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Observers for linear positive systems

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The paper is dedicated to Paul Fuhrmann on the occasion of his 70th birthday

Abstract

Observer synthesis for linear positive systems is treated. The concept of observability of a linear positive system is defined and a characterization of observability is provided. An observable canonical form is proposed for a linear positive system with respect to an equivalent relation defined by permutations of the state and of the output set. Observer synthesis is carried out for linear observers which are either globally asymptotically stable or whose error dynamics is assignable. An algorithm for the construction of a linear observer is stated which is based on the observable canonical form.

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1. Introduction

The purpose of this paper is the synthesis of observers for linear positive systems. The results are the existence of such an observer and the proof of its properties. In addition, it includes the concepts of observability and detectability of linear positive systems.

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The motivation for the observers of linear positive systems is their use in state estimation, in prediction, and in system identification. Such problems arise in medicine, biology, chemical engineering, and economics. Linear positive systems are often a first approximation to handling nonlinear positive systems. The long term interest of the authors in this investigation is primarily in observers for nonlinear positive systems which are models of biochemical reaction networks. The current paper is the first step towards observers of nonlinear positive systems.

The observer problem is to synthesize for either a discrete-time or for a continuous-time linear positive system a dynamic system, to be called an *observer*, which is driven by the output process of the linear positive system, such that the estimation error between the state of the linear positive system and the output of the observer is globally asymptotically stable. In the case of ordinary linear systems an equivalent condition for the existence of a globally asymptotically stable linear observer is that the system is detectable with observability as a sufficient condition. Therefore the observer problem includes the formulation of the concepts of observability and detectability for linear positive systems.

The concept of an observer for a dynamic system was introduced by Luenberger in [24,25] though it was clearly inspired by the concept of the Kalman filter introduced earlier by Kalman [19,20]. There are a few papers which discuss observers for positive systems. Van den Hof in [39] formulates and contributes to the observer problem for linear compartmental systems. A linear compartmental system is a linear positive system which satisfies the property of mass conservation. Chaves and Sontag in [8,9] solve the observer problem for a class of polynomial positive systems. Lemesle has developed observers for another particular class of nonlinear positive systems, see [23]. Below in this paper the observer problem for linear positive systems is treated which then extends the case of linear compartmental systems treated in [39]. Observability of linear positive systems is treated in several papers, see [7,29,30]. Controllability of the same class is treated in the paper [37], see also the references quoted in that paper. For realization of linear positive systems the reader is referred to the recent tutorial article [2].

The subclass of polynomial positive systems treated in [9] does not include the general case of linear positive systems treated below in this paper. Observability of polynomial systems, without the positivity constraint, is treated in [32]. Observability and observers for ordinary linear systems may be found in text books, for example [36]. An earlier draft of this paper appeared in a proceedings, [16].

The novelty of this paper is in: the characterization of observability of linear positive systems (Theorem 4.7 and Proposition 4.8), an observable canonical form for linear positive systems (Definition 4.9 and Theorem 4.12), observer synthesis based on the theory of observers for linear systems (Theorem 5.3 for not necessarily positive observers), and observer synthesis for linear positive systems with either an irreducible system matrix (Section 5.5) or a completely reduced system matrix (Algorithm 5.6 and Theorem 5.7).

The authors have been asked to clarify in the paper why they prefer an observer for a linear positive system which is a positive system itself. The reason for this is the interpretation of the observer states. Biologists, economists, and researchers in applied areas hesitate to use an observer which produce states with negative real numbers. For example, the second named author with his Ph.D. student Jacqueline van den Hof have cooperated with the government laboratory Rijksinstituut voor Volksgezondheid and Milieuhygiëne (National Institute of Public Health and Environmental Hygiene) on modeling and observers for compartmental systems. The institute had an application where the interpretation of the observers states as concentrations in particular organs of animals was important. Standards for the daily allowable uptake of dangerous chemicals are set upon the medically allowable largest concentrations in particular organs like the liver. For this reason there is a need for observers of linear positive systems which themselves are also positive systems. In addition, a linear positive system has an algebraic structure in terms of an interconnection of irreducible subsystems, as explained in Section 3.2 of the paper. This structure may also help in the interpretation of the observer state. It is true that in the paper [9] the authors do not require the observer to be positive. An argument, which is not stated in the above quoted paper, is that when the observer state, which is proven to be globally asymptotically stable, has closely approached the state of the system and when the system itself is positive then the observer state will also be positive. In most cases this may be true except if the state of the system generating the data moves on a facet of the positive orthant or close to it.

An outline of the paper follows. In the next section the problem of observer construction is formulated. Section 3 presents the concepts of a system graph and of a decomposition of a linear positive system into irreducible subsystems. Section 4 presents the concept of observability of linear positive systems and its characterization. The problem of observer synthesis is treated in Section 5. In Section 6 a linear positive observer is constructed for glycolysis in yeast. Concluding remarks are stated in Section 7. The corresponding results for continuous-time linear positive systems are stated in Appendix A.

2. Problem formulation

2.1. Terminology and notation

The notation for positive system used in this paper is known though scattered over several books and papers. The reader may want to consult the book on positive linear algebra [4] and its more recent edition [5]. Books on positive systems are [3,12] and also [26, chapter 6].

The *real numbers* are denoted by $\mathbb{R}$. The *positive real numbers* are denoted by $\mathbb{R}_+ = [0, \infty)$ and the *strictly-positive real numbers* by $\mathbb{R}_{s+} = (0, \infty)$. This terminology differs slightly from that used in certain books but is also used. The set of the positive real numbers is a *semiring*, it is closed with respect to addition and multiplication but does not have an inverse with respect to addition.

Denote the set of the *integers* by $\mathbb{Z}$ and the set of the *natural numbers* by $\mathbb{N}$. For $n \in \mathbb{Z}$ denote by $\mathbb{Z}_n = \{1, \ldots, n\}$ the set of the first n positive integers and by $\mathbb{N}_n = \{0, 1, \ldots, n\}$ the set of the first $n + 1$ natural numbers.

For $n \in \mathbb{Z}_+$, the set of *positive real vectors* with entries in $\mathbb{R}_+$ is denoted by $\mathbb{R}^n_+$ and the set of *strictly-positive real vectors* by $\mathbb{R}^n_{s+}$. The set of positive real vectors $\mathbb{R}^n_+$ is a positive vector space over the semiring $\mathbb{R}_+$.

For $n \in \mathbb{Z}_+$, denote the set of the *positive matrices* by $\mathbb{R}^{n\times n}_+$ which is defined as the set of matrices with elements in the positive real numbers. The set of positive matrices, say $\mathbb{R}^{n\times n}_+$ for $n \in \mathbb{Z}_+$, is a *dioid*, it does neither have an inverse with respect to addition nor with respect to multiplication, even for nonsingular positive matrices. Denote the set of permutation matrices of size $n \times n$ by $\mathbb{P}^{n\times n}$.

The spectrum of a matrix, say $A \in \mathbb{R}^{n\times n}_+$ is denoted by $\mathrm{spec}(A) \subset \mathbb{C}$. Denote,

$$\mathbb{D}_o = \{c \in \mathbb{C} | |c| < 1\}, \quad \mathbb{D}_c = \{c \in \mathbb{C} | |c| \leqslant 1\}, \quad \mathbb{C}^- = \{c \in \mathbb{C} | \mathrm{Re}(c) < 0\}.$$

2.2. Linear positive systems

Attention is restricted in this paper to linear positive systems without inputs. The extension to the presence of inputs is easy and analogous to the case of ordinary linear systems.

Definition 2.1. A *time-invariant discrete-time linear positive system* (without input) is a dynamic system of the form

$$x(t+1) = Ax(t), x(t_0) = x_0, \quad (1)$$
$$y(t) = Cx(t), \quad (2)$$
$$T = [t_0, \infty) \subset \mathbb{R},\ x : T \to \mathbb{R}_+^n,\ y : T \to \mathbb{R}_+^p,\ x_0 \in \mathbb{R}_+^n,$$
$$A \in \mathbb{R}_+^{n\times n},\ \text{and } C \in \mathbb{R}_+^{p\times n}.$$

Call then x the *state function* and y the *output function*.

An ordinary discrete-time dynamic system, not necessarily linear, say,

$$x(t+1) = f(x(t)), x(t_0) = x_0,$$
$$y(t) = h(x(t)), x : T \to \mathbb{R}^n, y : Y \to \mathbb{R}^p,$$

is called *positive* if the set of the positive real vectors $\mathbb{R}_+^n$ is forward invariant for the difference equation and if the output y remains positive: for all $x_0 \in \mathbb{R}_+^n$ and for all $t \in T$, $x(t) \in \mathbb{R}_+^n$ and $y(t) \in \mathbb{R}_+^p$. It is well known that the linear difference equation

$$x(t+1) = Ax(t), x(t_0) = x_0,$$
$$y(t) = Cx(t),$$
$$x : T \to \mathbb{R}^n, y : T \to \mathbb{R}^p, A \in \mathbb{R}^{n\times n}, C \in \mathbb{R}^{p\times n},$$

is positive if and only if $A \in \mathbb{R}_+^{n\times n}$. When the state is in the positive orthant then the output is a positive function if and only if the output matrix satisfies $C \in \mathbb{R}_+^{p\times n}$.

2.3. *Problem of observer synthesis*

The problem of observer synthesis for a linear positive system is formulated below but the problem formulation is preceded by the concept of an observer.

Definition 2.2. Consider the discrete-time linear positive system

$$x(t+1) = Ax(t), x(t_0) = x_0,$$
$$y(t) = Cx(t).$$

A *discrete-time linear observer* for this system is a dynamic system of the form

$$\hat{x}(t+1) = F\hat{x}(t) + Ky(t), \quad \hat{x}(t_0) = \hat{x}_0 \in \mathbb{R}_+^n, \quad \hat{x} : T \to \mathbb{R}^n, \quad (3)$$

for which the system matrices F and K are to be selected. It is called:

1. *globally asymptotically stable*: if the estimation error is globally asymptotically stable:
$$\forall x_0 \in \mathbb{R}_+^n,\ \forall \hat{x}_0 \in \mathbb{R}_+^n, \quad \lim_{t\to\infty} [\hat{x}(t) - x(t)] = 0; \quad (4)$$
2. *dynamically assignable*: if for any complex conjugate subset $\Lambda \subset \mathbb{D}_o$ there exists a gain matrix $K \in \mathbb{R}^{n\times p}$ such that the eigenvalues of the error system, for $\hat{x} - x$, have Λ as its eigenvalues; and

3. *a positive observer*: the observer is a positive system:

$$y : T \to \mathbb{R}_+^p, \quad \hat{x}_0 \in \mathbb{R}_+^n, \quad t \in T \Rightarrow \hat{x}(t) \in \mathbb{R}_+^n. \tag{5}$$

One can then define a positive observer which is globally asymptotically stable or a positive observer which is dynamically assignable.

For linear systems global asymptotic stability is equivalent to exponential stability but that terminology will not be used in this paper.

Problem 2.3. Consider a linear positive system, either discrete-time or continuous-time. Construct a linear observer for either system which is either globally asymptotically stable or which is dynamically assignable. A characterization of the existence of a positive observer will be useful. The problem includes formulating necessary and sufficient conditions for the existence of such observer and the observer synthesis as such.

3. Graphs and decomposition of linear positive systems into irreducible subsystems

The analysis and synthesis of positive systems is facilitated by the decomposition of a positive system into irreducible subsystems. Due to the algebraic properties of the positive real numbers these decompositions play a more important role in the system theory of positive systems than that of linear systems, systems over the real numbers.

The decomposition of a positive system into irreducible subsystems depends on the underlying graph associated to the system. Below both the graphs and the decompositions of matrices and of linear positive systems are formulated because these are used subsequently in the paper.

