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A Remark on the Modularity of Abelian Varieties of GL_2-type over $\mathbf{Q}$

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over the complex number field $\mathbf{C}$ implies that of A over $\mathbf{Q}$.

Key Words: abelian variety; GL_2-type; modularity.

1. INTRODUCTION AND A RESULT

Let A be an abelian variety of "GL_2-type over $\mathbf{Q}$." This means that A is an abelian variety defined over $\mathbf{Q}$ whose $\mathbf{Q}$-algebra of endomorphisms of A defined over $\mathbf{Q}$, denoted by $\mathrm{End}^0_{\mathbf{Q}}(A)$, is a number field E of degree equal to the dimension of A. From the results in [7], Ribet conjectured that any abelian variety of GL_2-type over $\mathbf{Q}$ is isogenous over $\mathbf{Q}$ to a $\mathbf{Q}$-simple factor of the jacobian variety $J_1(N)$ of the modular curve $X_1(N)$ for some integer $N \geqslant 1$, where a $\mathbf{Q}$-simple factor of $J_1(N)$ is a factor over $\mathbf{Q}$ which has no non-trivial abelian subvarieties defined over $\mathbf{Q}$. This conjecture is called the modularity conjecture and is a generalization of the Taniyama–Shimura conjecture on elliptic curves defined over $\mathbf{Q}$.

Shimura and Ribet gave the description of the $\mathbf{Q}$-simple factors of $J_1(N)$ in terms of cuspforms of weight two. More precisely, let $f = \sum_{n=1}^{\infty} a_n q^n$ ($q = e^{2\pi i z}$) be a normalized new form of weight two on $\Gamma_1(M)$, where M is a positive divisor of N. The Hecke ring $\mathbf{T} = \mathbf{T}_M$ is the subring of $\mathrm{End}_{\mathbf{Q}}(J_1(M))$ generated over $\mathbf{Z}$ by all Hecke operators T_n and all diamond automorphisms $\langle d \rangle$, where $\mathrm{End}_{\mathbf{Q}}(J_1(M))$ denotes the ring of endomorphisms of $J_1(M)$ defined over $\mathbf{Q}$ and n (resp. d) runs over the set of positive integers (resp. $(\mathbf{Z}/M\mathbf{Z})^{\times}$). Consider the homomorphism of rings $\lambda_f : \mathbf{T} \to \mathbf{C}$ such that $T_n \mapsto a_n$ and $\langle d \rangle \mapsto \varepsilon(d)$, where ε is the Nebentypus

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of f. Let $\mathbf{I}_f$ be the kernel of λ_f and J_f be the abelian variety over $\mathbf{Q}$ defined by $J_f = J_1(M)/\mathbf{I}_f J_1(M)$. Put $E_f = \mathbf{Q}(\{a_n \mid n \geqslant 1\})$. Then E_f is a number field and its degree is equal to the dimension of J_f. Moreover, the homomorphism of $\mathbf{Q}$-algebras $\theta : E_f \to \mathrm{End}^0_{\mathbf{Q}}(J_f)$ defined by $a_n \mapsto$ "the endomorphism of J_f induced by T_n" is an isomorphism. So J_f is a $\mathbf{Q}$-simple factor of $J_1(M)$, hence it is also a $\mathbf{Q}$-simple factor of $J_1(N)$ because of the canonical homomorphism from $J_1(M)$ to $J_1(N)$ is defined over $\mathbf{Q}$ and has a finite kernel. Conversely, any $\mathbf{Q}$-simple factor of $J_1(N)$ is isogenous over $\mathbf{Q}$ to J_f for some f as above.

We state some known results on the modularity conjecture. In the case of dimension one, Conrad *et al.* proved that any elliptic curves defined over $\mathbf{Q}$ whose conductor is not divided by 3^3 satisfy the modularity conjecture [1]. Recently a proof of the full Taniyama–Shimura conjecture was announced by Breuil, Conrad, Diamond and Taylor. In the case of higher dimension, Hasegawa *et al.* showed (by using the results of Taylor–Wiles–Diamond on the modularity on Galois representations) that for an abelian variety A of GL_2-type over $\mathbf{Q}$ without complex multiplication, if there exist an odd prime number p and a prime ideal of $\mathrm{End}^0_{\mathbf{Q}}(A)$ lying over p which satisfy some conditions, then the modularity conjecture for A is true [2].

