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Rigidity, global rigidity, and graph decomposition

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ABSTRACT

The recent combinatorial characterization of generic global rigidity in the plane by Jackson and Jordán (2005) [10] recalls the vital relationship between connectivity and rigidity that was first pointed out by Lovász and Yemini (1982) [13]. The Lovász–Yemini result states that every 6-connected graph is generically rigid in the plane, while the Jackson–Jordán result states that a graph is generically globally rigid in the plane if and only if it is 3-connected and edge-2-rigid.

We examine the interplay between the connectivity properties of the connectivity matroid and the rigidity matroid of a graph and derive a number of structure theorems in this setting, some well known, some new. As a by-product we show that the class of generic rigidity matroids is not closed under 2-sum decomposition. Finally we define the configuration index of the graph and show how the structure theorems can be used to compute it.

1. Introduction

Given a graph G together with an embedding of its vertices in the plane, for each edge e we consider the distance between the endpoints of e. These distances, together with the graph structure, are not usually enough to ensure a unique rendering in the plane up to congruence. In fact, for given generic edge lengths, even a rigid graph may have an exponential number of realizations as a rigid framework [3]. A natural question to ask is under what conditions does the set of distances together with the combinatorial information of G always produce congruent embeddings in the plane. Jackson and Jordán [10] showed that for generic embeddings this is the case exactly if G is edge-2-rigid and 3-connected. Should G lack one or both of these properties, we would like to describe the structure of G in terms of building blocks that do.

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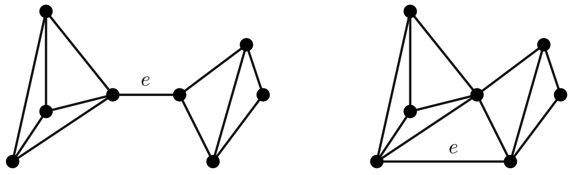

Fig. 1. A c-bridge and an r-bridge.

We first examine how decompositions of a graph induced by connectivity and by rigidity interact with each other. Finally, using Tutte's [17] canonical decomposition of a 2-connected graph into 3-blocks (see also [5]) and its generalization to matroids [4], we provide a canonical decomposition of an edge-2-rigid graph into globally rigid blocks.

1.1. Connectivity of graphs

We recall certain well-known definitions and theorems, mainly to stress the parallelism between various notions of connectivity among which graph connectivity is the most familiar.

A graph G is *connected* if it possesses a spanning tree, i.e. a subgraph $T = (V, E)$ such that $|E| = |V| - 1$ and

$$|F| \leq |V(F)| - 1 \quad \text{for all } F \subseteq E. \tag{1}$$

Edge sets all of whose subsets satisfy inequality (1) are called *c-independent*. Here the prefix c indicates that this notion of independence is related to connectivity. A *c-circuit*, also called a *cycle*, is a minimally c-dependent edge set and if a cycle consists of n edges, it spans n vertices. Note that we usually do not distinguish between the edge sets and the subgraphs they induce.

A graph is 2-*connected* if for any two of its edges a and b there is a cycle C containing both a and b. Note that this is equivalent to the standard definition of (vertex) 2-connectivity, namely that G remains connected after removal of any one of its vertices. An edge which is not contained in any cycle of G is called a *c-bridge* of G; see Fig. 1.

A connected graph is *edge-2-connected* if it remains connected after the removal of any edge. An edge-2-connected graph contains no bridges, i.e. every edge is contained in some cycle. The notion of edge-2-connectivity is weaker than 2-connectivity.

The *c-rank* of an edge set E is the cardinality of a maximal independent subset of E. The c-rank of a connected graph $G = (V, E)$ equals $|V| - 1$ and the c-rank of a disconnected graph equals the sum of the c-ranks of its connected components (maximal connected subgraphs). It is also equal to the number of bridges in G plus the sum over the ranks of the 2-connected components (maximal 2-connected subgraphs) of G.