3.1. Graphs of linear positive systems

The reader is assumed to be familiar with the concept of a *directed graph* denoted by $G = (V, E)$ where V denotes a finite set of *vertices* and $E \subseteq V \times V$ denotes a finite set of *edges*. A *path* is an ordered finite set of edges,

$$((v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)) \subset E.$$

Denote such a path as $v_0 \mapsto v_n$. Define the connectivity relation $\mathrm{R}_{\text{connect}} \subseteq V \times V$ as $(v_1, v_2) \in \mathrm{R}_{\text{connect}}$ if either (1) $v_1 = v_2$ or (2) there exists a path from vertex v_1 to vertex v_2 and a path from vertex v_2 to vertex v_1. The equivalence classes with respect to this connectivity relation are called strongly connected components. A *strongly connected component* of the graph G may be identified with a subgraph $G_1 = (V_1, E_1)$ of $G = (V, E)$ such that $V_1 \subseteq V$ and $E_1 \subseteq E$ and for all $v_1, v_2 \in V_1$ there exists a path $v_1 \mapsto v_2$ and a path $v_2 \mapsto v_1$. In general a directed graph has one or more disjoint strongly connected components. Note that a strongly connected component may also consist of a single vertex. If V_{S1} and V_{S2} are two distinct strongly connected components then there may exist an edge from a vertex of V_{S1} to a vertex of V_{S2} but, if so, then there does not exist an edge from a vertex of V_{S2} to a vertex of V_{S1} as the latter vertex would contradict the definition of the disjoint strongly connected components.

Below a directed graph will be associated with a linear system. Distinguish a dynamic system with inputs and outputs and a linear positive system. Associate with the linear dynamic system

$$x(t+1) = Ax(t), \quad x(t_0) = x_0 \in \mathbb{R}_+^n, \quad x : T \to \mathbb{R}_+^n,\ A \in \mathbb{R}_+^{n\times n},$$

the directed *matrix graph* $G = (V, E)$ where $V = \{1, 2, \ldots, n\}$ and $(i, j) \in E$ if the differential equation of $x_i(t)$ depends on the variable $x_j(t)$; thus if

$$x_i(t+1) = \sum_{j=1}^{n} A_{i,j} x_j(t), \quad \text{with } A_{i,j} \neq 0.$$

The same definition is used for a continuous-time linear dynamic system. Associate with a discrete-time linear positive system

$$\begin{aligned} &x(t+1) = Ax(t) + Bu(t), x(t_0) = x_0, \\ &y(t) = Cx(t) + Du(t), \end{aligned}$$

the directed *system graph*

$$\begin{aligned} &G = (V, E), V = V_X \times V_U \times V_y, \\ &E \subseteq V \times V, (i, j) \in E \text{ if} \\ &(1) \text{ either } i, j \in V_X \text{ and } A_{i,j} \neq 0; \\ &(2) \text{ or } i \in V_X,\ j \in V_U \text{ and } B_{i,j} \neq 0; \\ &(3) \text{ or } i \in V_y,\ j \in V_x, \text{ and } C_{i,j} \neq 0. \end{aligned}$$

The definition of a system graph of a continuous-time linear positive system is identical to that of a discrete-time linear positive system. The concept of a system graph was defined for example by Davison [10]. See [15] for graph theory.

3.2. Decomposition of linear positive systems

The following concepts are well known and are included for future reference.

The positive matrices $A_1, A_2 \in \mathbb{R}_+^{n\times n}$ are said to be *permutation similar* if there exists a permutation matrix $P \in \mathbb{P}^{n\times n}$ such that $A_1 = PA_2P^{\mathrm{T}}$; see [6]. Recall that for a permutation matrix $P^{-1} = P^{\mathrm{T}}$. Permutation similarity is an equivalence relation.

A positive matrix $A \in \mathbb{R}_+^{n\times n}$ is said to be *reducible* if $n \geqslant 2$ and it is permutation similar to a matrix with the following structure:

$$\begin{aligned} &\begin{pmatrix} A_{1,1} & 0 \\ A_{2,1} & A_{2,2} \end{pmatrix} \in \mathbb{R}_+^{n\times n}, \\ &\quad n_1, n_2 \in \mathbb{Z}_+,\ n_1 + n_2 = n,\ A_{1,1} \in \mathbb{R}_+^{n_1\times n_1},\ A_{2,1} \in \mathbb{R}_+^{n_2\times n_1},\ A_{2,2} \in \mathbb{R}_+^{n_2\times n_2}. \end{aligned}$$

It is called *irreducible* if (1) $n = 1$ or if (2) $n \geqslant 2$ and it is not reducible. A positive matrix $A \in \mathbb{R}_+^{n\times n}$ is said to be *completely reduced* if $n \geqslant 2$ and it is permutation similar to a matrix of the form,

$$\begin{pmatrix} A_{11,} & 0 & \cdots & 0 \\ A_{2,1} & A_{2,2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ A_{k,1} & A_{k,2} & \cdots & A_{k,k} \end{pmatrix} \in \mathbb{R}_+^{n\times n},$$

$$k \in \mathbb{Z}_+,\ n_1, \ldots, n_k \in \mathbb{Z}_+,\ n = n_1 + n_2 + \cdots + n_k;$$

$$\forall i, j \in \mathbb{Z}_k,\ A_{i,j} \in \mathbb{R}_+^{n_i \times n_j};\ \forall i \in \mathbb{Z}_k,\ A_{i,i} \text{ is irreducible.}$$

Every positive matrix is either a positive real number, an irreducible matrix of size 2×2 or larger, or a matrix which is permutation similar to a completely reduced matrix.

In this paper the well known Perron–Frobenius theorem will be used. The statement of this theorem is included for future reference.

Theorem 3.1 (Due to Perron and Frobenius [31,13]). *Consider an irreducible positive matrix* $A \in \mathbb{R}_+^{n \times n}$ *for* $n \in \mathbb{Z}_+$.

(a) *There exists a real strictly-positive eigenvalue of this matrix which has maximal modulus with respect to all other eigenvalues. Thus,*

$$\exists \lambda^* \in \operatorname{spec}(A) \cap \mathbb{R}_{s+},\ \text{such that } \forall \lambda \in \operatorname{spec}(A),\ |\lambda| \leqslant \lambda^*.$$

(b) *There exists also a strictly-positive real eigenvector corresponding to* λ^*:

$$\exists v^* \in \mathbb{R}_{s+}^n \text{ such that } Av^* = \lambda^* v^*.$$

A proof may be found in [27, Th. 1.4.1].

Consider a positive integer $n \in \mathbb{Z}_+$ and an irreducible positive matrix $A \in \mathbb{R}_+^{n \times n}$ with maximal eigenvalue $\lambda^* \in (0, \infty)$. The *index of imprimitivity* of A is defined as the value $k \in \mathbb{Z}_+$ if A has exactly k different eigenvalues of modulus λ^*. Denote the index of imprimitivity by Imprim $(A) \in \mathbb{Z}_+$. The positive matrix A is called *primitive* if Imprim $(A) = 1$ and called *imprimitive* otherwise. It is known that a positive matrix is primitive if and only if there exists an integer $k \in \mathbb{Z}_+$ such that $A^k \in \mathbb{R}_{s+}^{n \times n}$. If an irreducible positive matrix $A \in \mathbb{R}_+^{n \times n}$ has an index of imprimitivity k equal to two or larger then the matrix is permutation similar to the *Frobenius form*,

$$\begin{pmatrix} 0 & A_{12} & 0 & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & 0 & \cdots & 0 \\ 0 & 0 & & A_{34} & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & A_{k-1,k} \\ A_{k,1} & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$k \in \mathbb{Z}_+, n_1, n_2, \ldots, n_k \in \mathbb{Z}_n,\ n_1 + n_2 + \cdots + n_k = n,$$

$$\forall i \in \mathbb{Z}_k,\ A_{i,i+1} \in \mathbb{R}_+^{n_i \times n_{i+1}};\ \text{and } A_{k,1} \in \mathbb{R}_+^{n_k \times n_1},$$

$$A_{1,2} A_{2,3} \ldots A_{k-1,k} A_{k,1} \text{ is an irreducible matrix.}$$

See for these results [27], in particular for the Frobenius form see [27, Th. 3.4.1].

4. Observability of linear positive systems

For ordinary linear systems, not necessarily positive, there is an equivalence condition for the existence of a globally asymptotically stable observer. The equivalence condition is detectability, which is closely related to observability. In this section the concept of observability of a linear positive system is defined. A characterization of that concept is needed for actual computations. Moreover, every linear positive system can be transformed into an observable form which is needed for computations.

Observability of linear positive systems has been discussed elsewhere, see [7,29,30,39]. For a discussion of detectability of polynomial positive systems see [9]. Controllability of linear positive systems and algebraic decompositions of such systems including references to the literature may be found in the paper by Valcher [37].

The sequence of the topics of this section is: the concept of observability, characterization of observability of a linear positive systems, the observable form, and finally the concept of detectability.

4.1. Observability concepts of linear positive systems

In this subsection the concept of observability of discrete-time linear positive systems is formulated.

Definition 4.1. Consider the discrete-time linear positive system with representation,

$$\begin{aligned} &x(t+1) = Ax(t), \quad x(t_0) = x_0, \\ &y(t) = Cx(t), \quad T = [t_0, t_1] \text{ or } T = [t_0, \infty). \end{aligned}$$

Denote the trajectory of the output by $\{y(t; t_0, x_0), t \in T\}$. The system is called *observable as a linear positive system* on the interval T if the observability map is injective:

$$\text{obsmap} : \mathbb{R}_+^n \to (\mathbb{R}_+^p)^{\mathrm{T}}, \quad x_0 \mapsto y(.; t_0, x_0) : T \to \mathbb{R}_+^p.$$

In the remainder of the paper call a linear positive *observable* if it is observable as a linear positive system.

Problem 4.2. Characterize when a discrete-time linear positive system is observable.

Definition 4.3. For a discrete-time linear positive system with system matrices (A, C) define the *observability matrix* as

$$\text{obsm}(A, C) = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} \in \mathbb{R}_+^{np \times n}. \tag{6}$$

Then the *observability map* can be represented as

$$\begin{aligned} &x_0 \mapsto y(.; t_0, x_0) : T \to \mathbb{R}_+^p, \\ &\begin{pmatrix} y(t_0) \\ y(t_0+1) \\ \vdots \end{pmatrix} = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \end{pmatrix} x_0, \end{aligned} \tag{7}$$

$$\mathrm{proj}\left(\left(\begin{pmatrix} y(t_0) \\ y(t_0+1) \\ \vdots \\ y(t_0+n-1) \\ \vdots \end{pmatrix}\middle| \mathbb{R}_+^{pn}\right)\right) = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} x_0. \tag{8}$$

Definition 4.4. Consider the discrete-time linear positive system

$$x(t+1) = Ax(t), x(t_0) = x_0,$$
$$y(t) = Cx(t).$$

Call the system:

(a) *output connected* if for all $i \in \mathbb{Z}_n$ there exists a path in the system graph from state x_i to an output component y_k for a $k \in \mathbb{Z}_p$.
(b) *trajectory observable* if for all $x_0 \in \mathbb{R}_+^n$

$$\{y(t; t_0, x_0) = 0, \forall t \in T\} \Rightarrow \{x(s; t_0, x_0) = 0, \forall s \in T\}.$$

The concept of detectability of a linear positive system is defined in Section 4.4.

4.2. Examples of nonobservable systems

There are two aspects of the formulation of the concept of observability for linear positive systems. These aspects are illustrated by the following examples.

Example 4.5. Consider the linear positive system

$$x(t+1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x(t), \quad x(t_0) = x_0,$$
$$y(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} x(t).$$

Then part of the observability map of this system is

$$\begin{pmatrix} y(t_0) \\ y(t_0+1) \\ y(t_0+2) \\ y(t_0+3) \end{pmatrix} = \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{pmatrix} x_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} x_0,$$

while the remaining rows have a similar structure. It is then obvious that the map

$$x_0 \mapsto y(.; t_0, x_0) : T \to \mathbb{R}_+^p,$$

cannot be injective because the last component of the initial state x_0 never shows up in the output function.

The conclusion of the above example is that observability of the system requires that the state components of each irreducible diagonal block be connected via a path to an output component. This property is a necessary condition for the linear positive system to be observable.

Example 4.6. Consider the linear positive system,

$$x(t+1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} x(t), \quad x(t_0) = x_0,$$
$$y(t) = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} x(t).$$

Then the observability map is

$$\begin{pmatrix} y(t_0) \\ y(t_0+1) \\ y(t_0+2) \\ y(t_0+3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} x_0,$$
$$\text{rank}(\text{obsm}(A, C)) = 3 < 4. \text{ Define,}$$
$$x_0 = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}^{\mathrm{T}}, \ \bar{x}_0 = \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}^{\mathrm{T}}, \text{ then}$$
$$\text{obsm}(A, C)x_0 = \text{obsm}(A, C)\bar{x}_0, x_0 \neq \bar{x}_0.$$

Thus the system is not observable as will be proven below. Note that there is a nonzero element in every column of the observability matrix so the comment of the previous example does not apply.

