It is well known [12,13] that cyclic cohomology of this algebra are related to equivariant homology of loop space $L(M)$. More precisely, there exists a map of equivariant homology $H_{S^1}(L(M))$ into cyclic cohomology $HC(\Omega^*(M),d)$; if M is simply connected this map is an isomorphism.

Recall that the loop space LM is defined as a space of all continuous maps of the circle $S^1=\mathbf{R}/\mathbf{Z}$ into M; the group S^1 acts on LM in obvious way: $\gamma(t)\to\gamma(t+s)$. It will be convenient to modify the definition of LM considering only piecewise differential maps; this modification does not change the homology.

Instead of equivariant homology of LM one can consider homology of the space of closed curves (string space) SM obtained from LM by means of factorization with respect to S^1. The manifold M is embedded in LM and in $SM = LM/S^1$ as the space of constant loops; excluding constant loops from consideration we can identify S^1-equivariant homology of $LM\setminus M$ with homology of $SM\setminus M$. (In general S^1-equivariant homology of over real numbers can be identified with homology of quotient space if all stabilizers are finite.)[2]

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[2] The space $SM\setminus M$ is an infinite-dimensional orbifold with orbifold points corresponding to n-fold curves. See [14] for the first analysis of orbifold structure of SM and calculation of homology of SM in simple cases.

Following [3] we will use the term “string homology” and the notation $\mathcal{H}_*M$ for the homology of string space *SM*.

The homomorphism of $\mathcal{H}_*M$ into the space of observables of Chern–Simons theory can be described in the following way [2].

Let us consider the standard simplex $\Delta_n = \{(t_1, \dots, t_n) \in \mathbf{R}^n \mid 0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant 1\}$ and evaluation maps $ev_{n,k} : \Delta_n \times LM \to M$ that transform a point $(t_1, \dots, t_n, \gamma) \in \Delta_n \times LM$ in $\gamma(t_k)$ (here $1 \leqslant k \leqslant n$). Using these maps we can construct a differential form on *LM* by the formula

$$h(C) = \operatorname{Tr} \int_{\Delta_n} ev_{n,1}^* C \cdots ev_{n,n}^* C,$$

where $C \in \mathcal{E} = \Omega^*(M) \otimes \mathrm{Mat}_N$ is a differential form on M taking values in $N \times N$ matrices. We obtain a map of the space of fields of Chern–Simons theory into $\Omega^*(LM)$.

The form $h(C)$ descends to the string space *SM*. If a is a singular chain in *SM*, then

$$\rho_a(C) = \int_a h(C) \tag{3}$$

specifies a functional on the space $\mathcal{E}$ of fields (a preobservable of Chern–Simons theory). It follows from results of [2] that

$$\delta\rho_a = (-1)^n \rho_{\partial a}, \tag{4}$$

where ∂a stands for the boundary of the chain a. This means, in particular, that in the case when a is a cycle in the homology of *SM* (in the string homology) ρ_a is an observable and that two homologous cycles specify equivalent observables. We obtain a map of string homology $\mathcal{H}_*M$ into the space of observables of Chern–Simons theory on M.

4. String bracket

Let us describe some operations in homology of loop space *LM* and string space *SM* that were introduced in [3].

The most fundamental of these operation is the loop product on the loop space. It assigns (under some transversality assumptions) an $(i+j-d)$-dimensional chain $a \bullet b$ in *LM* to i-dimensional chain a and j-dimensional chain b. To construct $a \bullet b$ one first intersects in M the chain of marked points of a with the chain of marked points of b to obtain an $(i+j-d)$-dimensional chain in M along which the marked points of a coincides with the marked points of b. Now one defines the chain $a \bullet b$ by means of concatenation of the loops of a and the loops of b having common marked points.

The operator Δ on the chains of the loop space *LM* transforms an i-dimensional chain a into $(i+1)$-dimensional chain Δa obtained by means of circle action on *LM*.

The bracket $\{a, b\}$ of i-dimensional chain a in *LM* and j-dimensional chain b in *LM* is an $(i+j+1)$-dimensional chain that can be defined by the formula

$$\{a, b\} = (-1)^i \Delta(a \bullet b) - (-1)^i \Delta a \bullet b - a \bullet \Delta b. \tag{5}$$

All these operations descend to homology of *LM*; the homology becomes a Batalin–Vilkovisky algebra [15, 16] with respect to them.

A natural map of *LM* onto *SM* (erasing the marked point) determines a homomorphism proj of chain complexes. An i-dimensional chain in *SM* can be lifted to $(i+1)$-dimensional chain in *LM* (we insert marked points in all possible ways); corresponding homomorphism of chain complexes will be denoted by lift.