There are easy inductive constructions to generate all 2-connected graphs, namely G is 2-connected if and only if it can be built up from a cycle by sequentially adjoining edges (loops are not allowed) and subdividing edges. A graph is edge-2-connected if and only if it can be built up from a vertex by adding edges (loops are allowed) and subdividing edges; see [6].

1.2. Rigidity of graphs

A graph G is generically *rigid* in the plane if and only if it possesses a spanning *isostatic* subgraph, i.e. a subgraph $L = (V, E)$ such that $|E| = 2|V| - 3$ and we have

$$|F| \leq 2|V(F)| - 3 \quad \text{for all } F \subseteq E. \tag{2}$$

The inequalities (2) are called *Laman's condition*, and sets of edges E which satisfy it are called *r-independent*. Here the prefix r indicates that this notion of independence is related to rigidity. An *r-circuit* is a minimally r-dependent edge set and if an r-circuit consists of $2n - 2$ edges, it spans n vertices; see Fig. 2.

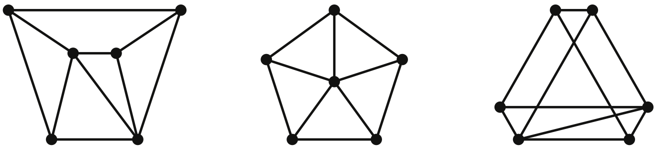

Fig. 2. Some r-circuits on six vertices.

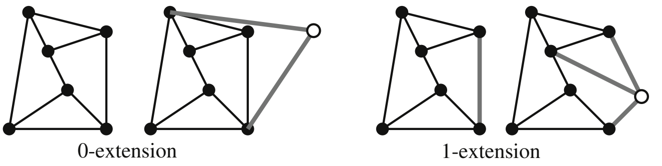

Fig. 3.

A graph G is 2-*rigid* if for any two of its edges a and b there is an r-circuit C containing both a and b. Note that this is *not* equivalent to requiring that G remains rigid after the removal of any one of its vertices. For example the removal of a vertex of degree larger than three will destroy rigidity of an r-circuit which is easily checked for the examples in Fig. 2. Vertex 2-rigidity was developed in [16]. An edge which is not contained in any r-circuit of G is called a r-*bridge* of G. A graph G is called *edge-*2-*rigid* (also called *redundantly rigid* in [10]), if it is rigid and every one of its edges is contained in an r-circuit, or, equivalently, if it contains no r-bridge.

The r-*rank* of an edge set E is the cardinality of a maximal r-independent subset of E. The r-rank of a rigid graph $G = (V, E)$ equals $2|V| - 3$ and the r-rank of a non-rigid graph equals the sum of the r-ranks of its rigid components (maximal rigid subgraphs). It is also equal to the number of r-bridges in G plus the sum over the ranks of the 2-rigid components (maximal 2-rigid subgraphs) of G.

There are several well-known inductive procedures to generate rigid graphs. A graph is rigid if and only if it can be obtained from an edge by a sequence of so called *Henneberg moves* [8,7]; see Fig. 3, or edge addition (no loops allowed). A graph is 2-rigid if and only if it can be obtained from tetrahedra by a sequence of 1-extension, edge addition and 2-sum [1,10]. We describe the 2-sum in Section 3.2.

2. Matroids on graphs

A matroid $\mathfrak{M}(E, \mathcal{I})$ is a finite set E, the *ground set*, together with a collection $\mathcal{I}$ of subsets of E, called *independent sets*, such that the following three axioms are satisfied:

(I1) $\emptyset \in \mathcal{I}$.
(I2) If $E_1 \in \mathcal{I}$ and $E_2 \subseteq E_1$, then $E_2 \in \mathcal{I}$.
(I3) If E and F are members of $\mathcal{I}$ with $|E| = |F| + 1$, then there exists $e \in E \setminus F$ such that $F \bigcup e \in \mathcal{I}$.

A subset of E not belonging to $\mathcal{I}$ is called *dependent*.