4.3. Characterization of observability concepts

Theorem 4.7. *Consider a time-invariant discrete-time linear positive system with representation,*

$$x(t+1) = Ax(t), x(t_0) = x_0,$$
$$y(t) = Cx(t), T = [t_0, t_1] \text{ or } T = [t_0, \infty).$$

The system is observable on the interval T if and only if

$$\text{rank}(\text{obsm}(A, C)) = n. \tag{9}$$

Proof. ($\Rightarrow$) If $\text{rank}(\text{obsm}(A, C)) < n$ then it follows from linear algebra that there exists a vector $x_0 \in \mathbb{R}^n$ such that $x_0 \neq 0$ and $\text{obsm}(A, C)x_0 = 0$. Decompose this vector as

$$x_0 = x_0^+ - x_0^-, \quad x_0^+ = \max\{x_0, 0\} \in \mathbb{R}_+^n, \quad x_0^- = \max\{-x_0, 0\} \in \mathbb{R}_+^n,$$
$$x_0 \neq 0 \Rightarrow x_0^+ \neq x_0^-; \text{ define,}$$
$$I_+(x_0^+) = \{i \in \mathbb{Z}_n | x_{0,i}^+ > 0\}, \quad I_+(x_0^-) = \{i \in \mathbb{Z}_n | x_{0,i}^- > 0\}; \text{ then}$$
$$I_+(x_0^+) \cap I_+(x_0^-) = \emptyset, \quad x_0^+ \neq x_0^-.$$

Hence $x_0^+, x_0^- \in \mathbb{R}_+^n$ are located on different faces of the positive orthant. Then

$$\begin{aligned}
0 &= \mathrm{obsm}(A, C)x_0 = \mathrm{obsm}(A, C)(x_0^+ - x_0^-) \\
&\Leftrightarrow \mathrm{obsm}(A, C)x_0^+ = \mathrm{obsm}(A, C)x_0^-, \quad \text{with } x_0^+, x_0^- \in \mathbb{R}_+^n, x_0^+ \neq x_0^-; \\
&\Leftrightarrow \mathrm{obsm}(A, C)(x_0^+ + v) = \mathrm{obsm}(A, C)(x_0^- + v), \forall v \in \mathbb{R}_+^n; \\
&\quad\; x_0^+ + v, x_0^- + v \in \mathbb{R}_+^n, x_0^+ + v \neq x_0^- + v; \\
&\Leftrightarrow y(t; (t_0, x_0^+)) = y(t; (t_0, x_0^-)), \forall t \in T, \\
&\quad\; \text{by the Cayley–Hamilton theorem.}
\end{aligned}$$

Hence the system is not observable. Because $\mathrm{rank}(\mathrm{obsm}(A, C)) \leqslant n$ it follows that this rank is equal to n.

($\Leftarrow$) If the system is not observable then by definition there exist two vectors

$$\begin{aligned}
&x_0, \bar{x}_0 \in \mathbb{R}_+^n, x_0 \neq \bar{x}_0, \quad \text{such that } y(t; t_0, x_0) = y(t; t_0, \bar{x}_0), \forall t \in T; \\
&\Rightarrow \mathrm{obsm}(A, C)x_0 = \mathrm{obsm}(A, C)\bar{x}_0, \text{ and } x_0 \neq \bar{x}_0, \\
&\Rightarrow 0 \neq x_0 - \bar{x}_0 \in \ker(\mathrm{obsm}(A, C)), \\
&\Rightarrow \mathrm{rank}(\mathrm{obsm}(A, C)) < n. \qquad \square
\end{aligned}$$

The following result establishes that the concept of output connectedness is strictly weaker than that of observability.

Proposition 4.8. *Consider the discrete-time linear positive system*

$$\begin{aligned}
&x(t + 1) = Ax(t), x(t_0) = x_0, \\
&y(t) = Cx(t).
\end{aligned}$$

(a) *The system is output connected if and only if every column of the observability matrix has at least one strictly positive element.*
(b) *The system is output connected if and only if it is trajectory-observable.*
(c) *If the system is observable then it is output connected.*
(d) *There exists an example of a discrete-time linear positive system which is output connected but not observable.*

Proof. (a) Consider for a linear positive system its associated system graph. For any $i \in \mathbb{Z}_n$ there exists a path from state x_i to a component of the output y_r for $r \in \mathbb{Z}_p$ if and only if there exists a $k \in \mathbb{Z}_{n-1}$ such that $\mathrm{obsm}(A, C)_{kp+r,i} > 0$. For example, if $k = 2$ then,

$$\begin{aligned}
&0 < \mathrm{obsm}(A, C)_{2p+r,i} = \sum_{s=1}^{n} C_{r,s}(A^2)_{s,i} = \sum_{s=1}^{n} C_{r,s}\left(\sum_{j=1}^{n} A_{s,j}A_{j,i}\right) \\
&\Leftrightarrow \exists s \in \mathbb{Z}_n, \exists j \in \mathbb{Z}_n \text{ such that } C_{r,s} > 0, A_{s,j} > 0, A_{j,i} > 0, \\
&\Leftrightarrow \exists \text{ path } x_i \mapsto y_r \text{ in the system graph.}
\end{aligned}$$

(b) ($\Rightarrow$) Consider an initial state $x_0 \in \mathbb{R}_+^n$. Note that the observability matrix is positive, $\text{obsm}(A, C) \in \mathbb{R}_+^{np \times n}$. The characterization of output connectedness proven in (a) shows that in every column of the observability matrix there exists a strictly positive element. Thus, if $y(t) = 0$ for all $t \in T$ then necessarily $x_i(t) = 0$ for all $i \in \mathbb{Z}_n$ and for all $t \in T$. Thus the system is trajectory-observable.

($\Leftarrow$) Suppose that the system is not output connected. From (a) follows that there exists a column, say $i \in \mathbb{Z}_n$, of the observability matrix which is identically zero. Take $x_0 \in \mathbb{R}_+^n$ such that $x_{0,i} > 0$ and $x_{0,j} = 0$ for all $j \in \mathbb{Z}_n \backslash \{i\}$. Then $\{x(t; t_0, x_0), t \in T\}$ is not identically zero. Because the system is not output connected there is no path $x_i \mapsto y_k$ for any $k \in \mathbb{Z}_p$. Consequently the trajectory $\{y(t; t_0, x_0), t \in T\}$ is identically zero (If $\exists t \in T$ and if $\exists k \in \mathbb{Z}_p$ such that $y_k(t; t_0, x_0) > 0$ then there exists a $j \in \mathbb{Z}_n$ and there exists a path $x_i \mapsto x_j \mapsto y_k$ contradicting the choice of x_i). Hence the system is not trajectory-observable.

(c) Suppose that the system is not output connected. By (a) there exists a column of the observability matrix $\text{obsm}(A, C)$ which is identical zero. Then $\text{rank}(\text{obsm}(A, C)) < n$ and from Theorem 4.7 follows that the system is not observable.

(d) See Example 4.6. Note that the observability matrix displayed there is such that every column has at least one strictly positive element. From (a) then follows that the system is output connected. But from the rank of the observability matrix and from Theorem 4.7 follow that the system is not observable. □

4.4. Towards an observable canonical form

For ordinary linear systems, not necessarily positive, there exists a canonical form for the system matrices with respect to the equivalence relation on system matrices defined by these realizing the same impulse response function, see [33, chapter 6] for a description.

In this subsection a canonical form of system matrices of linear positive systems with respect to permutations of states and of outputs will be formulated. Currently the realization theory of linear positive systems is incomplete. The main open question is the characterization of minimality and the characterization of equivalent minimal realizations of linear positive systems. Therefore a canonical form of the system matrices of a linear positive system will not be stated but rather a weaker property. The following results are related to the canonical forms stated for controllability of linear positive systems obtained by M.E. Valcher, see [37].

Consider a discrete-time linear positive system

$$x(t+1) = Ax(t) + Bu(t), x(t_0) = x_0,$$
$$y(t) = Cx(t) + Du(t).$$

For realization theory of linear positive systems see the paper by Van den Hof [38], and the references quoted in that paper. Two sets of system matrices (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2), not necessarily of the same state-space dimension, describe *equivalent realizations* if they correspond to the same impulse response function H by

$$H(0) = D_1 = D_2, \quad H(t) = C_1(A_1)^{t-1}B_1 = C_2(A_2)^{t-1}B_2, \quad \forall t \in \mathbb{Z}_+.$$

It is stated in the paper quoted above that if (A, B, C, D) are the system matrices of a linear positive system and if $M \in M^{n \times n}$ is a monomial matrix then $(MAM^{-1}, MB, CM^{-1}, D)$ are system matrices and these two tuples of system matrices represent equivalent realizations. However, there may exist state-space transformations $S \in \mathbb{R}^{n \times n}$ not necessarily positive for which both (A, B, C, D) and $(SAS^{-1}, SB, CS^{-1}, D)$ are system matrices of equivalent realizations as linear positive systems. Recall that any monomial matrix admits decompositions as $M = D_1 P_1 = P_2 D_2$ where $P_1, P_2 \in \mathbb{P}^{n \times n}$ are permutation matrices and $D_1, D_2 \in D_{+s}^{n \times n}$ are diagonal matrices with on the diagonal strictly positive elements.

Therefore below a canonical form for system matrices of equivalent realizations of a linear positive system will be stated with respect to permutations on the state set and on the output set. The permutation of the output set is usually not considered in realization theory but it is taken into consideration here.

Recall the notation

$$\left(A_{11} \oplus A_{22}\right) = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}.$$

Definition 4.9. (a) A linear positive system without inputs is said to be in *canonical form with respect to permutations of the state and of the output set* if the system matrices admit the decomposition,

$$x(t+1) = Ax(t), \quad x(t_0) = x_0, \tag{10}$$

$$y(t) = Cx(t), \tag{11}$$

$$A = \begin{pmatrix} \overline{A}_{1,1} & 0 \\ * & A_{k_2+1,k_2+1} \end{pmatrix}, \tag{12}$$

$$\overline{A}_{1,1} = A_{1,1} \oplus \cdots \oplus A_{k_1,k_1} \oplus A_{k_1+1,k_1+1} \oplus \cdots \oplus A_{k_2,k_2}, \tag{13}$$

$$C = \left(C_1 \oplus \cdots \oplus C_{k_1} \oplus C_{k_1+1} \oplus \cdots \oplus C_{k_2} \quad 0\right), \tag{14}$$

$$k_1, k_2 \in \mathbb{N}_n, n_1, \ldots, n_{k_2+1}, p_1, \ldots, p_{k_2} \in \mathbb{N},$$

$$\sum_{i=1}^{k_2+1} n_i = n, \quad \sum_{j=1}^{k_2} p_j = p, \tag{15}$$

$$A_{i,i} \in \mathbb{R}_+^{n_i \times n_i}, C_i \in \mathbb{R}_+^{p_i \times n_i}, i = 1, \ldots, k_2 + 1;$$

$$\operatorname{rank}(\operatorname{obsm}(A_{i,i}, C_i)) = n_i, \quad i = 1, \ldots, k_1; \tag{16}$$

$$\operatorname{rank}(\operatorname{obsm}(A_{i,i}, C_i)) < n_i, \quad i = k_1 + 1, \ldots, k_2; \tag{17}$$

$$A_{i,i} = \begin{pmatrix} F_{1,1} & 0 & 0 \\ \vdots & \ddots & 0 \\ * & * & F_{r_i,r_i} \end{pmatrix} \in \mathbb{R}_+^{n_i \times n_i}, \quad \text{block lower-triangular}, \tag{18}$$

$$F_{j,j} \in \mathbb{R}_+^{n_{i,j} \times n_{i,j}}, i = 1, \ldots, r_i, \quad \text{irreducible matrix},$$

$$C_i = \begin{pmatrix} H_{1,1} & 0 & 0 \\ \vdots & \ddots & 0 \\ * & * & H_{r_i,r_i} \end{pmatrix} \in \mathbb{R}_+^{p_i \times n_i}, \quad i = 1, \ldots, k_2. \tag{19}$$

Note that one or more block rows or block columns may be missing in the above defined matrices.

(b) The linear positive system is said to be in *observable canonical form with respect to permutations of the state and of the output set* if the system matrices admit the decomposition

$$A = \left(A_{1,1} \oplus \cdots \oplus A_{k_1,k_1}\right), \tag{20}$$

$$C = \left(C_1 \oplus \cdots \oplus C_{k_1}\right), \tag{21}$$

$$\text{rank}(\text{obsm}(A_{i,i}, C_i)) = n_i, \, i = 1, \ldots, k_1, \tag{22}$$

where $A_{i,i}$ and C_i are as stated in the Eqs. (18,19) and the relevant conditions stated there apply.

