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THE 2-QUASI-GREEDY ALGORITHM FOR CARDINALITY CONSTRAINED MATROID BASES

Fred GLOVER and Beth NOVICK
Management Science and Information Systems, School of Business, Box 419, University of Colorado, Boulder, CO 80309, USA

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The *quasi-greedy algorithm*, as proposed by Glover and Klingman [8], efficiently solves minimum weight spanning tree problems with a fixed (or bounded) number of edges incident to a specified vertex. As observed in [8], the results carry through to general matroid problems (where a base contains a bounded number of elements from a specified set). We extend this work to provide an efficient 2-*quasi-greedy* algorithm where a minimum weight base is constrained to have a fixed number of elements from two disjoint sets.

Our main results show that optimal bases for adjacent states may not themselves be adjacent. However, optimal solutions for adjacent states may be identified solely from information available in the current base, yielding a method whose efficiency rivals that of the quasi-greedy method. We also give theorems making it possible to jump over certain adjacent states, further increasing efficiency.

1. Introduction

Consider the two problems:

P: Find a minimum weight base B (in matroid M).

P(k): Find a minimum weight base B that contains k elements of an arbitrary set S.

Problem **P** is the classical minimum weight base problem solvable by the greedy algorithm. Problem **P**(k) does not admit of a greedy solution. For example, one cannot solve **P**(k) by first selecting k best elements from S, and then the remaining elements from outside S, subject to maintaining independence at each step.

Although **P**(k) represents a level of complexity above that of **P**, it can be solved very efficiently. Theorems giving primal and dual algorithms for **P**(k) in the graph theory setting, where B is a spanning tree and S is the set of edges incident to a specified node, were provided in Glover and Klingman [8]. These algorithms were collectively labeled *quasi-greedy* algorithms. In contrast to the greedy solutions, appropriate moves consist of "trading things between two buckets" instead of simply "putting things into a bucket" (in Jack Edmonds' terminology [1]).

The results for solving **P**(k) in the graph theory setting, as noted in [7], carry over

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directly to the matroid setting and apply as well to solving the problems where at least k or at most k elements of S must be contained in B.

A good deal of attention has focussed on the greedy algorithm for problem **P**. This has occurred particularly in reference to solving relaxations for the traveling salesman problem, as in the work of Held and Karp [10, 11], and in establishing the 'goodness' of the greedy solution for other problems, as in the work of Faigle [2, 3], Fisher [4], Jenkyns [12], Karp [13], Lawler [14] Korte et al., [15, 16], and others. The quasi-greedy algorithm clearly permits the solution of stronger traveling salesman relaxations than the greedy algorithms, with scarcely more effort, and invites speculation as to whether improved 'goodness of solution' results may be obtained by analyses related to those cited.

Our concern here is with establishing that a somewhat more difficult problem can also be solved highly efficiently. The form of this problem is:

$\mathbf{P}(k_1, k_2)$: Find a minimum weight base B that contains k_1 elements of S_1 and k_2 elements of S_2, for two arbitrary disjoint sets S_1 and S_2.

The increased combinatorial complexity of this two parameter problem is evidenced by the fact, as we show, that optimal bases for adjacent states (where exactly one parameter differs by a unit amount) may not themselves be adjacent, i.e., obtained from each other by a single base exchange step. By contrast, given an optimal base B for $\mathbf{P}(k)$, an optimal base for $\mathbf{P}(k+1)$ ($\mathbf{P}(k-1)$) can always be found among those adjacent to B, provided the latter problem has a solution [8]. Nevertheless, we establish that an optimal base B' for $\mathbf{P}(k_1, k_2+1)$ ($\mathbf{P}(k_1, k_2-1)$), can be obtained from an optimal base B for $\mathbf{P}(k_1, k_2)$ with no more information than in the quasi-greedy algorithm, making reference only to the structure of B. Moreover, we show that for the non-adjacent parameter states where k_1 and k_2 both differ by 1 unit, but in opposite directions, it is possible to generate B' from B with the same effort as in the case of adjacent parameter states.