The *rank function*, $\rho : 2^E \mapsto \mathbb{Z}$, is defined for $X \subseteq E$ by

$$\rho(X) = \max\left(|F| : F \subseteq X, F \in \mathcal{I}\right).$$

The rank of the matroid $\mathfrak{M}$ is the rank of the set E. A *base* of $\mathfrak{M}$ is a maximal independent subset of E. A *circuit* of $\mathfrak{M}$ is a minimal dependent subset of E, and a *bridge* of $\mathfrak{M}$ is an element that belongs to every base of $\mathfrak{M}$.

We are studying two matroids on the edge set E of a graph $G = (V, E)$, namely the *cycle matroid*, $\mathfrak{C}(G)$, defined by c-independent sets, whose circuits are cycles, see [19], and the (two-dimensional generic) *rigidity matroid*, $\mathfrak{R}(G)$, defined by r-independent edge sets, see [7].

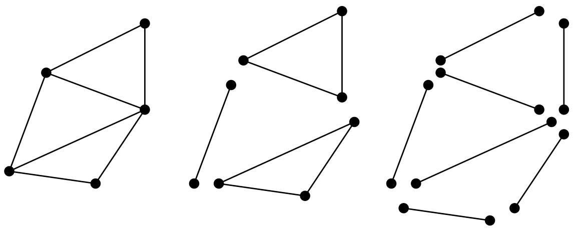

Fig. 4. Covers minimizing $\sum_i(2n_i - 3)$.

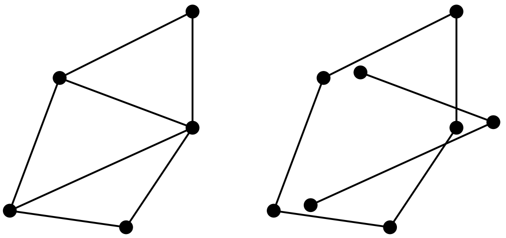

Fig. 5. $\sum_i(2n_i - 3) > \rho(E)$.

2.1. The connectivity matroid $\mathfrak{C}(G)$

If $G = (V, E)$ is a connected graph, the bases of $\mathfrak{C}(G)$ are the spanning trees of G. For any subset F of the edge set E, the rank of F in $\mathfrak{C}(G)$ is given by

$$\rho(F) = |V(F)| - c(F),$$

where $c(F)$ denotes the number of connected components of the subgraph of G induced by F. We stress that $\mathfrak{C}(G)$ is defined on the edge set E and the vertices are only indirectly used (via cardinalities of sets spanned by the edges) in the definition of $\mathfrak{C}(G)$. In fact, there are non-isomorphic graphs with isomorphic connectivity matroids, for example all trees on the same number of vertices. In the case of trees the matroids are totally free, so this is not surprising. But even if G is 2-connected, the matroid information is not enough to uniquely determine the graph, however, if G is 3-connected, $\mathfrak{C}(G)$ determines G uniquely up to isolated vertices; see for example [19].

2.2. The rigidity matroid $\mathfrak{R}(G)$

It was first pointed out in [13] that r-independent edge sets as defined by condition (2) are the independent sets of a matroid. Moreover, a useful formulation of the rank function in terms of edge covers is given there: Let $\{G_i\}$ be a cover of G by subgraphs G_i on $n_i \geq 2$ vertices, then the rank of $\mathfrak{R}(G)$ equals the minimum of $\sum_i(2n_i - 3)$ over all covers. If the edge set of G is rigid and r-independent, the cover may, at one extreme, be chosen to be the graph itself, or, at the other extreme, as $|E|$ singleton edges. Of course, other covers may work as well, see Fig. 4. If the subgraphs G_i are not rigid the minimum may not be achieved even if G is rigid and independent and the edges of G_i partition E; see Fig. 5.

It is well known that in the plane (but not in higher dimensions) r-circuits are rigid and, in fact, they remain rigid after the removal of any single edge. The only possible cover to compute the rank of an r-circuit is the r-circuit itself.