Example 4.10. A simple example of a linear positive system is presented which is in the above described canonical form. Consider the system matrices,

$$A_{1,1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \end{pmatrix}, \tag{23}$$

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}, \tag{24}$$

$$A_{2,2} = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 5 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{25}$$

$$A_{3,3} = \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}, \tag{26}$$

$$A = \left(A_{1,1} \oplus A_{2,2} \oplus A_{3,3}\right), \quad C = \begin{pmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \end{pmatrix}. \tag{27}$$

Note that the matrix tuple (A_{11}, C_1) is an observable pair, (A_{22}, C_2) is not an observable pair, and $(A_{33}, 0)$ is not an observable pair.

At this point a definition of detectability of linear positive systems can be formulated. Recall that for any ordinary linear system, not necessarily positive, there exists a nonsingular transformation matrix $S \in \mathbb{R}^{n\times n}$ such that

$$SAS^{-1} = \begin{pmatrix} A_{1,1} & 0 \\ A_{2,1} & A_{2,2} \end{pmatrix}, \quad CS^{-1} = \begin{pmatrix} C_1 & 0 \end{pmatrix},$$
$$n_1, n_2 \in \mathbb{N}, n_1 + n_2 = n, A_{i,j} \in \mathbb{R}^{n_i\times n_j}, C_1 \in \mathbb{R}^{p\times n_1},$$
$$(A_{1,1}, C_1) \text{ an observable pair.}$$

The linear system is then said to be detectable if $\mathrm{spec}(A_{2,2}) \subset \mathbb{D}_o$. Denote the unobservable submatrix of A by A_{uo} hence $A_{uo} = A_{22}$.

Definition 4.11. A linear positive system with system matrices (A, C) is said to be *detectable* if after transformation of the system matrices to the observable canonical form of Definition 4.9.(a) there holds,

$$(1)\ \mathrm{spec}(A_{k_2+1,k_2+1}) \subset \mathbb{D}_o; \tag{28}$$

$$(2)\ \mathrm{spec}(A_{i,i,uo}) \subset \mathbb{D}_o, \quad i = k_1 + 1, \ldots, k_2. \tag{29}$$

Theorem 4.12. *Consider a linear positive system without input and with system matrices* $(A, C) \in \mathbb{R}_+^{n\times n} \times \mathbb{R}_+^{p\times n}$.

(a) *There exist permutation matrices* $P_X \in \mathbb{P}^{n\times n}$ *and* $P_Y \in \mathbb{P}^{p\times p}$ *such that* $(P_X A P_X^{\mathrm{T}}, P_Y C P_X^{\mathrm{T}})$ *is in canonical form with respect to permutations of the state and of the output set. The two pairs* (A, C) *and* $(P_X A P_X^{\mathrm{T}}, P_Y C P_X^{\mathrm{T}})$ *of system matrices describe equivalent realizations of linear positive systems without inputs up to permutation of the output components.*

(b) *Any two sets of system matrices of system matrices* (A, C) *and* $(\overline{A}, \overline{C})$ *with the same canonical form with respect to permutations of the state and of the output set differ in one or more of the following ways*:
1. *permutation of the blocks* $(A_{i,i}, C_i)$ $i = 1, \ldots, k_1$;
2. *permutation of the blocks* $(A_{i,i}, C_i)$ $i = k_1 + 1, \ldots, k_2$;
3. *for any* $i = 1, \ldots, k_2$ *with the decomposition* (18), (19), *permutation of the* $F_{j,j}$ *blocks which permutations preserve the lower triangular block structure of* $(A_{i,i}, C_i)$ *and for any* $j = 1, \ldots, s_i$ *a permutation of the output set which preserves the block lower triangular structure of* C_i;
4. *for any* $i = 1, \ldots, k_2$ *and* $j = 1, \ldots, r_i$ *a permutation of the rows and columns of the block* $F_{j,j}$. *For any* $i = 1, \ldots, k_2$ *and* $j = 1, \ldots, r_i$, *a permutation of the rows and columns of the block* H_j; *and*
5. *any permutation of the block* A_{k_2+1,k_2+1}.

(c) *If the linear positive system is observable then there exists permutation matrices* $P_X \in \mathbb{P}^{n\times n}$ *and* $P_Y \in \mathbb{P}^{p\times p}$ *such that* $(P_X A P_X^{\mathrm{T}}, P_Y C P_X^{\mathrm{T}})$ *is in observable canonical form with respect to permutations of the state and of the output set. The uniqueness of this observable canonical form can easily be deduced from that described in* (b).

Proof. (a) (1) For each component y_k of the output $y \in \mathbb{R}_+^p$ with $k \in \mathbb{Z}_p$, define the index set of the subset of states for which there exists a path to y_k

$$X_{y_k} = \{i \in \mathbb{Z}_n | \exists i \mapsto k\}. \tag{30}$$

Thus, $i \in X_{y_k}$ if there exists a path in the system graph from state component x_i to output component y_k. Note that by definition of a strongly connected subset of states, X_{y_k} is a union of strongly connected subsets of states.

(2) Define the subsets

$$J_1 = \{k \in \mathbb{Z}_p | X_{y_1} \cap X_{y_k} \neq \emptyset\}, \tag{31}$$

$$I_1 = \{i \in \mathbb{Z}_n | \exists k \in J_1 \text{ such that } i \in X_{y_k}\}. \tag{32}$$

Thus J_1 contains all output components y_k with $k \in \mathbb{Z}_p$ whose index sets X_{y_k} have a nonempty intersection with X_{y_1}. The index set I_1 equals the index set of all state components for which there exists a path to an output component y_k with $k \in J_1$. For $j \in \mathbb{Z}_p \backslash J_1$, if $\mathbb{Z}_p \backslash J_1 \neq \emptyset$, define

$$J_2 = \{k \in \mathbb{Z}_p \backslash J_1 | X_{y_j} \cap X_{y_k} \neq \emptyset\}, \tag{33}$$

$$I_2 = \{i \in \mathbb{Z}_n | \exists k \in J_2 \text{ such that } i \in X_{y_k}\}. \tag{34}$$

Construct thus by induction the sequence of index sets

$$(I_1, J_1), (I_2, J_2), \ldots, (I_{k_2}, J_{k_2}), k_2 \in \mathbb{Z}_p,$$

till $\mathbb{Z}_p \backslash \cup_{j=1}^{k_2} J_j = \emptyset$ which will occur in at most p steps. Put then

$$I_{k_2+1} = \mathbb{Z}_n \Big\backslash \bigcup_{j=1}^{k_2} I_j,$$

which could be empty.

(3) Construct linear transformations in the form of permutation matrices $P_1 \in \mathbb{P}^{n \times n}$ and $P_2 \in \mathbb{P}^{p \times p}$ such that

$$P_1 : \mathbb{Z}_n \to \{I_1, \ldots, I_{k_2+1}\}, P_2 : \mathbb{Z}_p \to \{J_1, \ldots, J_{k_2}\},$$

where the image of P_1 goes to the elements of I_j for $j = 1, \ldots, k_2 + 1$ but where for any such j the ordering within I_j is arbitrary; and similarly for the image of P_2. By definition of the collection of sets $\{J_1, \ldots, J_{k_2}\}$, the sets of the collection are disjoint and form a partition of $\mathbb{Z}_p$. Similarly, the collection $\{I_1, \ldots, I_{k_2+1}\}$ form a partition of the set $\mathbb{Z}_n$. The resulting system matrices then have the form

$$P_1 A P_1^{\mathrm{T}} = \begin{pmatrix} A_{1,1} & \cdots & 0 & 0 \\ \vdots & \ddots & 0 & 0 \\ 0 & \cdots & A_{k_2,k_2} & 0 \\ * & * & * & A_{k_2+1,k_2+1} \end{pmatrix}, \tag{35}$$

$$P_2 C P_1^{\mathrm{T}} = \begin{pmatrix} C_1 & \cdots & 0 & 0 \\ \vdots & \ddots & 0 & 0 \\ 0 & \cdots & C_{k_2} & 0 \end{pmatrix}. \tag{36}$$

That the above decomposition holds is proven as follows. Suppose there exists $i, j \in \mathbb{Z}_n$ with $i > j$ with in Equation (35) $A_{i,j} \neq 0$ then there exists an element $(r, s) \in I_j \times I_i$ with $A_{r,s} > 0$ hence there exists a path $s \mapsto r$, from s to r. But $s \in I_j$ implies that there exists a path $s \mapsto y_{k_s}$ with $s \in X_{y_j}$ and $k_s \in J_j$. Similarly, $r \in I_i$ implies that there exists a path $r \mapsto y_{k_r}$ with $r \in X_{y_i}$ and $k_r \in J_i$. Thus there exist paths $x_s \mapsto y_{k_s}$ and $x_s \mapsto x_r \mapsto y_{k_r}$ with $k_s \in J_j$ and $k_r \in J_i$. Hence $s \in X_{y_j} \cap X_{y_i} \neq \emptyset$. But $i > j$ implies that J_i and J_j are disjoint hence $X_{y_j} \cap X_{y_i} = \emptyset$, This is a contradiction. The corresponding argument can be used if $i < j$. Because the subsets $J_1, \ldots, J_{k_2}$ are disjoint, the matrix $P_2 C P_1^{\mathrm{T}}$ has the structure displayed in Eq. (36).

(4) Next permute the blocks $(A_{i,i}, C_i)$ for $i = 1, \ldots, k_2$ such that,

$$\operatorname{rank}(\operatorname{obsm}(A_{i,i}, C_i)) = n_i, \quad i = 1, \ldots, k_1;$$

$$\operatorname{rank}(\operatorname{obsm}(A_{i,i}, C_i)) < n_i, \quad i = k_1 + 1, \ldots, k_2; k_1, k_2 \in \mathbb{N}_n.$$

Denote these permutations by P_i for $i = 1, \ldots, k_2$ and compose

$$P_3 = (P_{11} \oplus P_{22} \oplus \cdots \oplus A_{k_2,k_2} \oplus I) \in \mathbb{R}_+^{n \times n}.$$

From Theorem 4.7 then follows that the first k_1 subsystems are observable while the subsystems indexed by $k_1 + 1, \ldots, k_2$, if any, are not observable.

(5) For $i \in \mathbb{Z}_{k_2}$ consider the system matrices $(A_{i,i}, C_i) \in \mathbb{R}_+^{n_i \times n_i} \times \mathbb{R}_+^{p_i \times n_i}$ of the i-the subsystem. Either the matrix $A_{i,i}$ is irreducible or, according to the definition of a completely reduced matrix, there exists a permutation matrix $P_{i,i} \in \mathbb{R}^{n_i \times n_i}$ such that $P_{i,i} A_{i,i} P_{i,i}^{\mathrm{T}}$ is a completely reduced matrix hence it is lower block triangular with on the diagonal irreducible matrices. Next select a permutation matrix $P_{i,Y}$ such that

$$P_{i,Y} C P_i^{\mathrm{T}} = \begin{pmatrix} H_{1,1} & \cdots & 0 \\ * & \ddots & 0 \\ * & * & H_{r_i,r_i} \end{pmatrix},$$

which can always be done though one or more block rows may be missing. Denote the permutation matrices

$$\begin{aligned} P_4 &= \left(P_{1,1} \oplus P_{2,2} \oplus \cdots \oplus P_{k_2+1,k_2+1}\right), \\ P_5 &= \left(P_{1,Y} \oplus P_{2,Y} \oplus \cdots \oplus P_{k_2+1,Y}\right), \\ P_X &= P_4 P_3 P_1, \quad P_Y = P_5 P_2. \end{aligned}$$

It then follows that the matrix tuples (A, C) and $(P_X A P_X^{\mathrm{T}}, P_Y C P_X^{\mathrm{T}})$ described equivalent realizations of the impulse response function.

(b) This follows directly from (a).

(c) Because of the assumption that the linear positive system is observable it follows from (a) that only the first $k_1 \in \mathbb{N}_n$ blocks can be present. □

5. Observer synthesis for linear positive systems

The purpose of this section is to present the synthesis of observers for linear positive systems. First equivalent conditions are formulated for the existence of an observer. The observer synthesis is then formulated for discrete-time linear observers. For several special cases the construction will be explicitly sketched.

5.1. Equivalent conditions for the existence of observers

The following conditions are a reformulation of those stated in [39, p. 592].