In [5] Pyle gave necessary and sufficent conditions for an abelian variety defined over $\bar{\mathbf{Q}}$ to be a $\bar{\mathbf{Q}}$-simple factor of an abelian variety of GL_2-type over $\mathbf{Q}$, where $\bar{\mathbf{Q}}$ denotes a fixed algebraic closure of $\mathbf{Q}$. So if the modularity conjecture is true, then we can get a characterization of abelian varieties which are modular over $\bar{\mathbf{Q}}$.

In this paper, we will prove the following theorem:

THEOREM. *Let A be an abelian variety of GL_2-type over $\mathbf{Q}$ without complex multiplication. If there exists a non-zero homomorphism $\varphi: J_1(N) \to A$ defined over the complex number field $\mathbf{C}$ for some integer $N \geqslant 1$, then A is isogenous over $\mathbf{Q}$ to J_g for some normalized newform g of weight two on $\Gamma_1(M)$, where M is a suitable positive integer* (*which may be different from N*).

Here we say that an abelian variety A defined over $\bar{\mathbf{Q}}$ has complex multiplication, if A is isogenous over $\bar{\mathbf{Q}}$ to a product $A_1 \times \cdots \times A_s$ with abelian varieties A_i defined over $\bar{\mathbf{Q}}$ such that $\mathrm{End}^0_{\bar{\mathbf{Q}}}(A_i)$ is isomorphic to a CM-field of degree $2 \cdot \dim(A_i)$ for each i. This is so if and only if $\mathrm{End}^0_{\bar{\mathbf{Q}}}(A)$ contains a commutative semi-simple algebra of rank $2 \cdot \dim(A)$ over $\mathbf{Q}$ (see Section 5.1 in [9]). Shimura proved that if an abelian variety A of GL_2-type over $\mathbf{Q}$ has complex multiplication, then A is isogenous over $\bar{\mathbf{Q}}$ to a power of a CM elliptic curve (see Prop. 1.5 in [8]). So in this case, the structure of $\mathrm{End}^0_{\bar{\mathbf{Q}}}(A)$ is very simple. But the action of the absolute Galois group over $\mathbf{Q}$ on $\mathrm{End}^0_{\bar{\mathbf{Q}}}(A)$ is more complicated, because $\mathrm{End}^0_{\bar{\mathbf{Q}}}(A)$ is too big and therefore $\mathrm{End}^0_{\mathbf{Q}}(A)$ is not a maximal subfield of $\mathrm{End}^0_{\bar{\mathbf{Q}}}(A)$. Hence we exclude this case.

Finally, we remark that in the case where the dimension of A is one, this theorem is equivalent to the result of Mazur in [3] and some ideas of our proof can be seen in [3]. The essential new idea is to study the action of the absolute Galois group over $\mathbf{Q}$ on the full endomorphism algebra (see Section 2).

2. THE GALOIS ACTION ON THE FULL ENDOMORPHISM ALGEBRA

Let A be as in the theorem and n be the dimension of A. Put $E := \mathrm{End}^0_{\mathbf{Q}}(A)$. For any subfield k of $\bar{\mathbf{Q}}$, we denote by $\mathrm{End}_k(A)$ the ring of endomorphisms of A defined over k and put $\mathrm{End}^0_k(A) := \mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{End}_k(A)$. Pyle determines the structure of $\mathrm{End}^0_{\bar{\mathbf{Q}}}(A)$ as $\mathbf{Q}$-algebra in [5]: The center is a totally real subfield $F \subseteq E$ and

$$\mathrm{End}^0_{\bar{\mathbf{Q}}}(A) \simeq \mathrm{M}_m(D),$$

where D is F or a division quaternion algebra over F; $\mathrm{End}^0_{\bar{\mathbf{Q}}}(A)$ contains E as a maximal subfield, i.e.,

$$[E:F] = \sqrt{\dim_F \mathrm{M}_m(D)} = mt, \; t = \begin{cases} 1 & \text{if} \quad D = F, \\ 2 & \text{if} \quad \text{otherwise.} \end{cases}$$