The string bracket of two chains in *SM* can be defined by the formula

$$[a, b] = \operatorname{proj}(\operatorname{lift} b \bullet \operatorname{lift} a). \tag{6}$$

If $\dim a = i$, $\dim b = j$, then $\dim[a, b] = i + j - d + 2$. This bracket descends to homology of *SM* (to string homology), defining a graded Lie algebra. The above definition of bracket agrees with [2]; in the definition of [3] a and b are interchanged.

As we know, there exists a map of string homology into the space of observables. The main result of [2] is a theorem that this map is compatible with Lie algebra structures on string homology and on the space of observables:

$$\{\rho_a, \rho_b\} = \rho_{[a,b]}, \tag{7}$$

where $\{\,,\}$ stands for the Poisson bracket.

It is important to notice that (7) remains correct if a and b are arbitrary chains not necessary cycles obeying some transversality conditions. Then ρ_a and ρ_b are in general preobservables. This fact follows immediately from the considerations of [2].

Notice that the action of the group $\mathrm{Diff}(S^1)$ of orientation preserving diffeomorphims of circle S^1 determines an action of this group on *LM*. Factorizing *LM* with respect to this action we obtain a space SM_{new} that is homotopically equivalent to *SM*. (This follows from the fact that $\mathrm{Diff}(S^1)$ is homotopically equivalent to S^1.) Similarly, instead of *LM* we can consider a space LM_{new} obtained from *LM* by means of factorization with respect to the contractible group $\mathrm{Diff}_0\, S^1$ defined as a subgroup of $\mathrm{Diff}(S^1)$ consisting of maps leaving intact the point $1 \in \partial D$.

5. *A*-model and string bracket

In this section we review some results of Fukaya [4]. We will give also modification of these results to the form that allows us to relate them with the constructions of [2].

Let us consider a symplectic manifold V and a Lagrangian submanifold $M \subset V$. Correlation functions of A-model on V can be calculated by means of localization to moduli spaces of (psedo)holomorphic maps of Riemann surfaces; in the case of open strings one should consider maps of bordered surfaces transforming the boundary into M [1]. We restrict ourselves to the genus zero case; then one should consider holomorphic maps φ of the disk D into V obeying $\varphi(\partial D) \subset M$. Every such map specifies an element of $\pi_2(V,M)$. One denotes by $\tilde{\mathcal{M}}(M,\beta)$ the moduli space of holomorphic maps $\varphi : (D,\partial D) \to (V,M)$ that have a homotopy type $\beta \in \pi_2(V,M)$. We use the notations $\hat{\mathcal{M}}(M,\beta) = \tilde{\mathcal{M}}(M,\beta)/\operatorname{Aut}(D^2,1)$ and $\mathcal{M}(M,\beta)/PSL(2,\mathbf{R})$ where $PSL(2,\mathbf{R})$ is the group of fractional linear transformations identified with biholomorphic maps $D \to D$ and $\operatorname{Aut}(D,1)$ denotes its subgroup consisting of maps leaving intact the point $1 \in \partial D$. The spaces $\hat{\mathcal{M}}(M,\beta)$ and $\mathcal{M}(M,\beta)$ should be compactified by including stable maps from open Riemann surfaces of genus 0; we will use the same notation for compactified spaces.

Notice that $\mathcal{M}(M,\beta)$ specifies a chain $\mathcal{M}_\beta$ in the string space *SM*. (We define a map $\mathcal{M}(M,\beta) \to SM_{\mathrm{new}}$ restricting every map $\varphi : D \to V$ belonging to $\mathcal{M}(M,\beta)$ to the boundary of the disk D. We use in this construction the modified definition of *SM* discussed at the end of Section 4. To obtain a chain in *SM* we use a map of SM_{new} onto *SM* that specifies homotopy equivalence of these two spaces.) Similarly, $\hat{\mathcal{M}}(M,\beta)$ specifies a chain $\hat{\mathcal{M}}_\beta$ in the loop space *LM*; the chain $\hat{\mathcal{M}}_\beta$ can be considered as a lift of $\mathcal{M}_\beta$. (Again we are using modified definition of *LM* at the intermediate step.)

Fukaya [4,17] proved the following relation

$$\partial \hat{M}_\beta + \frac{1}{2} \sum_{\beta=\beta_1+\beta_2} \{\hat{M}_{\beta_1}, \hat{M}_{\beta_2}\} = 0, \tag{8}$$

where $\{,\}$ stands for the loop bracket in *LM*. We will derive from (8) the relation

$$\partial M_\beta + \frac{1}{2} \sum_{\beta=\beta_1+\beta_2} [M_{\beta_1}, M_{\beta_2}] = 0, \tag{9}$$

where $[\,,]$ denotes the string bracket in *SM*.