Proposition 5.1. *Consider a discrete-time linear positive system and the associated candidate linear observer*

$$\begin{aligned} x(t+1) &= Ax(t), x(t_0) = x_0, \\ y(t) &= Cx(t), \\ \hat{x}(t+1) &= (A - KC)\hat{x}(t) + Ky(t), \quad \hat{x}(t_0) = \hat{x}_0. \end{aligned} \tag{37}$$

The dynamic system (37) *is a globally asymptotically stable linear observer if and only if the following conditions all hold*:

$$(1)\ K \in \mathbb{R}^{n\times p}, \tag{38}$$
$$(2)\ \text{spec}(A - KC) \subset \mathbb{D}_o. \tag{39}$$

The dynamic system (37) *is a globally asymptotically stable linear positive observer if and only if the following conditions all hold*:

$$(1)\ K \in \mathbb{R}_+^{n\times p}, \tag{40}$$
$$(2)\ (A - KC) \in \mathbb{R}_+^{n\times n},\ \textit{and} \tag{41}$$
$$(3)\ \text{spec}(A - KC) \subset \mathbb{D}_o. \tag{42}$$

Proof. The observer (37) is a positive system if and only if $K \in \mathbb{R}_+^{n\times p}$ and $(A - KC) \in \mathbb{R}_+^{n\times n}$ as follows from the statements below Definition 2.1. Define the error signal $e(t) = x(t) - \hat{x}(t)$. Then

$$e(t+1) = Ax(t) - (A - KC)\hat{x}(t) - KCx(t) = (A - KC)e(t),$$
$$e(t_0) = x_0 - \hat{x}_0.$$

This error system is globally asymptotically stable if and only if $\text{spec}(A - KC) \subset \mathbb{D}_o$. □

Stability conditions for observers have been formulated in terms of the state-to-output stability concept by E.D. Sontag and co-workers, see [1,34,35].

5.2. Nonexistence of linear positive observers

A linear positive observer which is globally asymptotically stable need not exist.

Example 5.2. *Nonexistence of a globally asymptotically stable linear positive observer for a discrete-time linear positive system*. Consider the linear positive system,

$$x(t+1) = Ax(t), \quad x(t_0) = x_0, \tag{43}$$
$$y(t) = Cx(t), \tag{44}$$
$$A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \tag{45}$$

which system is observable. Attention is restricted to a linear observer of the form

$$\hat{x}(t+1) = A\hat{x}(t) + K[y(t) - C\hat{x}(t)], \quad \hat{x}(t_0) = \hat{x}_0, \tag{46}$$
$$(A - KC) = \begin{pmatrix} 1 - k_1 & 1 \\ 2 - k_2 & 3 \end{pmatrix}. \tag{47}$$

According to Proposition 5.1 there exists a positive observer of the form above if and only if there exists a matrix $K \in \mathbb{R}_+^{n\times p}$ such that

$$(A - KC) \in \mathbb{R}_+^{n\times n} \quad \text{and} \quad \text{spec}(A - KC) \subset \mathbb{D}_o.$$

Below a contradiction will be derived from the supposition of the existence of a linear positive observer for the linear positive system. Suppose then that a linear positive observer exists hence there exists a positive matrix $K \in \mathbb{R}_+^{n\times p}$ with the properties stated above. For the system matrices considered the conditions then become

$$\begin{aligned}
&K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in \mathbb{R}^{2\times 2} \Leftrightarrow 0 \leqslant k_1, \quad 0 \leqslant k_2;\\
&(A-KC) \in \mathbb{R}_+^{2\times 2} \Leftrightarrow k_1 \leqslant 1, \quad k_2 \leqslant 2;\\
&\operatorname{spec}(A-KC) \subset \mathbb{D}_o.
\end{aligned}$$

The characteristic polynomial of the matrix $(A - KC)$ is

$$s^2 - s[4-k_1] + (1 - 3k_1 + k_2) = 0.$$

If the two eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ are both real and in $\mathbb{D}_o$ then the characteristic polynomial of $(A - KC)$ satisfies

$$\begin{aligned}
&0 = s^2 - (\lambda_1+\lambda_2) + \lambda_1\lambda_2;\\
&\qquad \lambda_1, \lambda_2 \in \mathbb{D}_o\\
&\ \Leftrightarrow -2 < \lambda_1+\lambda_2 < 2, -1 < \lambda_1\lambda_2 < 1,\\
&\ \Rightarrow -2 < 4-k_1 < 2, -1 < 1-3k_1+k_2 < 1,\\
&\ \Leftrightarrow 2 < k_1 < 6, -2+3k_1 < k_2 < 3k_1.
\end{aligned}$$

These conditions are incompatible because of the combined condition $2 < k_1 \leqslant 1$. Hence no linear positive observer exists in this case.

In case the two eigenvalues λ_1, λ_2 of $A - KC$ are complex conjugate then

$$\begin{aligned}
&\lambda_1+\lambda_2 = 2\mathrm{Re}(\lambda_1), \lambda_1\lambda_2 = \mathrm{Re}(\lambda_1)^2 + \mathrm{Im}(\lambda_1)^2.\\
&\lambda_1, \lambda_2 \in \mathbb{D}_o\\
&\Leftrightarrow -2 < \lambda_1+\lambda_2 < 2, 0 < \lambda_1\lambda_2 < 1,\\
&\Leftrightarrow -2 < 4-k_1 < 2, 0 < 1-3k_1+k_2 < 1,\\
&\Leftrightarrow 2 < k_1 < 6, 3k_1 - 1 < k_2 < 3k_1,
\end{aligned}$$

which conditions are incompatible because $2 < k_1 \leqslant 1$. Hence in this case no positive observer exists either.

A globally asymptotically stable linear observer exists though it is not a linear positive system. Consider,

$$K = \begin{pmatrix} 3.6 \\ 9.48 \end{pmatrix} \in \mathbb{R}_+^2, \tag{48}$$

$$A - KC = \begin{pmatrix} -2.6 & 1 \\ -7.48 & 3 \end{pmatrix} \in \mathbb{R}^{2\times 2}, \tag{49}$$

$$\hat{x}(t+1) = A\hat{x}(t) + K[y(t) - C\hat{x}(t)] = (A-KC)\hat{x}(t) + Ky(t) \tag{50}$$

$$= \begin{pmatrix} -2.6 & 1 \\ -7.48 & 3 \end{pmatrix} \hat{x}(t) + \begin{pmatrix} 3.6 \\ 9.48 \end{pmatrix} y(t), \tag{51}$$

$$\operatorname{spec}(A-KC) = \{0.8, -0.4\} \subset \mathbb{D}_o. \tag{52}$$

Thus, if $\hat{x}_0 = (1,0)^{\mathrm{T}}$ and $y(t_0) = 0$ then $\hat{x}(1) = (-2.6, -7.48)^{\mathrm{T}} \in \mathbb{R}_-^2$.

The examples of nonexistence of a linear positive observer for a linear positive system are motivated by a result of Patrick De Leenheer on nonexistence of a positive control law to stabilize a linear positive system, see [21,22]. However, those references do not contain examples.

A consequence of the above examples is that for a subclass of linear positive systems the reader has to accept that an observer is not necessarily positive. Thus for a particular initial condition of the system the observer may be such that one or more of the state components of the observer state $\hat{x}(t)$ are negative for certain times: $\exists t \in [t_0, \infty)$ and $i \in \mathbb{Z}_n$ such that $\hat{x}_i(t) < 0$. However, if the state of the linear positive system x remains in the interior of the positive orthant and if the observer state $\hat{x}$ converges to the state of the system then from a particular time $t_1 \in [t_0, \infty)$ onwards the state of the observer will be in the positive orthant and remain so. But in case the state of the linear positive observer moves in a facet of the positive orthant then the observer state may be negative for an infinitely long interval of time.

5.3. Observer synthesis

In case attention is restricted to a linear observer which is not necessarily positive then the result for observer synthesis of linear systems can be applied and this is then formulated in the following theorem.

Theorem 5.3. *Consider a linear positive system*,

$$x(t+1) = Ax(t), \quad x(t_0) = x_0, \tag{53}$$

$$y(t) = Cx(t). \tag{54}$$

(a) *There exists a linear observer*, *not necessarily positive*, *of the form*

$$\hat{x}(t+1) = A\hat{x}(t) + K[y(t) - C\hat{x}(t)], \quad \hat{x}(t_0) = \hat{x}_0 \in \mathbb{R}_n, \tag{55}$$

whose error system is globally asymptotically stable if and only if the linear positive system is detectable.

(b) *There exists a linear observer of the form* (55) *such that for any complex conjugate set* $\Lambda \subset \mathbb{C}$ *of precisely n eigenvalues there exists a matrix* $K \in \mathbb{R}^{n \times p}$ *with* $\mathrm{spec}(A - KC) = \Lambda$ *if and only if the linear system is observable*.

Proof. See [33, Th. 7.31] and [36]. □

The formulation of the above theorem is not always stated in the literature in the above stated form. Yet, the reader will not have major difficulties to deduce the proof of the above theorem from the literature.

5.4. Observer synthesis based on the decomposition of a linear positive system

Theorem 5.3 presents observer synthesis based on observer synthesis for linear systems without positivity considerations. This does not provide information on the existence of a positive observer. Moreover, it does not use the decomposition of a linear positive system into the observable canonical form with respect to permutations of the state and of the output set and that based on permutation similarity. Below observer synthesis is described based on the decompositions mentioned.

Distinguish the cases:

1. The matrix tuple (A, C) is observable and the system matrix $A \in \mathbb{R}_+^{n\times n}$ is irreducible. This case in turn is distinguished into the cases:
   (a) The index of imprimitivity of the system matrix A is one. A particular case of this is when the system matrix A is strictly positive.
   (b) The index of imprimitivity of the system matrix A is the same as the size of the matrix, in which case the matrix graph of the system matrix A consists of a single cycle.
   (c) The system matrix A is irreducible with the index of imprimitivity being in the range of $2, \ldots, n-1$.
2. The matrix tuple (A, C) is observable and the system matrix $A \in \mathbb{R}_+^{n\times n}$ is permutation similar to a completely reduced matrix.

The problem is to carry out observer synthesis for each of the above cases and based on the appropriate decomposition of the linear positive systems, and to establish whether or not a linear positive observer exists.

5.5. *Observer synthesis in case of an irreducible system matrix*

This subsection refers to Case 1.b described above. The following result is closely related to that of Patrick De Leenheer, see [22], though part (c) of the following theorem is new.

Theorem 5.4. *Observer synthesis in case the system matrix is irreducible with the index of imprimitivity being equal to the size of the matrix.*

Consider the specific discrete-time linear positive system of the form,

$$x(t+1) = Ax(t),\ x(t_0) = x_0,$$
$$y(t) = Cx(t),$$
$$A = \begin{pmatrix} 0 & a_{1,2} & 0 & \cdots & 0 \\ 0 & 0 & a_{2,3} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n} \\ a_{n,1} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

which is an irreducible matrix,

$$C = \begin{pmatrix} c_1 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}_+^{1\times n},$$
$$\forall i \in \mathbb{Z}_{n-1} a_{i,i+1} \in \mathbb{R}_{s+};\ a_{n,1} \in \mathbb{R}_{s+};\ c_1 \in \mathbb{R}_{s+}.$$

Note that then

$$\mathrm{Imprim}(A) = n, \quad \mathrm{rank}(\mathrm{obsm}(A, C)) = n.$$

Hence the system matrix A is irreducible with imprimitivity index equal to the size of the matrix and the system is observable.

(a) *The spectrum of the system matrix A has n distinct eigenvalues all satisfying* $|\lambda(A)| = a_c^{1/n}$ *with*

$$a_c = a_{1,2}a_{2,3}\ldots a_{n-1,n}a_{n,1} \in \mathbb{R}_{s+}.$$

(b) *There exists a matrix* $K \in \mathbb{R}_+^{n\times p}$ *such that*

$$(A - KC) \in \mathbb{R}_+^{n\times n} \text{ and } \operatorname{spec}(A - KC) \subset \mathbb{D}_o,$$

in fact the spectrum of $(A - KC)$ *has n eigenvalues all with a modulus* $\bar{a}_c^{1/n} < 1$, *hence the dynamic system*

$$\hat{x}(t+1) = A\hat{x}(t) + K(y(t) - C\hat{x}(t)), \quad \hat{x}(t_0) = \hat{x}_0,$$

is a linear positive observer for the above system.