In order to analyze the structure of $\mathfrak{R}(G)$ we want to further examine decompositions of the edge set and their relation to the rank function.

3. Matroid connectivity

A partition $\{E_1, E_2\}$ of E is called a *k-separator* of a matroid $\mathfrak{M}$ on E if $|E_i| \geq k$ and $\rho(E_1) + \rho(E_2) \leq \rho(E) + k - 1$. Tutte [18] calls $\mathfrak{M}$ *n-connected*, if there is no k separator for $k < n$. With this definition every matroid is 1-connected.

A matroid is 2-connected if there is no partition of E into two sets E_1 and E_2 such that $|E_i| \geq 1$ and $\rho(E_1) + \rho(E_2) \leq \rho(E)$, i.e. if it is not the direct sum of its restrictions to the E_i's. It is clear that every matroid can be uniquely decomposed into a direct sum such that each of the summands is 2-connected. Note that many authors call a matroid *connected* if it is 2-connected in the Tutte sense. We choose to use Tutte's 2-connectivity, so that 2-connectivity of the graph G is equivalent to 2-connectivity of its cycle matroid $\mathfrak{C}(G)$. If $\mathfrak{R}(G)$ is 2-connected we say G is 2-*rigid*.

It is well known, see for example [14] or [15], that a matroid is 2-connected if and only if for any partition of the ground set into two sets, there is a circuit C intersecting both of them. In fact an even stronger conclusion holds, namely a matroid is 2-connected if and only if any pair of its edges is contained in a circuit.

3.1. *r-bridges and c-bridges*

Every edge of E is either a c-bridge, or is contained in a maximal edge-2-connected subgraph. Our first structure theorem is an elementary observation.

Theorem 1. *Let $G = (V, E)$ be a connected graph with c-bridges B, and let $E - B$ be partitioned*

$$E - B = A_1 \coprod \dots \coprod A_k$$

into edge sets of the connected components of the subgraph of G induced by $E - B$. Then $A_1, \dots, A_k$ induce the maximal edge-2-connected subgraphs of G.

Moreover, for each pair (i, j), $1 \leq i < j \leq k$, there is a bridge $b_{i,j}$ such that A_i and A_j are contained in different connected components of $G - b_{i,j}$.

Proof. First we note that none of the subgraphs induced by the A_i's contains a c-bridge, since that edge would be a direct summand of $\mathfrak{C}(A_i)$, and hence a direct summand of $\mathfrak{C}(E)$, and hence a c-bridge for G. So each A_i is edge-2-connected, and is contained in the set of edges of some maximal edge-2-connected subgraph, D_i of G. Since D_i is bridgeless, we have $D_i \subseteq A_1 \coprod \dots \coprod A_k$, and, since it induces a connected graph, it is contained in one of the summands of the partition, so it must coincide with A_i.

To prove the second claim, choose a spanning tree T for G. T contains all bridges, and $T \bigcap A_i$ is connected for each i. There is a unique shortest path P in T connecting an endpoint of an edge in A_i to an endpoint of an edge in A_j. The path in T must contain bridges, since none of the A_i's share a vertex. Any bridge contained in P will separate A_i and A_j. $\square$

For $\mathfrak{R}(G)$ the statement is analogous, but looking at a rigid graph and two of its vertices not contained in a rigidity circuit, it might not be so obvious that there exists a single edge whose removal allows a motion changing the distance between the two given vertices.

Theorem 2. *Let G be a rigid graph with r-bridges B, and let $E - B$ be partitioned*

$$E - B = A_1 \coprod \dots \coprod A_k$$

into the edge sets of the rigid components of the subgraph of G induced by $E - B$. Then $A_1, \dots, A_k$ induce the maximal edge-2-rigid subgraphs of G.

Moreover, for each pair (i, j), $1 \leq i < j \leq k$, there is a bridge $b_{i,j}$ such that A_i and A_j are contained in different rigid components of $G - b_{i,j}$.