We now provide more formal definitions and notations, leading to a precise statement of our results.

2. Notation and background

Relative to a matroid M, with a weight function $w(e)$ defined on elements e, we define the weight of a set B by $w(B) = \sum_{e \in B} w(e)$.

Then we define the problem:

$\mathbf{P}(k)$: Find a base B to
Minimize $w(B)$
subject to $|B \cap S| = k$
for an arbitrary specified set **S**.

(The problem $\mathbf{P}(k)$ evidently has a feasible solution only if the rank of S is at least k.)

Similarly, we define

$\mathbf{P}(k_1,k_2)$: Find a base B to
Minimize $w(B)$
subject to $|B\cap S_1|=k_1$ and $|B\cap S_2|=k_2$
for arbitrary disjoint sets S_1 and S_2.

We call an ordered pair (e,e') a *B exchange* if $e\in B$, $e'\notin B$, and $B+e'-e$ is a base. The fundamental results for solving $\mathbf{P}(k)$ are given by Theorems 1 and 2 of [8], expressed in terms of matroids as follows.

Theorem 1 (Primal Theorem). *Assume B is feasible for* $\mathbf{P}(k)$. *Then B is optimal if and only if:*

(1) *There is no B exchange (e,e') such that $w(e')<w(e)$ and either $e,e'\in \mathbf{S}$ or $e,e'\notin \mathbf{S}$; and*

(2) *There are no two distinct B exchanges (e_1,e_1') and (e_2,e_2') such that $w(e_1')+w(e_2')<w(e_1)+w(e_2)$, and $e_1,e_2'\in S$, $e_1',e_2\notin S$.*

Theorem 2 (Dual Theorem). *Assume B is optimal for* $\mathbf{P}(k)$. *Then the base $B'=B+e'-e$ is optimal for $P(k+1)$ ($P(k-1)$) if and only if (e,e') is a least weight B exchange such that $e\notin S$ and $e'\in S$ ($e\in S$ and $e'\notin S$).*

By implication, if there is no exchange of the specified type in Theorem 2, then the problem $\mathbf{P}(k+1)$ (*or* $\mathbf{P}(k-1)$) has no feasible solution. It may be noted that the solution generated by Theorem 2 for $\mathbf{P}(k+1)$ assures that all elements of S that are in B are also contained in B'. Thus, when solving a succession of problems $\mathbf{P}(k)$, $\mathbf{P}(k+1)$, $\mathbf{P}(k+2)$, etc., each element added to B stays in B thereafter, and each element dropped from B stays out thereafter. (Similar remarks apply, in reverse, for solving the succession of problems $\mathbf{P}(k)$, $\mathbf{P}(k-1)$, $\mathbf{P}(k-2)$, etc.) In developing theorems for $\mathbf{P}(k_1,k_2)$ we will see that similar easily specified monotonicity properties are generated.

Another simple consequence of the results of [8] is that the optimum objective value Z_k for $\mathbf{P}(k)$ becomes increasingly bad (larger) moving in either direction away form the solution to the unconstrained problem $\mathbf{P}$. In general, for any k, the Z_k values satisfy the convexity property $Z_{k+2}-Z_{k+1}\ge Z_{k+1}-Z_k$. (The papers by Gabow [5] and Gabow and Tarjan [6] exploit such properties to provide worst case bounds for the quasi-greedy algorithm in special matroid settings.) The convexity property, like the monotonicity property for $\mathbf{P}(k)$, follows as a special case of analogous properties for the problem $\mathbf{P}(k_1,k_2)$. The most general form of these properties and their implications are characterized in [7].

3. Results for the 2-quasi-greedy algorithm

We will first state the theorems that provide a method (actually, a collection of methods) for solving $\mathbf{P}(k_1, k_2)$, and reserve proofs until a subsequent section. To facilitate the statement of our results, we use the following conventions:

(C1) S_0 denotes the set complement of $S_1 + S_2$ (hence S_0, S_1 and S_2 partition the elements of M).