(c) *There does not exist a positive observer with the gain matrix K having, besides the nonzero element* k_n *a second strictly positive element. Therefore if attention is restricted to a linear positive observer then the spectrum of the error system is not dynamically assignable.*

Proof. (a) Note that

$$\begin{aligned}
\det(sI - A) &= \det\left(\begin{pmatrix} s & -a_{1,2} & 0 & \cdots & 0 \\ 0 & s & -a_{2,3} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & 0\ldots & -a_{n-1,n} \\ -a_{n,1} & 0 & 0 & 0\ldots & s \end{pmatrix}\right) \\
&= s\det\begin{pmatrix} s & -a_{2,3} & 0 & \cdots & 0 \\ 0 & s & -a_{3,4} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & 0\ldots & -a_{n-1,n} \\ 0 & 0 & 0 & 0\ldots & s \end{pmatrix} \\
&\quad + (-1)^{n-1}(-a_{n,1})\det\begin{pmatrix} -a_{1,2} & 0 & \cdots & 0 & 0 \\ s & -a_{2,3} & \cdots & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & -a_{n-2,n-1} & 0 \\ 0 & 0 & \cdots & s & -a_{n-1,n} \end{pmatrix} \\
&= s^n - a_c,\ a_c = a_{1,2}a_{2,3}\ldots a_{n-1,n}a_{n,1} \in \mathbb{R}_{s+}.
\end{aligned}$$

Thus the characteristic polynomial of this matrix has n eigenvalues all satisfying $|\lambda(A)| = a_c^{1/n}$.

(b) If $a_c \in (0, 1)$ then $|\lambda(A)| = a_c^{1/n} \in (0, 1)$ for all eigenvalues. In this case one may take $K = 0 \in \mathbb{R}_+^{n\times p}$. The observer will function more efficiently if the eigenvalues of the error system are placed closer to the origin. Take

$$\begin{aligned}
&\bar{a}_c \in (0, \min\{1, a_c\}), \\
&k_n = \left[a_{n,1} - \frac{\bar{a}_c}{a_{1,2}\ldots a_{n-1,n}}\right]\Big/ c_1, \\
&K = (0\ldots 0\ k_n)^{\mathrm{T}} \in \mathbb{R}^n.
\end{aligned}$$

Then

$$\begin{aligned}&\bar{a}_c/a_{1,2}\ldots a_{n-1,n} < a_c/a_{1,2}\ldots a_{n-1,n} = a_{n,1},\\ \Rightarrow &k_n = [a_{n,1} - \bar{a}_c/a_{1,2}\ldots a_{n-1,n}]/c_1 \in (0,\infty) = \mathbb{R}_{s+},\\ &K \in \mathbb{R}_+^{n\times p}.\end{aligned}$$

Then also

$$A - KC = \begin{pmatrix} 0 & a_{1,2} & 0 & \cdots & 0\\ 0 & 0 & a_{2,3} & \cdots & 0\\ \vdots & & & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & a_{n-1,n}\\ \bar{a}_{n,1} & 0 & 0 & \cdots & 0\end{pmatrix},$$

$$\begin{aligned}\bar{a}_{n,1} &= a_{n,1} - k_n c_1 = a_{n,1} - \left(a_{n,1} - \frac{\bar{a}_c}{a_{1,2}\ldots a_{n-1,n}}\right)\\ &= \frac{\bar{a}_c}{a_{1,2}\ldots a_{n-1,n}} \in \mathbb{R}_{s+},\\ &\quad a_{1,2}\ldots a_{n-1,n}\bar{a}_{n,1} = \bar{a} \in (0,1).\end{aligned}$$

From (a) then follows that $|\lambda(A-KC)| = \bar{a}_c^{1/n} < 1$ hence $\mathrm{spec}(A-KC) \subset \mathbb{D}_o$.

(c) This follows directly from the expression for the matrix $(A-KC)$. □

Observer synthesis is considered for the case in which the system matrix $A \in \mathbb{R}_+^{n\times n}$ is irreducible with Indeximprim (A)= 1. Recall from Example 5.2 that in this case there may not exist a linear positive observer. A problem in this case is therefore to characterize when a linear positive observer exists. Equivalent conditions for the existence of a linear positive observer follow from Proposition 5.1. The conditions that $K \in \mathbb{R}_+^{n\times p}$ and that $(A-KC) \in \mathbb{R}_+^{n\times n}$ are linear constraints on the elements of the unknown gain matrix K. The stability condition that $\mathrm{spec}(A-KC) \subset \mathbb{D}_o$ relates via the characteristic polynomial of the matrix to the elements of the matrix K. An equivalent condition for $\mathrm{spec}(A-KC) \subset \mathbb{D}_o$ is that the maximal real eigenvalue satisfies $\lambda^*(A-KC) < 1$ because the modulus of all other eigenvalues is less than or equal to λ^*. The problem of determining a gain matrix K of a linear positive observer then becomes an algebraic-geometric problem for which no simple mathematical solution seems possible to the authors. This line of research is therefore left unexplored at this stage.

In a particular case necessary conditions for the gain matrix can be obtained. If a linear positive observer is required then there follow conditions on the elements of the gain matrix K. If there exist $i, j \in \mathbb{Z}_n$ and $r \in \mathbb{Z}_p$ such that $A_{i,j} = 0$ and $C_{r,j} > 0$ then the existence of a positive observer implies that $K_{i,r} = 0$. This condition is illustrated in the following example.

Example 5.5. Consider a linear positive system with the system matrices,

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0\\ a_{21} & a_{22} & a_{23} & 0\\ 0 & a_{32} & a_{33} & a_{34}\\ 0 & 0 & a_{43} & a_{44}\end{pmatrix}, \quad a_{i,j} \in \mathbb{R}_{s+}, \forall i,j \in \mathbb{Z}_n;$$

$$C = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix}.$$

If there exist linear positive observer, $i, j \in \mathbb{Z}_n$, and $r \in \mathbb{Z}_p$ such that $A_{i,j} = 0$ and $C_{r,j} > 0$ then it follows from the formula

$$A_{i,j} - \sum_{j=1}^{p} K_{i,r} C_{r,j} \in \mathbb{R}_+,$$

that $K_{i,r} = 0$. When this condition is applied to the above defined matrices then the following gain matrix K is obtained,

$$K = \begin{pmatrix} K_{11} & 0 \\ K_{21} & 0 \\ 0 & K_{32} \\ 0 & K_{42} \end{pmatrix}, \quad K_{11}, K_{21}, K_{32}, K_{42} \in \mathbb{R}_+.$$

The experience with this example was used to design an observer for the example of Section 6.

5.6. Observer synthesis in the completely reduced case

Next attention is focused on the case where the system matrix is completely reduced. Suppose that the system matrices A, C admit the following decomposition:

$$x(t+1) = Ax(t), \quad x(t_0) = x_0, \tag{56}$$

$$y(t) = Cx(t), \tag{57}$$

$$A = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ A_{k-1,1} & A_{k-1,2} & \cdots & 0 \\ A_{k,1} & A_{k,2} & \cdots & A_{k,k} \end{pmatrix} \in \mathbb{R}_+^{n \times n}, \tag{58}$$

$$C = \begin{pmatrix} C_{11} & 0 & \cdots & 0 \\ C_{21} & C_{22} & & 0 \\ \vdots & & \ddots & \\ C_{k,k} & C_{k,2} & \cdots & C_{k,k} \end{pmatrix}, \tag{59}$$

$$n_1, n_2, \ldots, n_k \in \mathbb{Z}_n, \; p_1, p_2, \ldots, p_k \in \mathbb{Z}_p,$$

$$A_{i,j} \in \mathbb{R}_+^{n_i \times n_j}, C_{i,j} \in \mathbb{R}_+^{p_i \times n_j}, \sum_{j=1}^{k} n_j = n, \sum_{j=1}^{k} p_i = p,$$

$$A_{i,i}, \; i \in \mathbb{Z}_k \text{ are irreducible,}$$

$$(A, C), \text{ observable pair.}$$

Below an algorithm is presented for observer synthesis for the linear positive system described above. A motivation of the algorithm precedes it.

A naive approach is to carry out observer synthesis for each irreducible block of the system matrix separately. However, this is not possible, for example for the case in which the system matrix has the form,

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix},$$
$$C = \begin{pmatrix} 0 & C_2 \end{pmatrix}.$$

It thus becomes clear that the observer synthesis must combine several irreducible subsystems and output components.

Yet another special case is the following:

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix},$$
$$C = \begin{pmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{pmatrix},$$
$$(A_{11}, C_{11}), (A_{22}, C_{22}) \text{ observable pairs.}$$

It is then possible to carry out observer synthesis for the matrix tuples (A_{11}, C_{11}) and (A_{22}, C_{22}) separately though information about the first subsystem is also available in the second output components. According to this approach the resulting matrices are then,

$$K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix},$$
$$A - KC = \begin{pmatrix} A_{11} - K_1 C_{11} & 0 \\ A_{21} - K_2 C_{21} & A_{22} - K_2 C_{22} \end{pmatrix}.$$

An alternative is to consider a block gain matrix of the form,

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix},$$
$$A - KC = \begin{pmatrix} A_{11} - K_{11}C_{11} - K_{12}C_{21} & -K_{12}C_{22} \\ A_{21} - K_{21}C_{11} - K_{22}C_{21} & A_{22} - K_{22}C_{22} \end{pmatrix}.$$

If a linear positive observer is wanted then from the (1,2) block follows that in general $K_{12} = 0$ is necessary. In addition, the condition

$$(A_{21} - K_{21}C_{11} - K_{22}C_{22}) \in \mathbb{R}_+^{n_2 \times n_2}$$

is necessary besides the positiveness and the stability conditions of the two diagonal block matrices.

Below an algorithm is presented in which observer synthesis is carried out for a sequence of subsystems.

Algorithm 5.6. Observer synthesis in case the system matrices are of the form displayed in Eqs. (58) and (59).

1. Determine the least integer $k_1 \in \mathbb{Z}_k$ such that the following matrix tuple is an observable pair,
$$F_{11} = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ A_{k_1,1} & A_{k_1,2} & \cdots & A_{k_1,k_1} \end{pmatrix} \in \mathbb{R}_+^{\bar{n}_1 \times \bar{n}_1}, \tag{60}$$
$$H_{11} = \begin{pmatrix} C_{11} & 0 & \cdots & 0 \\ C_{21} & C_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ C_{k_1,1} & C_{k_1,2} & \cdots & C_{k_1,k_1} \end{pmatrix} \in \mathbb{R}_+^{\bar{p}_1 \times \bar{n}_1}. \tag{61}$$
2. Determine a matrix $K_1 \in \mathbb{R}^{\bar{n} \times \bar{p}}$ such that $\text{spec}(F_{11} - K_1 H_{11}) \subset \mathbb{D}_o$. In case the linear positive observer is required then determine a matrix $K_1 \in \mathbb{R}_+^{\bar{n} \times \bar{p}}$ such that
$(F_{11} - K_1 H_{11}) \in \mathbb{R}_+^{\bar{n}_1 \times \bar{n}_1} \text{spec}(F_{11} - K_1 H_{11}) \subset \mathbb{D}_o$.
In case no such positive matrix exists then stop and output that no positive observer exists.
3. Denote then
$$A = \begin{pmatrix} F_{11} & 0 \\ \overline{A}_{21} & \overline{A}_{22} \end{pmatrix}, \tag{62}$$
$$C = \begin{pmatrix} H_{11} & 0 \\ \overline{C}_{21} & \overline{C}_{22} \end{pmatrix}. \tag{63}$$
4. Proceed with the matrix tuple $(\overline{A}_{22}, \overline{C}_{22})$ according to Algorithm 5.6 thus start at Step 1 with the matrix tuple $(\overline{A}_{22}, \overline{C}_{22})$ in stead of (A, C). Thus select matrices (F_{22}, H_{22}) and construct a matrix $K_2 \in \mathbb{R}^{\bar{n}_2 \times \bar{p}_2}$ such that
$\text{spec}(F_{22} - K_2 H_{22}) \subset \mathbb{D}_o$.
This is possible because of the assumption that the matrix tuple (A, C) is observable and an extension argument proven in the theorem below. If a positive observer is required then determine a matrix $K_2 \in \mathbb{R}_+^{\bar{n}_2 \times \bar{p}_2}$ such that
$$(F_{22} - K_2 H_{22}) \in \mathbb{R}_+^{\bar{n}_2 \times \bar{n}_2}, (F_{21} - K_2 H_{21}) \in \mathbb{R}_+^{\bar{n}_2 \times \bar{n}_2}, \tag{64}$$
$$\text{spec}(F_{22} - K_2 H_{22}) \subset \mathbb{D}_o. \tag{65}$$
In case no such positive matrix exists then stop and output that no positive observer exists.
5. Proceed by induction computing a sequence of gain matrices $(K_1, K_2, \ldots, K_s)$.
6. Compose
$$K = \begin{pmatrix} K_1 & 0 & \cdots & 0 \\ 0 & K_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_s \end{pmatrix} \in \mathbb{R}^{n \times p}. \tag{66}$$
7. The observer is then
$$\hat{x}(t+1) = A\hat{x}(t) + K[y(t) - C\hat{x}(t)], \tag{67}$$

which is a globally asymptotically stable linear observer. Moreover, it is a positive observer if the conditions in the Steps 2 and 4 for each step of the algorithm are met.