Proof. For the first statement, the argument is the same as in the proof of Theorem 1, with prefix c replaced by prefix r.

For the second statement, let L be a spanning r-independent subgraph of G. The edges of L consist of the set of r-bridges as well as the edges of spanning r-independent subgraphs L_i for each of the

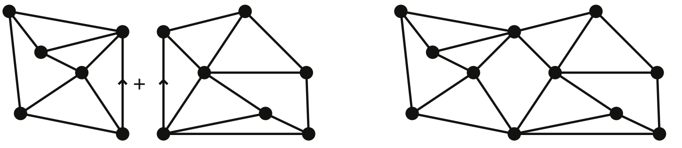

Fig. 6. The 2-sum of two r-circuits.

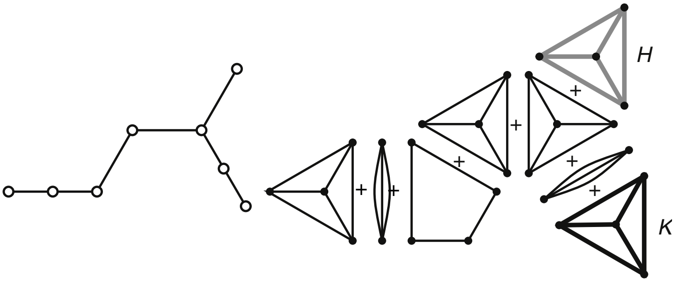

Fig. 7. A 2-sum tree.

subgraphs induced by A_i. Consider P, an edge minimal rigid subgraph of L containing both L_i and L_j. A simple counting argument shows that P is in fact the intersection of all the rigid subgraphs of L containing both L_i and L_j. Since $L_1 \coprod \dots \coprod L_k$ does not induce a rigid graph, P must contain some bridges, and the removal of any of these bridges from P, say b, has L_i and L_j in two distinct rigid components of $P - b$ by the minimality of P. Moreover, if $L - b$ had L_i and L_j in the same rigid component, then b would not be in the intersection of all rigid subgraphs containing these two sets. So A_i and A_j belong to two distinct rigid components of $L - b$. Thus A_i and A_j belong to two distinct rigid components of $G - b$. $\square$

3.2. The 2-sum

The 2-*sum*, $M_1 \bigoplus_{2/e} M_2$, of two matroids M_1 and M_2, both containing at least 3 elements and having exactly one element e in common, where e is neither dependent (a loop) or a bridge in either of the M_i, is a matroid on the union of the ground sets of M_1 and M_2 excluding e and the circuits of $M_1 \bigoplus_{2/e} M_2$ consist of circuits of M_i not containing e and of sets of the form $(C_1 \bigcup C_2) \setminus e$ where C_i is a circuit of M_i containing e.

A matroid is 3-connected if and only if it cannot be written as a 2-sum.

The 2-sum is also defined for graphs, but here one cannot identify two edges without specifying which pairs of endpoints are to be identified, in other words, without specifying an orientation on the edges to be amalgamated; see Fig. 6. Note that the 2-sum of two cycles is a cycle.

3.3. The 2-sum and 2-connectivity

Clearly the 2-sum of graphs is associative provided that the edges to be amalgamated are distinct, and so it is convenient to represent the result of a succession of 2-sums as a tree in which the nodes of the tree encode the graphs to be joined, and the edges of the tree encode the (oriented) edges of the graphs to be amalgamated; see Fig. 7. If all the graphs corresponding to the nodes in the amalgamation tree are 2-connected, then the graph which is the result of the joins encoded by the tree is also 2-connected. We consider the case when each of the graphs corresponding to the nodes in the tree is a 3-*block*, that is, either 3-connected, a simple cycle with at least 3 edges, or a k-*link* which is a graph consisting solely of two vertices and $k \geq 3$ parallel edges. If all the graphs corresponding to the nodes are 3-blocks with the restriction that no adjacent nodes correspond to cycles, and no adjacent nodes

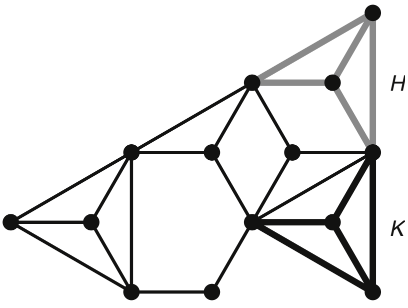

Fig. 8. The 3-block tree in Fig. 7 encodes this graph.