The derivation is based on relation $\hat{\mathcal{M}}_\beta = \operatorname{lift} \mathcal{M}_\beta$. We notice that

$$\partial \hat{M}_\beta = \partial(\operatorname{lift} M_\beta) = \operatorname{lift}(\partial M_\beta). \tag{10}$$

From the other side

$$\begin{aligned}
\partial \hat{\mathcal{M}}_\beta &= -\frac{1}{2} \sum_{\beta_1+\beta_2=\beta} \{\hat{\mathcal{M}}_{\beta_1}, \hat{\mathcal{M}}_{\beta_2}\} \\
&= -\frac{1}{2} \sum_{\beta_1+\beta_2=\beta} \{\operatorname{lift}\mathcal{M}_{\beta_1}, \operatorname{lift}\mathcal{M}_{\beta_2}\} \\
&= -\frac{1}{2} \sum_{\beta_1+\beta_2=\beta} \Big((-1)^{(\dim \mathcal{M}_{\beta_1}+1)} \\
&\quad \times \Delta(\operatorname{lift}\mathcal{M}_{\beta_1} \bullet \operatorname{lift}\mathcal{M}_{\beta_2}) \\
&\quad - (-1)^{(\dim \mathcal{M}_{\beta_1}+1)} \Delta(\operatorname{lift}\mathcal{M}_{\beta_1}) \bullet \operatorname{lift}\mathcal{M}_{\beta_2} \\
&\quad - (\operatorname{lift}\mathcal{M}_{\beta_1}) \bullet \Delta(\operatorname{lift}\mathcal{M}_{\beta_2})\Big) \\
&= \operatorname{lift}\left(-\frac{1}{2} \sum_{\beta_1+\beta_2=\beta} [\mathcal{M}_{\beta_1}, \mathcal{M}_{\beta_2}]\right).
\end{aligned} \tag{11}$$

In the derivation of this formula we used (5)–(7) and relations $\Delta \bullet \operatorname{lift} = 0$, $\Delta = \operatorname{lift} \bullet \operatorname{proj}$.

We obtain (9) comparing (8) and (11).

Let us fix a ring Λ and a map $\alpha : H_2(V,M) \to \Lambda$ obeying $\alpha(\beta_1+\beta_2) = \alpha(\beta_1) \cdot \alpha(\beta_2)$.

We can construct a Λ-valued chain $\mathcal{M}$ on *SM* taking

$$\mathcal{M} = \sum_\beta \alpha_\beta \mathcal{M}_\beta. \tag{12}$$

It follows immediately from (11) that

$$\partial\mathcal{M}+\frac{1}{2}[\mathcal{M},\mathcal{M}]=0. \tag{13}$$

Usually one takes as Λ the Novikov ring (a ring of formal expressions of the form $\sum a_i T^{\lambda_i}$ where $a_i\in\mathbf{R}$, $\lambda_i\in\mathbf{R}$, $\lambda_i\to+\infty$). The map α should be fixed in a way that guarantees finiteness of all relevant expressions. Our considerations will be completely formal; we refer to [18] for an appropriate choice of α.

6. *A*-model and Chern–Simons theory

Let us start with the chain $\mathcal{M}$ on SM constructed at the end of Section 5.

We can construct the corresponding preobservable of generalized Chern–Simons theory using (13). It follows immediately from (7) and (13) that the preobservable $\rho=\rho_{\mathcal{M}}$ obeys

$$\delta\rho+\frac{1}{2}\{\rho,\rho\}=0.$$

We can modify the Chern–Simons functional adding ρ. The new functional $\mathcal{S}+\rho$ verifies

$$\{\mathcal{S}+\rho,\mathcal{S}+\rho\}=0.$$

This means that $\mathcal{S}+\rho$ can be considered as a solution of classical master equation (an action functional in BV formalism) if $\dim M$ is odd and as a BRST generator if $\dim M$ is even. In the case $\dim M=3$ the functional ρ represents instanton corrections to the Chern–Simons action; one can argue that this is true in any dimension.

The above consideration is not completely rigorous. We used the results of [2,3,12] about the string bracket on the space of chains in SM. These papers use different definitions of string bracket; all of them agree on homology, however, it is essential for us to consider the bracket of chains that are not necessarily cycles. To give a rigorous proof one has to check that all results we are using can be verified with the same definition of string bracket; this should not be a problem.