We fix an isomorphism $i\colon \mathrm{M}_m(D) \to \mathrm{End}^0_{\bar{\mathbf{Q}}}(A)$ and denote by the same notation E the inverse image $i^{-1}(\mathrm{End}^0_{\mathbf{Q}}(A))$. The absolute Galois group $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ over $\mathbf{Q}$ acts on $\mathrm{End}^0_{\bar{\mathbf{Q}}}(A)$ by the action on coefficients of endomorphisms. Hence for every element σ of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, there exists a unique isomorphism of F-algebras $\eta_\sigma\colon \mathrm{M}_m(D) \to \mathrm{M}_m(D)$ such that the following diagram commutes:

$$\begin{array}{ccc} \varphi & \longmapsto & {}^{\sigma}\varphi \\ \cap & & \cap \\ \mathrm{End}^0_{\bar{\mathbf{Q}}}(A) & \longrightarrow & \mathrm{End}^0_{\bar{\mathbf{Q}}}(A) \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} \\ \mathrm{M}_m(D) & \xrightarrow[\eta_\sigma]{} & \mathrm{M}_m(D) \end{array}$$

By the Noether–Skolem Theorem and the facts that $\eta_\sigma(x) = x$ for all $x \in E$ and E is a maximal subfield of $\mathrm{M}_m(D)$, there exists a non-zero element $\alpha(\sigma)$ of E such that

$$\eta_\sigma(x) = \alpha(\sigma)^{-1} x \alpha(\sigma) \qquad \text{for all} \quad x \in \mathrm{M}_m(D),$$

where $\alpha(\sigma)$ is uniquely determined up to a multiple of non-zero elements of F. The following two propositions are shown in [5]:

PROPOSITION 2.1. *The field E is generated over F by the $\alpha(\sigma)$ for all $\sigma \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$.*

PROPOSITION 2.2. *The field E is an abelian Galois extension of F.*

We define a homomorphism $\tilde{\alpha}\colon \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to E^{\times}/F^{\times}$ by $\sigma \mapsto \alpha(\sigma) \bmod F^{\times}$. We denote by K the fixed field of the kernel of $\tilde{\alpha}$. Then K is the smallest field such that $\mathrm{End}_K^0(A) = \mathrm{End}_{\mathbf{Q}}^0(A)$. By the theory of simple algebras, we can take an E-basis $\{a_\tau\}_{\tau \in G_1}$ of $\mathrm{M}_m(D)$, where $G_1 := \mathrm{Gal}(E/F)$, such that $a_e = 1$ and every a_τ satisfies the following relations:

$$a_\tau x = \tau(x)\, a_\tau \qquad \text{for all} \quad x \in E$$

(see Lemma (i), (ii) in [4, p. 251]). For every element τ of G_1, we define a homomorphism $\beta_\tau : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to E^{\times}$ by

$$\sigma \mapsto \frac{\tau(\alpha(\sigma))}{\alpha(\sigma)}.$$

The following lemma can be easily proved:

LEMMA 2.3. *For every element τ of G_1, we have*

$${}^{\sigma}i(a_\tau) = i(\beta_\tau(\sigma)) \circ i(a_\tau) \qquad \text{for all} \quad \sigma \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}).$$

By this lemma, we can fully understand how $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ acts on $\mathrm{End}_{\mathbf{Q}}^0(A)$.

3. THE PROOF OF THE THEOREM

Let the notation be as in Section 2. We suppose that there exists a non-zero homomorphism $\varphi\colon J_1(N) \to A$ defined over $\mathbf{C}$ for some integer $N \geqslant 1$. Since $J_1(N)$ and A are defined over $\mathbf{Q}$, φ is defined over $\bar{\mathbf{Q}}$. So we may assume that φ is defined over a subfield L of $\bar{\mathbf{Q}}$ such that $L/\mathbf{Q}$ is a finite Galois extension and L contains K.