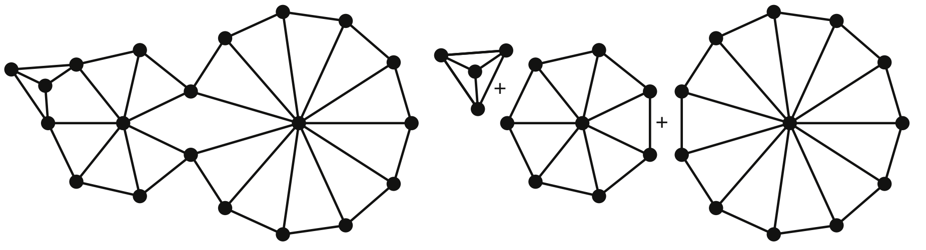

Fig. 9. A circuit in $\mathfrak{R}(G)$ decomposed in $\mathfrak{C}(G)$.

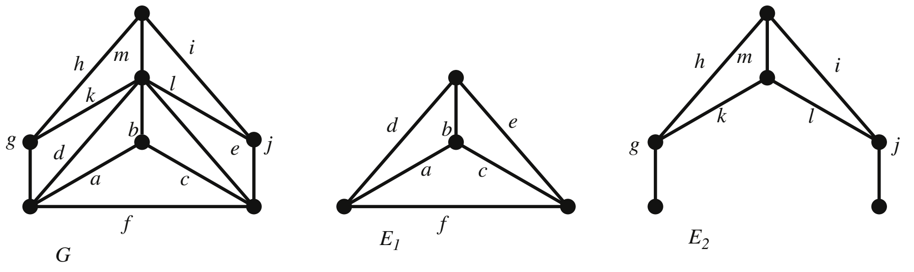

Fig. 10. A non-graphic 2-sum decomposition of $\mathfrak{R}(G)$.

correspond to k-links, then the resulting 2-sum tree is called a 3-*block tree*; see Fig. 7. Tutte proved the following deep theorem characterizing finite 2-connected graphs; see [17,5] (Fig. 8).

Theorem 3 (*[17]*). *A* 2-*connected graph G is uniquely encoded by its* 3-*block tree.*

This result has been generalized to matroids. Every 2-connected matroid has a unique encoding as a 3-block tree in which the 3-blocks are 3-connected matroids, bonds (matroids in which every 2-element subset is a circuit) and polygons (matroids consisting of a single circuit,) such that no two bonds are adjacent, nor two polygons; see [4] Theorem 18.

Note that in forming the 3-block decomposition of a matroid each circuit must be considered indecomposable since any non-trivial partition of the edge set forms a matroid 2-separation. So the r-circuit of Fig. 9 is a 3-block of $\mathfrak{R}(G)$ decomposable under $\mathfrak{C}(G)$. In general, the 3-block decomposition of $\mathfrak{R}(G)$ will involve matroid 2-sums which do not correspond to 2-sums of the graph, such as the separation in Fig. 10, and, moreover, will involve 3-blocks which are not r-graphic, that is, not the generic rigidity matroid of any graph. See Fig. 11, in which the $\mathfrak{R}(G)$ from Fig. 10 has been decomposed into its 3-blocks, B_i. The matroids B_1 and B_2 are both circuits and the matroid B_3 is the 3-connected matroid in which every three-element subset is a basis. Neither B_1 nor B_3 are r-graphic.