Theorem 5.7. *Consider a linear positive system with the system matrices described in the equations* (58) *and* (59). *Assume that the linear positive system is observable. Algorithm* 5.6 *is correct, thus it results in a globally asymptotically stable linear observer and, in addition, in a positive observer if the conditions of the Steps* 2 *and* 4 *of the algorithm are satisfied.*

Proof. The steps of the algorithm will be followed closely.

(1) An integer $k_1 \in \mathbb{Z}_k$ such that (F_{11}, H_{11}) is an observable pair exists because (A, C) is an observable pair, the linear positive is assumed to be observable.

(2) It follows from observer synthesis of linear systems that there exists a matrix $K_1 \in \mathbb{R}^{\bar{n}_1 \times \bar{p}_1}$ such that $\mathrm{spec}(F_{11} - K_1 H_{11}) \subset \mathbb{D}_o$. For a positive observer one determines a positive matrix $K_1 \in \mathbb{R}_+^{\bar{n}_1 \times \bar{p}_1}$ such that $(F_{11} - K_1 H_{11}) \in \mathbb{R}_+^{\bar{n}_1 \times \bar{n}_1}$ and $\mathrm{spec}(F_{11} - K_1 H_{11}) \subset \mathbb{D}_o$. If no such positive matrix exists then there does not exist a positive observer. The latter statement follows from Proposition 5.1.

(3) It will be proven that the matrix tuple (F_{22}, H_{22}) is an observable pair. Recall that (A, C) being an observable pair, satisfies the Hautus condition,

$$\mathrm{rank}\begin{pmatrix} A - \lambda I \\ C \end{pmatrix} = n, \quad \forall \lambda \in \mathbb{C}.$$

But

$$\begin{aligned}
n = \mathrm{rank}\begin{pmatrix} A - \lambda I \\ C \end{pmatrix} &= \mathrm{rank}\begin{pmatrix} F_{11} - \lambda I & 0 \\ F_{21} & F_{22} - \lambda I \\ H_{11} & 0 \\ H_{21} & H_{22} \end{pmatrix} \\
&= \mathrm{rank}\begin{pmatrix} F_{11} - \lambda I & 0 \\ H_{11} & 0 \\ F_{21} & F_{22} - \lambda I \\ H_{21} & H_{22} \end{pmatrix} = \bar{n}_1 + \bar{n}_2, \\
\Leftrightarrow \mathrm{rank}&\begin{pmatrix} F_{11} - \lambda I \\ H_{11} \end{pmatrix} = \bar{n}_1 \quad \text{and} \quad \mathrm{rank}\begin{pmatrix} F_{22} - \lambda I \\ H_{22} \end{pmatrix} = \bar{n}_2.
\end{aligned}$$

Thus (F_{22}, H_{22}) is an observable pair.

(4) Proceed with the matrix tuple (F_{22}, H_{22}) as the tuple (A, C) in Step 1 of this algorithm. To simplify the proof, suppose that the matrix tuple (F_{22}, H_{22}) does not admit a further decomposition of the form used in Step 3 of the algorithm with strictly smaller (F_{ii}, H_{ii}) block. Then the observability of the matrix tuple (F_{22}, H_{22}) implies that there exists a matrix $K_2 \in \mathbb{R}^{\bar{n}_2 \times \bar{p}_2}$ such that $\mathrm{spec}(F_{22} - K_2 H_{22}) \subset \mathbb{D}_o$. In case a positive observer is required then one determines, if one exists, a matrix $K_2 \in \mathbb{R}_+^{\bar{n}_2 \times \bar{p}_2}$ such that

$$\begin{aligned}
&(F_{22} - K_2 H_{22}) \in \mathbb{R}_+^{\bar{n}_2 \times \bar{n}_2}, \quad (F_{21} - K_2 H_{21}) \in \mathbb{R}_+^{\bar{n}_2 \times \bar{n}_1}, \\
&\mathrm{spec}(F_{22} - K_2 H_{22}) \subset \mathbb{D}_o.
\end{aligned}$$

In case no such matrix exists then, as stated above, one concludes that no positive observer exists.

(5) In general, K is defined as indicated.

(6) It is proven that the dynamic system of equation (67) is a globally asymptotically stable observer. Note that for the case of only two blocks,

$$
\begin{aligned}
(A-KC) &= \begin{pmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{pmatrix} - \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \begin{pmatrix} H_{11} & 0 \\ H_{21} & H_{22} \end{pmatrix} \\
&= \begin{pmatrix} F_{11}-K_1H_{11} & 0 \\ F_{21}-K_2H_{21} & F_{22}-K_2H_{22} \end{pmatrix}, \\
\text{spec}(A-KC) &= \text{spec}(F_{11}-K_1H_{11}) \cup \text{spec}(F_{22}-K_2H_{22}) \subset \mathbb{D}_o,
\end{aligned}
$$

by the Steps 2 and 4 of the algorithm. Note that the second components of the state of the system and that of the observer are,

$$
\begin{aligned}
x_2(t+1) &= F_{22}x_2(t) + F_{21}x_1(t), \\
\hat{x}_2(t+1) &= F_{22}\hat{x}_2(t) + F_{21}\hat{x}_1(t) + K_2[y_2(t) - H_{22}\hat{x}_2(t) - H_{21}\hat{x}_1(t)], \\
e_2(t) &= x_2(t) - \hat{x}_2(t), \quad e_1(t) = x_1(t) - \hat{x}_1(t), \\
e_2(t+1) &= (F_{22}-K_2H_{22})e_2(t) + (F_{21}-K_2H_{21})e_1(t).
\end{aligned}
$$

Thus also the second component of the error system is globally asymptotically stable with the indicated error dynamics. If the conditions of the Steps 2 and 4 of the algorithm for the existence of positive K_1 and K_2 are satisfied than,

$$
K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \in \mathbb{R}_+^{n\times p}, \quad (A-KC) \in \mathbb{R}_+^{n\times n}, \quad \text{spec}(A-KC) \subset \mathbb{D}_o.
$$

Thus the linear observer is a positive observer.

In case of three or more blocks in the A matrix one proceeds by induction. □

Example 5.8. Consider the discrete-time linear positive system

$$
\begin{aligned}
&x(t+1) = Ax(t), x(t_0) = x_0, \\
&y(t) = Cx(t), \\
&A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \in \mathbb{R}_+^{n\times n}, \quad A_{11} \in \mathbb{R}_+^{n_1\times n_1}, \ A_{22} \in \mathbb{R}_+^{n_2\times n_2}, \\
&C = \begin{pmatrix} 0 & C_2 \end{pmatrix} \in \mathbb{R}_+^{p\times(n_1+n_2)}, \\
&(A, C) \text{ observable pair.}
\end{aligned}
$$

Then Algorithm 5.6 produces an observer by Step 2 of the form

$$
\begin{aligned}
&K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \in \mathbb{R}^{n\times p}, \\
&(A-KC) = \begin{pmatrix} A_{11} & -K_1C_2 \\ A_{21} & A_{22}-K_2C_2 \end{pmatrix} \in \mathbb{R}^{n\times n}, \\
&\text{spec}(A-KC) \subset \mathbb{D}_o.
\end{aligned}
$$

If $\text{spec}(A_{11}) \not\subset \mathbb{D}_o$ then no globally asymptotically stable *positive* observer can exist. Because if a positive observer existed which is globally asymptotically stable and if $\text{spec}(A_{11}) \not\subset \mathbb{D}_o$ then the matrix $K_1 \in \mathbb{R}_+^{n_1\times p}$ must have at least one strictly positive element. (Proven by contradiction: If $K_1 = 0$ then $\text{spec}(A-KC) \not\subset \mathbb{D}_o$.) Because (A, C) is an observable pair, the matrix C_2 must have a strictly positive element. Hence K_1C_2 has at least one strictly positive element thus $(A - KC) \notin \mathbb{R}_+^{n\times n}$. This is a contradiction of the supposition that a globally asymptotically stable positive observer exists.

6. Positive observer for glycolysis in yeast

The development of the theorems in this article is motivated by their use for models of biochemical systems. To illustrate this, an observer for the model of glycolysis in yeast has been constructed. Glycolysis is the name of a cellular process in which glucose is converted into different carbon compounds in a number of reaction steps, see Fig. 1. This process plays an important role in the metabolism of most organisms, and its function is partly to produce energy rich molecules and partly to produce carbon compounds that can be used for biosynthesis. The glycolysis has been extensively explored in many organisms and several mathematical models of it have been made, see for example [28,18,17,14,11].

Here, a model of glycolysis made by Teusink et al. [11] has been used. In order to make this model suitable for our purpose, a few modifications were done by changing state variables to constants and certain constants into state variables. After these changes the model contains 13 state variables, $x_1, \ldots, x_{13}$, each representing the concentration of a carbon compound, except from one, x_5, which represents a module of two interconvertable carbon compounds, see Fig. 1. As output variables four of the state variables have been chosen, those which are easier to measure than the others, namely x_1, x_{12}, x_{10}, and x_{13}. Since the biological process is nonlinear, the system was linearized around its steady state, which resulted in the system

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t), x(t_0) = x_0,\\
&y(t) = Cx(t), \text{ where,}\\
&x = (x_1, \ldots, x_{13})^{\mathrm{T}}, \text{ is the vector of the 13 state variables,}\\
&y = (y_1, \ldots, y_4)^{\mathrm{T}} = (x_1, x_{12}, x_{10}, x_{13})^{\mathrm{T}}, \text{ is the vector of the 4 output variables,}
\end{aligned}
$$

$$
A = \begin{pmatrix}
-472.8 & 1.416 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
408.3 & -90.04 & 487.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 88.62 & -1127 & 51.52 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 639.9 & -295.3 & 308.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 487.5 & -787.3 & 3.361\times 10^4 & 0 & 0 & 0 & 0 & 0 & 6.238 & 0\\
0 & 0 & 0 & 0 & 161.4 & -1.212\times 10^6 & 875.3 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1.179\times 10^6 & -1947 & 6881 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1072 & -9408 & 722.0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2527 & -2137 & 3.938 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1415 & -10.52 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.582 & -2480 & 0 & 7.645\\
0 & 0 & 0 & 0 & 8.984 & 0 & 0 & 0 & 0 & 0 & 0 & -126.2 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2438 & 0 & -10.24
\end{pmatrix},
$$