By considering the Weil restriction from L to $\mathbf{Q}$ of φ, we get the homomorphism

$$\Phi\colon J_1(N) \to R_{L/\mathbf{Q}}(A_{/L})$$

defined over $\mathbf{Q}$, where $R_{L/\mathbf{Q}}(A_{/L})$ is the Weil restriction from L to $\mathbf{Q}$ of $A_{/L}$. So $R_{L/\mathbf{Q}}(A_{/L})$ is an abelian variety defined over $\mathbf{Q}$ and it is isomorphic over L to $A^{[L:\mathbf{Q}]}$, where we write $A^r = A \times \cdots \times A$ (r terms). Since Φ is not a zero map, we can take a non-zero $\mathbf{Q}$-simple factor C of $\mathrm{Im}(\Phi)_{/\mathbf{Q}}$ and fix it. Then there exists a new form $f = \sum_{n=1}^{\infty} a_n q^n$ whose level divides N such that J_f is isogenous over $\mathbf{Q}$ to C, that is expressed by $J_f \sim_{\mathbf{Q}} C$. We put $H := \mathbf{Q}(\{a_n \mid n \geqslant 1\})$. By the Shimura–Ribet theory explained in Section 1, we have the canonical isomorphism $\theta\colon H \to \mathrm{End}^0_{\mathbf{Q}}(J_f)$.

Put $M := \mathrm{Hom}_{\bar{\mathbf{Q}}}(A, J_f) \otimes_{\mathbf{Z}} \mathbf{Q}$, where $\mathrm{Hom}_{\bar{\mathbf{Q}}}(A, J_f)$ denotes the additive group of homomorphisms from A to J_f defined over $\bar{\mathbf{Q}}$. Then $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ acts on M as well as the case of $\mathrm{End}^0_{\bar{\mathbf{Q}}}(A)$. Moreover, M has the structure of a left H- and right $\mathrm{M}_m(D)$-module by considering a composition of homomorphisms. Then the action of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ on M is H-linear.

LEMMA 3.1. *We have* $\dim_H M = [E:F] = mt$.

Proof. Let s be the dimension of C. Since $\mathrm{End}^0_{\bar{\mathbf{Q}}}(A) = \mathrm{End}^0_K(A) \cong \mathrm{M}_m(D)$, we have

$$A \sim_K B \times \cdots \times B \qquad (m \text{ terms}),$$

where B is a $\bar{\mathbf{Q}}$-simple abelian variety defined over K such that $\mathrm{End}^0_{\bar{\mathbf{Q}}}(B) = \mathrm{End}^0_K(B) \cong D$. Since C is a $\mathbf{Q}$-factor of $R_{L/\mathbf{Q}}(A_{/L})$ and $R_{L/\mathbf{Q}}(A_{/L})$ is isomorphic over L to $A^{[L:\mathbf{Q}]}$, there exists a positive integer r such that $C \sim_L B^r$. By comparing the dimensions, we have $s = \frac{rn}{m}$. Since $J_f^m \sim_L A^r$, it follows that

$$\begin{aligned} M^{\oplus m} &\cong \mathrm{Hom}_{\bar{\mathbf{Q}}}(A, J_f^m) \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathrm{Hom}_{\bar{\mathbf{Q}}}(A, A^r) \otimes_{\mathbf{Z}} \mathbf{Q} \\ &\cong \mathrm{M}_m(D)^{\oplus r} \end{aligned}$$

as $\mathbf{Q}$-vector space. So we have

$$\begin{aligned} m \dim_{\mathbf{Q}} M = r \dim_{\mathbf{Q}} \mathrm{M}_m(D) &= r\,[E:F]^2\,[F:\mathbf{Q}] \\ &= r\,[E:\mathbf{Q}]\,[E:F] \\ &= s\,m\,[E:F]. \end{aligned}$$

Hence we obtain $\dim_{\mathbf{Q}} M = s\,[E:F]$. Since $[H:\mathbf{Q}] = s$, we get the assertion. ∎

Let ℓ be a prime number. We denote by $T_\ell(A)$ the Tate module of A and put $V_\ell(A) := T_\ell(A) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$. Now we consider the module $M \otimes_{\mathrm{M}_m(D)} V_\ell(A)$ on which $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ acts by diagonal and H acts by the action on M. We define a homomorphism

$$v\colon M \otimes_{\mathrm{M}_m(D)} V_\ell(A) \to V_\ell(J_f), \qquad \eta \otimes x \mapsto \eta(x).$$

PROPOSITION 3.2. *v is an isomorphism of (left) $H \otimes_{\mathbf{Q}} \mathbf{Q}_\ell[\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})]$-modules, where $H \otimes_{\mathbf{Q}} \mathbf{Q}_\ell[\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})]$ denotes the group algebra of* $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ *over $H \otimes_{\mathbf{Q}} \mathbf{Q}_\ell$.*