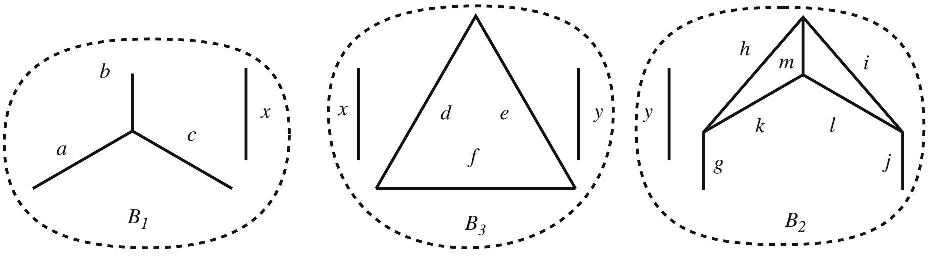

Fig. 11.

Given a graph G, the 3-block decomposition of $\mathfrak{C}(G)$ only involves graphic matroids and we say that the class of graphic matroids is closed under 2-sum decomposition. While $\mathfrak{C}(G_1) \bigoplus_{2/e} \mathfrak{C}(G_2) = \mathfrak{C}(G_1 \bigoplus_{2/e} G_2)$, the matroid $\mathfrak{C}(G)$ may be written as a 2-sum not corresponding to the 2-sum of two graphs yielding G, which occurs because there are non-isomorphic graphs with the same connectivity matroid. A simple criterion to decide whether a particular 2-separator of $\mathfrak{C}(G)$ corresponds to a 2-separator of G, is to check whether it decomposes G into two connected subgraphs. For $\mathfrak{R}(G)$ the situation is as follows.

Theorem 4. *Let $G = (V, E)$ be a graph with 2-connected rigidity matroid $\mathfrak{R}(G)$. Let $\{E_1, E_2\}$ be a 2-separator of $\mathfrak{R}(G)$. Then both E_1 and E_2 induce rigid subgraphs G_1 and G_2 of G if and only if an edge e can be added to both G_1 and G_2 so that $\mathfrak{R}(G_1 + e) \bigoplus_{2/e} \mathfrak{R}(G_2 + e) = \mathfrak{R}(G)$.*

Proof. We have $E = E_1 \bigcup E_2$, $E_1 \bigcap E_2 = \emptyset$, and $\rho(E) + 1 = \rho(E_1) + \rho(E_2)$. Let n_i be the cardinality of the support of E_i, $n = |V|$, and $n_1 + n_2 = n - x$, so x is the number of vertices in which G_1 and G_2 intersect. Since $\mathfrak{R}(G)$ is 2-connected, $\rho(E) = 2n-3$. Now $2n-2 = \rho(E_1)+\rho(E_2) \leq 2n_1-3+2n_2-3 = 2n + 2x - 6$ with equality if and only if both G_1 and G_2 are rigid. In that case we get $x = 2$, which means that G_1 and G_2 intersect in precisely two vertices, yielding the endpoints of the edge e required for the 2-sum. Note that $G_i + e$ might contain a doubled edge, which is considered an r-circuit.

For the converse, if one of the E_i's, say E_1 induces a non-rigid graph G_1, then it is not possible to add an edge e so that $G_1 + e$ is 2-rigid, so $\mathfrak{R}(G_1 + e)$ is not 2-connected and cannot be a 2-summand of a 2-connected matroid. □

The Cunningham and Edmonds decomposition theory, [4], includes fast algorithms, but as our example in Fig. 11 shows, using these algorithms on the rigidity matroid would, in general, yield 3-blocks which do not correspond to rigidity matroids of graphs. Theorem 4 shows, that if we want to decompose a 2-connected rigidity matroid of a graph into 2-summands corresponding to rigidity matroids of graphs, we need only to decompose the graph into its 3-blocks. The rigidity matroids on the graphic 3-blocks need not be 3-connected matroids (or cycles resp. multilinks), but are indecomposable in the sense that they cannot be written as the 2-sum of two r-graphic matroids.