$$
C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

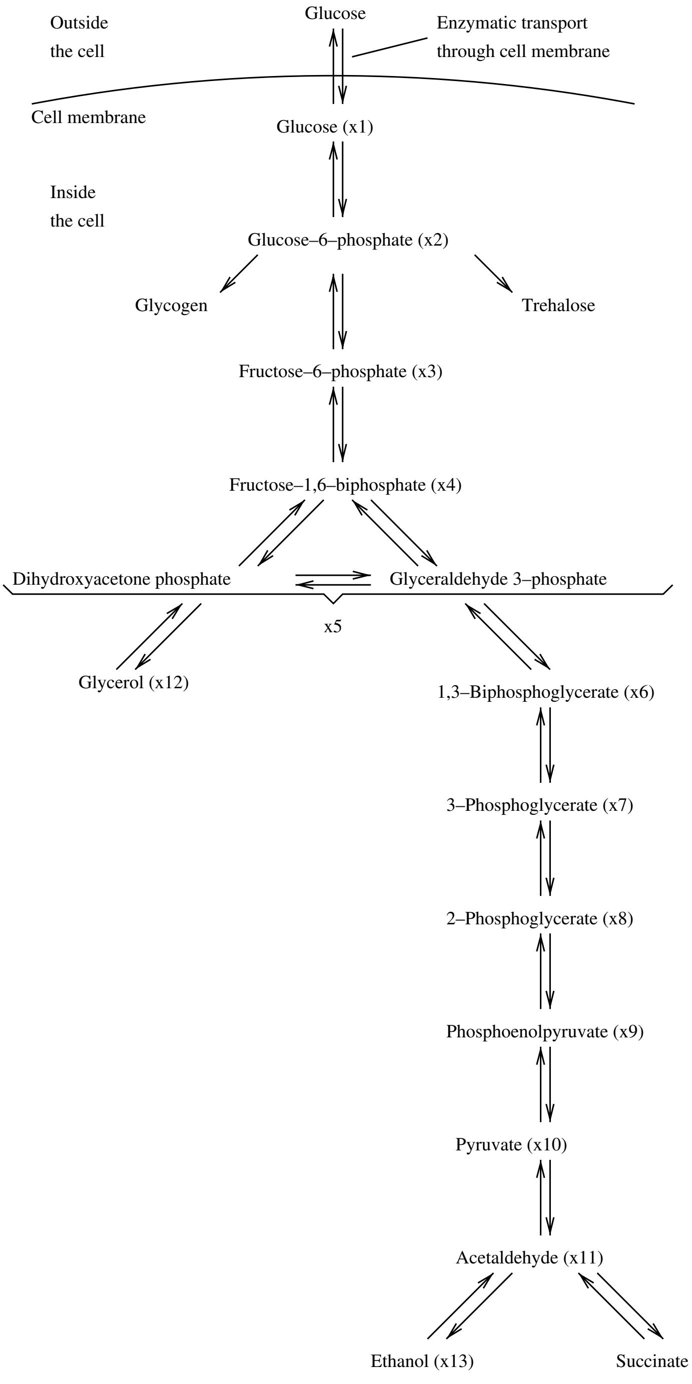

Fig. 1. Reaction diagram of glycolysis in yeast as it was described in Teusink et al. [11]. Arrows represent reactions from one compound to another and double arrows represent reversible reactions. The state variables x_1 to x_{13}, representing the concentrations of compounds, are indicated.

Because A is a Metzler matrix and all its eigenvalues lie in the open left-half complex plane, see Table 1, the system is a stable positive system. An observer for the system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t) = (A - KC)\hat{x}(t) + Ky(t), \hat{x}(t_0) = \hat{x}_0 \tag{68}$$

where K is chosen as

$$K = \begin{pmatrix} 1000 & 0 & 0 & 0 \\ 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 20 & 0 \\ 0 & 0 & 5 & 6.5 \\ 0 & 1000 & 0 & 0 \\ 0 & 0 & 0 & 20 \end{pmatrix}.$$

The matrix $A - KC$ is Metzler and its eigenvalues are negative, see Table 1. Thus the system described by Eq. (68) is a linear positive observer according to Proposition 5.1. The reason to choose $K \neq 0$ is to make the observer more stable. As seen in Table 1, most of the eigenvalues of $A - KC$ are located in the left-half complex plane further to the left than those of A.

When using this observer one has to keep in mind that it is an observer for the linearized system. If measurements of the system have been made when the state is far away from the steady state at which it is linearized, then the observer might give bad estimates of the state. However, as long as the system is close to steady state, the use of a linear observer has the advantage that it requires less design work than the design of a nonlinear observer.

Table 1
Eigenvalues of the matrices A and $A - KC$, respectively

A	$A - KC$
−2.724	−22.03
−6.498	−29.11
−21.68	−29.60
−80.34	−80.98
−126.4	−205.4
−206.3	−967.8
−474.1	−1126
−967.9	−1222
−1222	−1473
−2021	−2019
−2487	−2481
-1.042×10^4	-1.042×10^4
-1.213×10^6	-1.213×10^6

7. Concluding remarks

The paper formulates the problem of observer synthesis for linear positive systems. The concept of observability of a linear positive system is formulated and characterized by a rank condition for the matrix pair (A, C). An observable canonical form is formulated and interpreted. A dynamic system for a linear positive system is defined and it is proven for several cases that the dynamic system is indeed a linear observer which in particular cases can be a linear positive observer. For a mathematical model of glycolysis in yeast in the form of a continuous-time linear positive systems, a linear positive observer is synthesized.

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Appendix A. Observer synthesis for continuous-time linear positive systems

In this appendix the topics of this paper are treated for *continuous-time* linear positive systems. In the main body of the paper only observer synthesis for discrete-time linear positive systems is described.

A.1. Continuous-time linear positive systems

The matrix $A \in \mathbb{R}^{n \times n}$ is called a *Metzler* matrix if for all $i, j \in \mathbb{Z}_n$ with $i \neq j$, $A_{i,j} \in \mathbb{R}_+$; or, equivalently, if the off-diagonal elements are positive.

Definition A.1. A *time-invariant continuous-time linear positive system* (without input) is a dynamic system as defined in, for example [33], with the equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t), \quad x(t_0) = x_0, \tag{69}$$

$$y(t) = Cx(t), \tag{70}$$

$$T = [t_0, \infty) \subset \mathbb{R}, x : T \to \mathbb{R}_+^n, y : T \to \mathbb{R}_+^p, x_0 \in \mathbb{R}_+^n,$$

$$A \in \mathbb{R}^{n \times n} \text{ is a Metzler matrix and } C \in \mathbb{R}_+^{p \times n}.$$

The linear system

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t), \quad x(t_0) = x_0, \tag{71}$$

$$y(t) = Cx(t), \tag{72}$$

$$T = [t_0, \infty) \subset \mathbb{R}, x : T \to \mathbb{R}^n, y : T \to \mathbb{R}_+^p, x_0 \in \mathbb{R}_+^n,$$

$$A \in \mathbb{R}^{n \times n} C \in \mathbb{R}^{p \times n}.$$

has the positive orthant as an invariant set if and only if $A \in \mathbb{R}^{n\times n}$ is a Metzler matrix. If the state is in the positive orthant then the output is positive function if and only if the output matrix satisfies $C \in \mathbb{R}_+^{p\times n}$.

Definition A.2. Consider the continuous-time linear positive system

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t), \quad x(t_0) = x_0,$$
$$y(t) = Cx(t).$$

A *continuous-time linear observer* for this system is a dynamic system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t) = F\hat{x}(t) + Ky(t), \quad \hat{x}(t_0) = \hat{x}_0 \in \mathbb{R}_+^n, \quad \hat{x} : T \to \mathbb{R}_n, \tag{73}$$

for which the system matrices F and K are to be selected. It is called:

1. *globally asymptotically stable*: if the estimation error is globally asymptotically stable:
$$\forall x_0 \in \mathbb{R}_+^n, \forall \hat{x}_0 \in \mathbb{R}_+^n, \lim_{t\to\infty} [\hat{x}(t) - x(t)] = 0; \tag{74}$$
2. *dynamically assignable* if for any complex conjugate subset $\Lambda \subset \mathbb{C}^-$ there exists a gain matrix $K \in \mathbb{R}^{n\times p}$ such that the eigenvalues of the error system, for $\hat{x} - x$, have Λ as its eigenvalues; and
3. *a positive observer*: the observer is a positive system:
$$y : T \to \mathbb{R}_+^p, \hat{x}_0 \in \mathbb{R}_+^n, t \in T \Rightarrow \hat{x}(t) \in \mathbb{R}_+^n. \tag{75}$$

One can then define a positive observer which is globally asymptotically stable or a positive observer which is dynamically assignable.

A.2. *Observability*

The observability concepts and their characterization for continuous-time linear positive systems are quite analogous to those for discrete-time linear positive systems.

A system matrix $A \in \mathbb{R}^{n\times n}$ of a continuous-time linear positive system can be transformed into a system matrix of a discrete-time linear positive system by the following transformation. Recall that a continuous-time linear system is positive if and only if $A \in \mathbb{R}^{n\times n}$ is a Metzler matrix and $C \in \mathbb{R}_+^{p\times n}$. By definition of a Metzler matrix there exists a constant $a \in \mathbb{R}_+$ such that $(A + aI) \in \mathbb{R}_+^{n\times n}$, for example,

$$a = \max\left\{0, -\min_{i\in\mathbb{Z}_n} A_{i,i}\right\}; \quad \text{then } A_{i,i} + a \geqslant 0,\ \forall i \in \mathbb{Z}_n.$$

Call the transformation

$$A \in \mathbb{R}^{n\times n} \text{ Metzler} \mapsto (A + aI) \in \mathbb{R}_+^{n\times n},$$

the *Metzler-to-positive-matrix transform*.

Definition A.3. Consider the continuous-time linear positive system

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t), x(t_0) = x_0,$$
$$y(t) = Cx(t), \quad T = [t_0, t_1], \text{ or } T = [t_0, \infty).$$

The system is called *observable as a continuous-time linear positive system* on the interval T if the observability map is injective:

$$\text{obsmap} : \mathbb{R}_+^n \to (\mathbb{R}_+^n)^{\mathrm{T}}, x_0 \mapsto \{y(.; t_0, x_0) : T \to \mathbb{R}_+^p\} \text{ is injective.}$$

From now on the term *observable* will be used to describe *observable as a continuous-time linear positive system.*

Theorem A.4. *The continuous-time linear positive system of Definition* A.3 *is observable if and only if*

$$\text{rank}(\text{obsm}(A, C)) = n.$$

Proof. As is well known from observability for ordinary linear systems, the following relations hold.

$$\left.\begin{pmatrix} y(s) \\ \frac{\mathrm{d}}{\mathrm{d}s}y(s) \\ \vdots \\ \frac{\mathrm{d}^k}{\mathrm{d}s^k}y(s) \\ \vdots \end{pmatrix}\right|_{s=t_0} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \\ \vdots \end{pmatrix} x_0.$$

The projection of this infinitely long vector to a finite vector of length pn on the first pn components then produces on the right-hand side the observability matrix of the matrix pair (A, C).

Next apply the Metzler-to-positive-matrix-transformation. With the use of elementary linear algebra it follows thats

$$\begin{aligned} &\text{rank}(\text{obsm}((A + aI), C)) \\ &= \text{rank}\begin{pmatrix} C \\ C(A + aI) \\ \vdots \\ C(A + aI)^{n-1} \end{pmatrix} = \text{rank}\left(S\begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}\right) \\ &\quad S \in \mathbb{R}^{pn \times pn}, \text{ nonsingular}, \\ &= \text{rank}(\text{obsm}(A, C)). \end{aligned}$$

The result then follows from Theorem 4.7. □

A.3. Observer synthesis – equivalent conditions

Proposition A.5. *Consider a continuous-time linear positive system and the associated candidate linear observer*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t), \quad x(t_0) = x_0,\\
&y(t) = Cx(t),\\
&\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t) = (A - KC)\hat{x}(t) + Ky(t), \quad \hat{x}(t_0) = \hat{x}_0.
\end{aligned} \tag{76}$$

The dynamic system (76) *is a globally asymptotically stable linear observer if and only if the following conditions all hold*:

$$(1)\ K \in \mathbb{R}^{n\times p}, \tag{77}$$

$$(2)\ \mathrm{spec}(A - KC) \subset \mathbb{C}^-. \tag{78}$$

The dynamic system (76) *is an globally asymptotically stable linear positive observer if and only if the following conditions all hold*:

$$(1)\ K \in \mathbb{R}_+^{n\times p}, \tag{79}$$

$$(2)\ (A - KC) \in \mathbb{R}^{n\times n} \quad \textit{is a Metzler matrix; and} \tag{80}$$

$$(3)\ \mathrm{spec}(A - KC) \subset \mathbb{C}^-. \tag{81}$$

A.4. Nonexistence of a continuous-time linear positive observer

Example A.6. *Nonexistence of a continuous-time linear positive observer for a linear positive system*. Consider the linear positive system,

$$\dot{x}(t) = Ax(t), \quad x(t_0) = x_0, \tag{82}$$

$$y(t) = Cx(t), \tag{83}$$

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \tag{84}$$

which system is observable. Attention is restricted to a linear observer of the form

$$\mathrm{d}\hat{x}(t)/\mathrm{d}t = A\hat{x}(t) + K[y(t) - C\hat{x}(t)], \hat{x}(t_0) = \hat{x}_0. \tag{85}$$

Using Proposition A.5 one then shows that a linear positive observer does not exist.

A.5. Observer synthesis

The observer synthesis for a continuous-time linear observer or a linear positive observer can now easily be deduced from the associated observer problem for discrete-time linear positive systems via the Metzler-to-positive-matrix transformation described in Section A.2.

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