(C2) Elements will be given the same subscripts as the sets they belong to. For example, $e_0, e_0' \in S_0$, $e_1, e_1' \in S_1$, etc.

(C3) When reference is made to a base B, elements without a prime (′) mark denote elements in B, and those with a prime mark denote those not in B. For example, $e \in B$, $e' \notin B$, and by (C2), $e_i \in S_i \cap B$, $e_i' \in S_i - B$.

(C4) A B exchange (e_i, e_j') will be called a *least weight exchange of its type* if $w(e_j') - w(e_i) \le w(g_j') - w(g_i)$ for all $g_i \in S_i \cap B$, $g_j' \in S_j - B$.

We begin by stating a primal theorem that may be viewed as an analog of Theorem 1.

Theorem 3 (Primal Theorem). *Assume B is feasible for $\mathbf{P}(k_1, k_2)$ (and the notational conventions* (C1)–(C3) *apply; hence, $e_i \in S_i \cap B$, $e_i' \in S_i - B$). Then B is optimal if and only if none of the following conditions hold:*

(1) *There is a B exchange of the form (e_i, e_i') for some $i = 0, 1, 2$, such that $w(e_i') < w(e_i)$.*

(2) *There are two B exchanges of the form (e_i, e_j'), (e_j, e_i') for some i, j $(i \ne j)$ such that $w(e_i') + w(e_j') < w(e_i) + w(e_j)$.*

(3) *There are three B exchanges of the form $(e_0, e_i')(e_i, e_j')(e_j, e_0')$ for some i, j, $j \ne i$, and $i, j \ne 0$ such that $w(e_0') + w(e_i') + w(e_j') < w(e_0) + w(e_i) + w(e_j)$.*

Moreover, if condition (1) *cannot be met, $B + \{e_i', e_j'\} - \{e_i, e_j\}$ is a base for every two B exchanges satisfying* (2)*; and if conditions* (1) *and* (2) *cannot be met, then $B + \{e_0', e_i', e_j'\} - \{e_0, e_i, e_j\}$ is a base for every three B exchanges satisfying* (3).

The complexity of the conditions stated in the primal theorem does not make it highly appealing as a computational tool. More appealing are the conditions of the dual theorem, stated next.

Theorem 4 (Dual Theorem). *Assume B is optimal for $\mathbf{P}(k_1, k_2)$ and assume (e_0, e_2'), (f_0, f_1'), and (f_1, f_2') are least weight B-exchanges of their respective types. (Possibly $e_0 = f_0$ or $e_2' = f_2'$.) Then,*

$$B + e_2' - e_0 \text{ is optimal for } \mathbf{P}(k_1, k_2 + 1)$$

if

$$w(e_2') - w(e_0) \le w(f_2') + w(f_1') - w(f_1) - w(f_0)$$

and otherwise

$$B+f_2'+f_1'-f_1-f_0 \text{ is optimal for } \mathbf{P}(k_1,k_2+1).$$

(If neither of the indicated candidates for an optimal $\mathbf{P}(k_1,k_2+1)$ *solution exists, then* $\mathbf{P}(k_1,k_2+1)$ *has no feasible solution.)*

Theorem 4 has several interesting features. Note first that $B+f_2'+f_1'-f_1-f_0$ is implied to be a base by the assertion that it is optimal for $\mathbf{P}(k_1,k_2+1)$, although no check is required to that effect. (Normally, the fact that (f_0,f_1') and (f_1,f_2') are both B exchanges does not imply that they can be carried out sequentially to yield a new base.) Also, it should be noted that the evaluation for selecting the three least weight B exchanges of their types involves looking only at a subject of the total B exchanges. In a graph theory setting, this corresponds to evaluating a subset of 'nonbasic edges'. The labeling procedure given in [8] for expediting such an evaluation for the problem $\mathbf{P}(k)$ is applicable in the present context as well. However, the evaluation is also automatically restricted in solving the sequence of problems $\mathbf{P}(k_1,k_2), \mathbf{P}(k_1,k_2+1), \mathbf{P}(k_1,k_2+2), \ldots,$ by noting that each element of S_2 added to B remains in B thereafter, and each element of S_0 dropped from B remains out of B thereafter. (The labeling scheme can be organized to further capitalize on this and other structural features of successive problems.)