We have seen that A-model instanton corrections to Chern–Simons functional can be generalized very naturally to any dimension. This is a strong indication that Chern–Simons functional by itself also appears in multidimensional A-model. Indeed, analyzing Witten's arguments [1] based on the application of string field theory one can reach a conclusion that A-model on T^*M is equivalent to the generalized Chern–Simons theory on M. (One can understand from Witten's paper, that he was aware of possibility of multidimensional generalization of his constructions.)

It seems that the mathematical justification of this statement can be based on the idea that a stack of N coinciding D-branes can be replaced by N Lagrangian submanifolds that depend on some parameter and coincide when the parameter tends to 0. This situation was studied by Fukaya–Oh [5] and Kontsevich–Soibelman [6].

Let us consider N transversal Lagrangian submanifolds $M_1,\ldots,M_N$ in symplectic manifold V. One can construct corresponding A_∞-category (Fukaya category) [18]. The construction of operations in this category is based on the consideration of moduli spaces of pseudoholomorphic maps of a disk D into V. (One assumes that V is equipped with almost complex structure J; in the case when $V=T^*M$ one assumes that almost complex structure is induced by a metric on M.) One fixes the intersection points $x_i\in M_i\cap M_{i+1}$ for $1\leqslant i\leqslant N-1$ and $x_N\in M_N\cap M_1$. The Fukaya category is defined in terms of moduli spaces $M_J^z(V,M_i,x_i)$ of J-holomorphic maps $v:D\to V$ transforming given points $z_i\in\partial D$ into points x_i.

One should consider also the union of all spaces $\mathcal{M}_J^z$ where z_i run over all cyclically ordered subsets of ∂D and factorize this union with respect to the group $PSL(2,\mathbf{R})$ acting as a group of biholomorphic automorphisms of the disk; one obtains the moduli spaces $\mathcal{M}_J(V,M_i,x_i)$. The definition of operations in Fukaya category involves summation over $\mathcal{M}_J$.

Following [5] we can consider the case when $V=T^*M$ and the Lagrangian submanifolds M_i are defined as graphs $M_i=(x,\xi)\in T^*M|\xi=\varepsilon df_i(x)$ where $f_1,\ldots,f_N$ are such functions on M that difference between any two of them is a Morse function; then the corresponding Lagrangian submanifolds are transversal and intersection points $x_i\in M_i\cap M_{i+1}$ are critical points of functions f_i-f_{i+1}. Fukaya and Oh [5] have studied the moduli spaces $\mathcal{M}_J(V,M_i,x_i)$ for this choice of Lagrangian submanifolds. They have proved that for small ε these moduli spaces are diffeomorphic to moduli spaces $\mathcal{M}_g(M, f_i, p_i)$ of graph flows. (An element of moduli spaces $\mathcal{M}_g(M, f_i, p_i)$ where p_i are critical points of $f_i - f_{i+1}$ is a map of a metric graph γ into M transforming edges of the graph γ into trajectories of negative gradient flow of the difference of two of the functions. It is assumed that the graph γ is a rooted tree embedded into the disk D and the exterior vertices are mapped into ∂D.)

This picture is very close to the Witten's picture [1] where graphs appear as degenerate instantons. It is clear from it that A-model on T^*M can be reduced to quantum field theory—summation over embedded holomorphic disks can be replaced by the summation over graphs. However, it is not clear yet that this quantum field theory coincides with Chern–Simons theory. To establish this one can apply the results of [6].

The papers [5,6] use the language of A_∞-categories. In this language the results of [5] can be formulated in the following way: Fukaya A_∞-category constructed by means of Lagrangian submanifolds of T^*M is equivalent to Morse A_∞-category of smooth functions on M. It is proved in [6] under certain conditions that the Morse A_∞-category is equivalent to de Rham category. All A_∞-categories (or, more precisely, A_∞-precategories) in question are equipped with inner product; the equivalence is compatible with inner product.

The minimal model of Fukaya A_∞-category is related to tree level string amplitudes; the relation of these amplitudes to Chern–Simons theory can be derived from the remark that quasiisomorphic A_∞-algebras with inner product specify equivalent Chern–Simons theories.

It is important to emphasize that A-model for any genus is related to Chern–Simons theory. It was mentioned in [5] that not only moduli spaces of pseudoholomorphic disks on T^*M but also moduli spaces of higher genus pseudoholomorphic curves can be described in terms of graphs. Again, this is consistent with equivalence of A-model to quantum field theory. In simplest case the relation to Chern–Simons theory was studied in [19].

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