Proof. It is clear that v is a homomorphism of $H \otimes_{\mathbf{Q}} \mathbf{Q}_\ell[\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})]$-modules. So we will prove that v is bijective. We consider

$$\begin{array}{llcl} v^{\oplus m}\colon & (M \otimes_{\mathbf{M}_m(D)} V_\ell(A))^{\oplus m} & \longrightarrow & V_\ell(J_f)^{\oplus m} \\ & \wr\| & & \wr\| \\ & (M^{\oplus m}) \otimes_{\mathbf{M}_m(D)} V_\ell(A) & & V_\ell(J_f^m). \\ & \wr\| & & \\ & (\mathrm{Hom}_{\bar{\mathbf{Q}}}(A, J_f^m) \otimes_{\mathbf{Z}} \mathbf{Q}) \otimes_{\mathbf{M}_m(D)} V_\ell(A) & & \end{array}$$

Since $J_f^m \sim_L A^r$, there exists an isogeny $\psi : J_f^m \to A^r$ defined over L. Then,

$$\psi^* : \mathrm{Hom}_{\bar{\mathbf{Q}}}(A, J_f^m) \otimes_{\mathbf{Z}} \mathbf{Q} \to \mathrm{Hom}_{\bar{\mathbf{Q}}}(A, A^r) \otimes_{\mathbf{Z}} \mathbf{Q}, \qquad \eta \otimes a \mapsto \psi \circ \eta \otimes a,$$

is an isomorphism of right $\mathrm{M}_m(D)$-modules. So we have the commutative diagram:

$$\begin{array}{ccc} (\mathrm{Hom}_{\bar{\mathbf{Q}}}(A, J_f^m) \otimes_{\mathbf{Z}} \mathbf{Q}) \otimes_{\mathbf{M}_m(D)} V_\ell(A) & \xrightarrow{v^{\oplus m}} & V_\ell(J_f^m) \\ \downarrow{\scriptstyle \psi^* \otimes 1} & & \downarrow{\scriptstyle \psi} \\ (\mathrm{Hom}_{\bar{\mathbf{Q}}}(A, A^r) \otimes_{\mathbf{Z}} \mathbf{Q}) \otimes_{\mathbf{M}_m(D)} V_\ell(A) & \xrightarrow[\widetilde{v^{\oplus m}}]{} & V_\ell(A^r), \end{array}$$

where $\widetilde{v^{\oplus m}}$ is a $\mathbf{Q}_\ell$-linear map defined by $\eta' \otimes x' \mapsto \eta'(x')$ ($\eta' \in \mathrm{Hom}_{\bar{\mathbf{Q}}}(A, A^r) \otimes_{\mathbf{Z}} \mathbf{Q}$, $x' \in V_\ell(A)$) and the vertical maps are isomorphisms of $\mathbf{Q}_\ell$-vector spaces.

For $1 \leqslant i \leqslant r$, we put

$$q_i\colon A \to A^r = A \times \cdots \times A\,,\ x \mapsto (0, ..., 0, \underset{\hat{i}}{x}, 0, ..., 0)\,.$$

Then any element y of $V_\ell(A^r)$ can be written uniquely in the form $y = \sum_{i=1}^r q_i(x_i)$ ($x_i \in V_\ell(A)$). So we have

$$\widetilde{v^{\oplus m}}\left(\sum_{i=1}^r q_i \otimes x_i\right) = y.$$

Therefore, $\widetilde{v^{\oplus m}}$ is surjective. So $\widetilde{v^{\oplus m}}$ is bijective because of the equality of the dimensions over $\mathbf{Q}_\ell$. Hence v is bijective. ∎

Put $\bar{M} := \bar{H} \otimes_H M$. We give $\bar{M}$ the structure of a $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$-module by the action on M. Next we will study how $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ acts on $\bar{M}$. Since M

is a left H- and right $\mathrm{M}_m(D)$-module, $\bar{M}$ is a left $\bar{H}$- and right $\mathrm{M}_m(D)$-module. In particular, $E\,(\subseteq \mathrm{M}_m(D))$ acts $\bar{H}$-linearly on $\bar{M}$ on the right. Since E is commutative, this action corresponds to a homomorphism of **Q**-algebras

$$j : E \to \mathrm{M}_{mt}(\bar{H})$$

by taking a $\bar{H}$-basis of $\bar{M}$. E is generated over **Q** by some single element a as a **Q**-algebra. Since the minimal polynomial of $j(a)$ divides the minimal polynomial of a over **Q**, the minimal polynomial of $j(a)$ has no multiple roots. So $j(a)$ is diagonalizable. Therefore j is equivalent to a direct sum of mt isomorphisms of E into $\bar{H}$. Take an isomorphism $\imath$ which appears in this sum and hereafter we see E as a subfield of $\bar{H}$ by $\imath$. Then we can determine the other isomorphisms appearing in this sum:

LEMMA 3.3. *j is equivalent to $\sum_{\tau \in G_1} \tau$.*

Proof. We can take a element $\eta\ (\neq 0) \in \bar{M}$ such that $\eta \circ i(x) = x\,\eta$ for all $x \in E$. Then for any $x \in E$ and any $\tau \in G_1$, we have

$$\begin{aligned}(\eta \circ i(a_\tau)) \circ i(x) = \eta \circ i(a_\tau x) &= \eta \circ i(\tau(x)\, a_\tau)\\ &= (\eta \circ i(\tau(x))) \circ i(a_\tau)\\ &= \tau(x)\,\eta \circ i(a_\tau).\end{aligned} \tag{3.3}$$

So the isomorphism $\tau: E \to E \subseteq \bar{H}$ appears in the direct sum. Since $|G_1| = mt$, we have $j \cong \sum_{\tau \in G_1} \tau$. ∎

We put $\eta_\tau := \eta \circ i(a_\tau)$ for any $\tau \in G_1$. Then $\{\eta_\tau\}_{\tau \in G_1}$ is a $\bar{H}$-basis of $\bar{M}$.

LEMMA 3.4. *There exists a Dirichlet character*

$$\chi: \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \bar{H}^\times$$

such that ${}^\sigma\eta = \chi(\sigma)\,\eta$ for all $\sigma \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$.

Proof. For any $x \in E$, we have

$${}^\sigma\eta \circ i(x) = {}^\sigma\eta \circ {}^\sigma i(x) = {}^\sigma(\eta \circ i(x)) = {}^\sigma(x\,\eta) = x\,{}^\sigma\eta.$$

So ${}^\sigma\eta$ must be a scalar multiple of η. Hence the assertion holds. ∎

LEMMA 3.5. *For any $\tau \in G_1$, we have ${}^\sigma\eta_\tau = \chi(\sigma)\,\beta_\tau(\sigma)\,\eta_\tau$ $(\forall \sigma \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}))$.*

Proof. We have

$$
\begin{aligned}
{}^{\sigma}\eta_{\tau} = {}^{\sigma}(\eta \circ i(a_{\tau})) = {}^{\sigma}\eta \circ {}^{\sigma}i(a_{\tau}) &= \chi(\sigma)\, \eta \circ i(\beta_{\tau}(\sigma)) \circ i(a_{\tau}) \\
&= \chi(\sigma)\, \beta_{\tau}(\sigma)\, \eta \circ i(a_{\tau}) \\
&= \chi(\sigma)\, \beta_{\tau}(\sigma)\, \eta_{\tau}.
\end{aligned}
$$

Hence we get the assertion. ∎

We denote by $\bar{H}(\chi^{-1})$ the (left) $\bar{H}[\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})]$-module which is isomorphic to $\bar{H}$ as $\bar{H}$-module and on which $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ acts by $\sigma x = \chi(\sigma)^{-1} x$ for all $\sigma \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ and $x \in \bar{H}$. We define an isomorphism of $\bar{H}$-vector spaces

$$
\rho \colon \bar{H}(\chi^{-1}) \otimes_{\bar{H}} \bar{M} \to \bar{H} \otimes_{E} \mathrm{M}_m(D), \qquad \sum_{\tau \in G_1} b_{\tau} \otimes \eta_{\tau} \mapsto \sum_{\tau \in G_1} b_{\tau} \otimes a_{\tau}.
$$

Proposition 3.6. *ρ is a homomorphism of* $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$*-modules.*