The alternate statement of Theorem 4 that specifies an optimal solution to $\mathbf{P}(k_1,k_2-1)$ simply occurs by reversing the roles of the primed and unprimed elements (replacing (e_0,e_2') by (e_2,e_0'), etc.). Thus, the theorem may be viewed as a result for generating optimal solutions for adjacent parameter states, where exactly one of k_1 or k_2 differs by 1 unit from its current value.

In this view, Theorem 4 offers a variety of algorithmic possibilities for solving the problem $\mathbf{P}(k_1,k_2)$. For example, one can first solve the unconstrained problem (ignoring the cardinality constraints) by the greedy algorithm. Then k_1 and k_2 can alternately be held fixed while the other is moved toward its desired value by Theorem 4. One can choose to alternate more or less frequently.

Another approach is to solve $\mathbf{P}(k_1)$ (or $\mathbf{P}(k_2)$) by the quasi-greedy algorithm, (i.e. using Theorem 2), and then apply Theorem 4 to solve $\mathbf{P}(k_1,k_2)$. It may in fact be noted that the quasi-greedy algorithm is a special case of the more general 2-Quasi-Greedy Algorithm. In particular, Theorem 2 is the special case of Theorem 4 where S_1 is the empty set and $k_1=0$.

But it is possible to solve $\mathbf{P}(k_1,k_2)$ in an accelerated fashion if the desired k_1 and k_2 values lie respectively below and above their current values (or vice versa). This results from the following theorem for jumping over adjacent parameter states.

Theorem 5. *Assume B is optimal for* $\mathbf{P}(k_1,k_2)$ *and assume* (e_1,e_2'), (f_1,f_0'), (f_0,f_2') *are least weight B exchanges of their respective types. Then*

$$B+e_2'-e_1 \text{ is optimal for } \mathbf{P}(k_1-1,k_2+1)$$

if

$$w(e_2') - w(e_1) \le w(f_0') + w(f_2') - w(f_1) - w(f_0)$$

and otherwise

$$B + f_0' + f_2' - f_1 - f_0 \text{ is optimal for } \mathbf{P}(k_1 - 1, k_2 + 1).$$

The striking resemblance of Theorem 5 to Theorem 4 derives from the fact that it is actually an alternative statement of the same fundamental result, as we show subsequently. Observations concerning elements that stay permanently in or out of B when solving a succession of problems follow the same lines as those made earlier.

4. Justification of the method

In this section we provide a series of results that establish the theorems of Section 3. We begin with two results that were used in the derivation of the original quasi-greedy algorithm. The first has been known for many years (see, e.g., [17]), and is easily proved from fundamental principles.

Lemma 1. *Let B and B' be distinct bases. Then for each $e \in B - B'$ there is an $e' \in B' - B$ such that (e, e') is a B exchange and (e', e) is a B' exchange.*

Lemma 2. *If B, B' are bases and $E = B - B'$, $E' = B' - B$, then the elements of E may be put in one-one correspondence with those of E' so that each pair (e, e') defined by this correspondence is a B exchange.*

Two proofs of Lemma 2, each giving a different method of constructing the indicated correspondence, are given in [9]. (The correspondence may not be unique.) The next two lemmas generalize results of [8] to handle broader concerns of the present context.

Lemma 3. *Given a set B, $E \subset B$, $E' \cap B = \emptyset$, $f \in B$, $f' \notin B$. If both $B - E + E'$ and $B - f + f'$ are bases and $B - E + E' - f + f'$ is not a base, then:*

(1) *There is an element $e \in E$ such that $B + E' - (E - e + f)$ and $B + f' - e$ are both bases, and also*

(2) *There is an element $e' \in E'$ such that $B + (E' - e' + f') - E$ and $B + e' - f$ are both bases.*

Proof. If either f or f' is contained in either E or E', the problem is trivial, so assume the contrary. Since $f \in (B - E + E') - (B - f + f')$, by Lemma 1 there exists an element e in $(B - f + f') - (B - E + E')$ such that $B - E + E' - f + e = B + E' - (E - e + f)$ is a base and $B - f + f' - e + f = B + f' - e$ is a base. Since e cannot be f' (or $B - E + E' - f + f'$ would be a base), $e \in E$.

Likewise, since $f' \in (B-f+f')-(B-E+E')$ there must be an element $e' \in (B-E+E')-(B-f+f')$ such that $B-f+f'+e'-f'=B+e'-f$ and $B-E+E'-e'+f'=B+(E'-e'+f')-E$ are bases. Since e' cannot be f it must belong to E'. This completes the proof.

A useful special case of the foregoing is expressed in the following.

Corollary 1. *Given a set B, $e \in B$, $e' \notin B$, $f \in B$, $f' \notin B$. If $B-e+e'$ and $B-f+f'$ are both bases, but $B+e'+f'-e-f$ is not a base, then $B+e'-f$ and $B+f'-e$ are both bases.*

The next result is closely related to Lemma 3.

Lemma 4. *Let B be a base, $E \subset B$, $E' \cap B = \emptyset$, $f \in B$, $f' \notin B$. If not both $B+E'-E$ and $B+f'-f$ are bases, but $B'=B-E+E'-f+f'$ is a base, then*

(1) There is an element $e \in E$ such that $B+E'-(E-e+f)$ and $B+f'-e$ are both bases, and also

(2) There is an element $e' \in E'$ such that $B+(E'-e'+f')-E$ and $B+e'-f$ are both bases.

Proof. Again, if either f or f' is contained in either E or E', the problem is trivial, so assume the contrary. Since $f' \in B'-B$, by Lemma 1 there exists an element $e \in B-B'$ such that $B+f'-e$ and $(B-E+E'-f+f')-f'+e=B-(E-e+f)+E'$ are both bases. The fact that not both $B-f+f'$ and $B-E+E'$ are bases implies $e \in E$, given $e \neq f$.

Similarly, since $f \in B-B'$, there exists an element $e' \in B'-B$ such that $B-f+e'$ and $(B-E+E'-f+f')=B-E+(E'-e'+f')$ are bases. Since e' cannot be f', $e' \in E'$, completing the proof.

Lemma 4, like Lemma 3, has a useful special case.

Corollary 2. *Given a base B, $e \in B$, $e' \notin B$, $f \in B$, $f' \notin B$. If (e,e') and (f,f') are not both B exchanges, but $B+e'+f'-e-f$ is a base, then both (e,f') and (f,e') are B exchanges.*

We are now ready to establish our main results. We do this by proving Theorem 4, and then observe how the remaining results follow by analogous arguments.

Proof of Theorem 4. Let $E=B-B'$, $E'=B'-B$. By Lemma 2, E and E' may be put into a correspondence to yield a set Z of exchanges. Delete from Z the smaller weight collection (1) or (2) below. Choose (1) if the weights are the same.

(1) The least weight exchange in Z of the form (e_0, e_2').

(2) The least weight exchange in Z of the form (f_0, f_1') and the least weight exchange of the form (f_1, f_2').

In either case, the remaining set of exchanges Z' leaves the total number of elements in each of S_0, S_1, and S_2 constant. Any exchange in Z' of the form (e_i, e_i') cannot be improving, i.e., cannot yield $w(e_i') < w(e_i)$, or B would not have been optimal. Similarly, if there is a pair of exchanges in Z' of the form $(e_i, e_j'), (e_j, e_i')$ for which $w(e_i') + w(e_j') < w(e_i) + w(e_j)$, then $B + e_i' + e_j' - e_i - e_j$ cannot be a base. Exchanges (e_i, e_j') in Z' that can not thus be paired with an (e_j, e_i') exchange must be part of a set of three exchanges of the form $(e_i, e_j'), (e_j, e_k'), (e_k, e_i')$. Assume $w(e_i') + w(e_j') + w(e_k') < w(e_i) + w(e_j) + w(e_k)$. This implies that $B - e_i - e_j - e_k + e_i' + e_j' + e_k'$ is not a base. It also implies that $B - e_i - e_j + e_j' + e_k'$ must be a base, since if not Corollary 1 would imply that both (e_i, e_k') and (e_j, e_j') would be B exchanges. But (e_j, e_j') cannot be improving nor can the pair of exchanges $(e_i, e_k'), (e_k, e_i')$.