Proof. For any $\sigma \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, we have

$$
\begin{aligned}
\rho\left({}^{\sigma}\left(\sum_{\tau \in G_1} b_{\tau} \otimes \eta_{\tau}\right)\right) &= \rho\left(\sum_{\tau \in G_1} (\chi(\sigma)^{-1} b_{\tau}) \otimes {}^{\sigma}\eta_{\tau}\right) \\
&= \rho\left(\sum_{\tau \in G_1} (\chi(\sigma)^{-1} b_{\tau}) \otimes (\chi(\sigma)\, \beta_{\tau}(\sigma)\, \eta_{\tau})\right) \\
&= \rho\left(\sum_{\tau \in G_1} (b_{\tau}\beta_{\tau}(\sigma)) \otimes \eta_{\tau}\right) \\
&= \sum_{\tau \in G_1} (b_{\tau}\beta_{\tau}(\sigma)) \otimes a_{\tau} \\
&= \sum_{\tau \in G_1} b_{\tau} \otimes (\beta_{\tau}(\sigma)\, a_{\tau}) \\
&= \sum_{\tau \in G_1} b_{\tau} \otimes {}^{\sigma}a_{\tau} \ \text{(because of } {}^{\sigma}i(a_{\tau}) = i(\beta_{\tau}(\sigma)\, a_{\tau})\text{)} \\
&= {}^{\sigma}\left(\sum_{\tau \in G_1} b_{\tau} \otimes a_{\tau}\right) \\
&= {}^{\sigma}\rho\left(\sum_{\tau \in G_1} b_{\tau} \otimes \eta_{\tau}\right).
\end{aligned}
$$

So we get the assertion.

Proposition 3.7. *ρ is a homomorphism of right* $\mathrm{M}_m(D)$*-modules.*

Proof. For any $x \in E$, we have

$$\begin{aligned}\rho\left(\left(\sum_{\tau\in G_1} b_\tau \otimes \eta_\tau\right)\cdot x\right) &= \rho\left(\sum_{\tau\in G_1} b_\tau \otimes (\eta_\tau \circ i(x))\right)\\ &= \rho\left(\sum_{\tau\in G_1} (b_\tau \tau(x)) \otimes \eta_\tau\right)\\ &= \sum_{\tau\in G_1} (b_\tau \tau(x)) \otimes a_\tau\\ &= \sum_{\tau\in G_1} b_\tau \otimes (\tau(x)\, a_\tau)\\ &= \sum_{\tau\in G_1} b_\tau \otimes (a_\tau x)\\ &= \left(\sum_{\tau\in G_1} b_\tau \otimes a_\tau\right)\cdot x = \rho\left(\sum_{\tau\in G_1} b_\tau \otimes \eta_\tau\right)\cdot x.\end{aligned}$$

Now we remark that for any $\tau, \tau' \in G_1$, there exists a unique $c(\tau, \tau') \in E^\times$ such that $a_\tau a_{\tau'} = c(\tau, \tau')\, a_{\tau\tau'}$. Then for any $\tau' \in G_1$, we have

$$\begin{aligned}\rho\left(\left(\sum_{\tau\in G_1} b_\tau \otimes \eta_\tau\right)\cdot a_{\tau'}\right) &= \rho\left(\sum_{\tau\in G_1} b_\tau \otimes (\eta_\tau \circ i(a_{\tau'}))\right)\\ &= \rho\left(\sum_{\tau\in G_1} b_\tau \otimes (\eta \circ i(a_\tau) \circ i(a_{\tau'}))\right)\\ &= \rho\left(\sum_{\tau\in G_1} b_\tau \otimes (\eta \circ i(c(\tau, \tau')\, a_{\tau\tau'}))\right)\\ &= \rho\left(\sum_{\tau\in G_1} b_\tau \otimes (c(\tau, \tau')\, \eta_{\tau\tau'})\right)\\ &= \rho\left(\sum_{\tau\in G_1} (b_\tau c(\tau, \tau')) \otimes \eta_{\tau\tau'}\right)\\ &= \sum_{\tau\in G_1} (b_\tau c(\tau, \tau')) \otimes a_{\tau\tau'}\\ &= \sum_{\tau\in G_1} b_\tau \otimes (c(\tau, \tau')\, a_{\tau\tau'})\\ &= \sum_{\tau\in G_1} b_\tau \otimes (a_\tau a_{\tau'})\\ &= \left(\sum_{\tau\in G_1} b_\tau \otimes a_\tau\right)\cdot a_{\tau'} = \rho\left(\sum_{\tau\in G_1} b_\tau \otimes \eta_\tau\right)\cdot a_{\tau'}\,.\end{aligned}$$

Since $\mathrm{M}_m(D) = \bigoplus_{\tau\in G_1} E\, a_\tau$, we obtain the assertion. ∎

PROPOSITION 3.8. *We have*

$$\bar{H} \otimes_E V_\ell(A) \cong \bar{H}(\chi^{-1}) \otimes_H V_\ell(J_f)$$

as a (left) $\bar{H} \otimes_{\mathbf{Q}} \mathbf{Q}_\ell[\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})]$*-module.*

Proof. By Proposition 3.2, $M \otimes_{\mathrm{M}_m(D)} V_\ell(A) \cong V_\ell(J_f)$. So we have

$$\begin{array}{c} \bar{H} \otimes_H (M \otimes_{\mathrm{M}_m(D)} V_\ell(A)) \cong \bar{H} \otimes_H V_\ell(J_f). \\ \wr\| \\ (\bar{H} \otimes_H M) \otimes_{\mathrm{M}_m(D)} V_\ell(A) \\ \| \\ \bar{M} \otimes_{\mathrm{M}_m(D)} V_\ell(A) \end{array}$$

By considering the tensor product with $\bar{H}(\chi^{-1})$ over $\bar{H}$, we get

$$\bar{H}(\chi^{-1}) \otimes_{\bar{H}} (\bar{M} \otimes_{\mathrm{M}_m(D)} V_\ell(A)) \cong \bar{H}(\chi^{-1}) \otimes_{\bar{H}} (\bar{H} \otimes_H V_\ell(J_f)).$$

From Propositions 3.6 and 3.7, the left-hand side is isomorphic to

$$\begin{aligned} (\bar{H}(\chi^{-1}) \otimes_{\bar{H}} \bar{M}) \otimes_{\mathrm{M}_m(D)} V_\ell(A) &\cong (\bar{H} \otimes_E \mathrm{M}_m(D)) \otimes_{\mathrm{M}_m(D)} V_\ell(A) \\ &\cong \bar{H} \otimes_E (\mathrm{M}_m(D) \otimes_{\mathrm{M}_m(D)} V_\ell(A)) \\ &\cong \bar{H} \otimes_E V_\ell(A). \end{aligned}$$

On the other hand, the right hand side is isomorphic to $\bar{H}(\chi^{-1}) \otimes_H V_\ell(J_f)$. Hence the assertion is proved. ∎

Take any prime number p satisfying the conditions: (i) A has good reduction at p; (ii) $(p, \ell N_1) = 1$, where N_1 is the level of f; (iii) p does not divide the conductor of χ. Let $\sigma_p \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ be a Frobenius element at p. Since $V_\ell(A)$ is free of rank 2 over $E \otimes_{\mathbf{Q}} \mathbf{Q}_\ell$, we can consider the trace (resp. determinant) of σ_p acting on $V_\ell(A)$ over $E \otimes_{\mathbf{Q}} \mathbf{Q}_\ell$. By Proposition 3.8, the trace (resp. determinant) of σ_p over $E \otimes_{\mathbf{Q}} \mathbf{Q}_\ell$ is equal to $\chi(\sigma_p)^{-1} a_p$ (resp. $\chi(\sigma_p)^{-2} \varepsilon(p)\, p$), where $f = \sum_{n=1}^{\infty} a_n q^n$ and ε is the Nebentypus of f. By the theory of twists of modular forms ([6]), there is a unique newform $g = \sum_{n=1}^{\infty} b_n q^n$ such that $b_p = \chi^{-1}(\sigma_p) a_p$ for all p satisfying the above condition (iii). It is also known that the Nebentypus of g coincides with $\chi^{-2}\varepsilon$. Hence we see that the characteristic polynomial of σ_p acting on $V_\ell(A)$ over $E \otimes_{\mathbf{Q}} \mathbf{Q}_\ell$ is equal to that of σ_p acting on $V_\ell(J_g)$ over $E \otimes_{\mathbf{Q}} \mathbf{Q}_\ell$. By Isogeny Theorem, this shows that $A \sim_{\mathbf{Q}} J_g$. We have finished the proof of the theorem.

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