Thus, Lemma 2 implies that at least one of the following cases is true:

(C1) $B' - e_i - e_j + e_i' + e_j'$ and $B - e_k + e_k'$ are both bases,

(C2) $B - e_i - e_j + e_i' + e_k'$ and $B - e_k + e_j'$ are both bases. Since neither (e_k, e_k') nor the combination of (e_i, e_j') and (e_j, e_i') may be improving, case (C2) must be true. Since (e_j, e_k') is a B exchange by assumption, (e_k, e_j') and (e_j, e_k') may not together be improving. Thus (e_i, e_i') cannot be a B exchange. Corollary 2 thus implies that both (e_i, e_k') and (e_j, e_i') are B exchanges in addition to (e_k, e_j'). Pair (e_i, e_k') with (e_k, e_i'), (e_j, e_i') with (e_i, e_j'), and (e_k, e_j') with (e_j, e_k') to obtain three sets of paired exchanges none of which may yield an improvement, contradicting our assumption that $w(e_i') + w(e_j') + w(e_k') < w(e_i) + w(e_j) + w(e_k)$. Thus none of Z' may be improving.

Whenever (1) is true, $B - f_0 - f_1 + f_1' + f_2'$ is a base, since if it were not, then Corollary 1 would imply that (f_0, f_2') and (f_1, f_1') are both B exchanges. Since (f_1, f_1') cannot be improving, this would imply that (f_0, f_2') weights no more than (f_0, f_2') and (f_1, f_1') together. A similar argument implies that if (2) is true, then $B - f_0 - f_1 + f_1' + f_2'$ is a base.

Assume all exchanges in Z' together increase the objective function. If (1) is true, then $B - e_0 + e_2'$ must be an improvement over B'; if (2) is true, $B - f_0 - f_1 + f_1' + f_2'$ is an improvement over B'. In either case the optimality of B' is contradicted. Thus all exchanges in Z' together leave the weight constant and $w(B') = w(B - f_0 - f_1 + f_1' + f_2')$ or $w(B - e_0 + e_2')$, whichever is smaller. This completes the proof.

Theorem 5 may be derived from the preceding proof simply by exchanging the roles of S_0 and S_1. The primal result, Theorem 3, follows by noting that the arguments which insured the weight of all B exchanges in Z' are nonnegative are countered exactly by means of the stipulations of Theorem 3. The assertions about the exchanges that yield bases also follow from arguments analogous to those of the proof of Theorem 4 (employing Lemmas 3 and 4).

5. Conclusion

We have focussed in this paper on the fundamental results for a method that solves a 'two-parameter' generalization of the quasi-greedy algorithm. Our purpose has not been to identify the most efficient implementations or order bounds for specific settings, although the basic efficiency of the method is evident. The ability to solve adjacent problems solely by reference to simple exchanges available to the present base is the feature that makes such efficiency possible. At every step it is necessary to evaluate only a subset of available exchanges. In a vector space setting, for example, the method requires at most the amount of work to evaluate adjacent extreme points - analogous to the type of effort in applying the simplex method. But in contrast to the simplex method, the 2-quasi-greedy method applies a discrete (non-continuous) problem and includes 'non-adjacent moves', involving two exchanges to be performed in a block.

The question of the degree of further generalization naturally arises. The sequel [7] extends the present development to a broader problem class, and characterizes the structural features that establish the limits of generality.

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