3.4. 2-sum and edge-2-rigidity

Edge-2-rigid graphs in $\mathfrak{R}(G)$ are the analogues of edge-2-connected graphs in $\mathfrak{C}(G)$. It may happen that G is a 3-connected graph, so $\mathfrak{C}(G)$ is 3-connected as well, with trivial 3-block decomposition, while $\mathfrak{R}(G)$ is not even 2-connected, for example $K_{3,3}$ or the triangular prism. Fig. 17 shows the famous 5-connected 5-regular G from [13] containing r-bridges. On the other hand if we consider a graph G with 2-connected rigidity matroid $\mathfrak{R}(G)$, $\mathfrak{C}(G)$ is necessarily 2-connected, since r-circuits are rigid and therefore necessarily vertex 2-connected.

Jackson and Jordán proved in [10] that a graph on more than 3 vertices is globally rigid if and only if it is both 3-connected and edge-2-rigid. With this characterization of global rigidity we can easily describe the 3-blocks of a graph G whose rigidity matroid is 2-connected and we obtain the following reformulation of their Theorem 3.7.

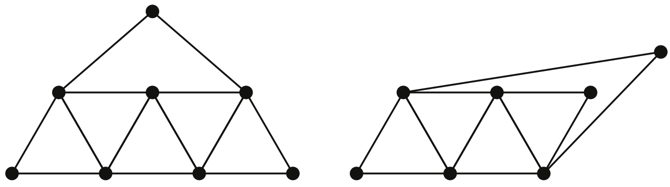

Fig. 12. Bases with 3 vertices of degree 2, all others of degree 4.

Theorem 5 (*[10]*). *Let G be a rigid graph with* 2*-connected rigidity matroid* $\mathfrak{R}(G)$. *Then the* 3*-blocks of G are multilinks or globally rigid graphs on at least four vertices.*

Proof. We note that if a rigidity matroid $\mathfrak{R}(A)$ is not 2-connected, then the matroid $\mathfrak{R}(A) \bigoplus_{2/e} \mathfrak{R}(B)$ is also not 2-connected, therefore, cycles do not occur as 3-blocks of $\mathfrak{C}(G)$. Also, the 3-blocks which are 3-connected must be 2-rigid, hence globally rigid. □

For a 3-connected graph, 2-rigidity and edge-2-rigidity are the same. In [10] an even stronger result, (Theorem 3.7), is proved, relaxing the condition of 3-connectivity. We provide here another slightly stronger variant.

Theorem 6. *A* 3*-connected graph G which is not* 2*-rigid contains at least three r-bridges.*

Proof. Suppose there are k 2-connected components of $\mathfrak{R}(G)$, $k > 1$. Every non-trivial 2-connected component of the matroid $\mathfrak{R}(G)$ induces a subgraph which, because of 3-connectivity of G, is attached to the rest of the graph by at least 3 vertices. We may, without loss of generality, assume that each of these non-trivial components is a complete graph, since the additional edges, if any, neither change the rank or connectivity of $\mathfrak{R}(G)$, or decrease the connectivity of G. We may also assume that each non-trivial component has at least 5 vertices. Then each non-trivial component has a basis in which the vertices of attachment to other components have degree 2 or more, and all other vertices have degree 4 or more; see Fig. 12.

A basis for $\mathfrak{R}(G)$ is obtained by the union of the bases for the non-trivial 2-rigid components together with the r-bridges. The only vertices of degree 3 in this basis occur as endpoints of r-bridges. Since, by Laman's condition, every isostatic graph with no vertices of degree 2 must have at least 6 vertices of degree 3, G must contain at least 3 r-bridges. □

Corollary 1. *A graph G on more than* 3 *vertices is globally rigid if and only if* $\mathfrak{C}(G)$ *is* 3*-connected and* $\mathfrak{R}(G)$ *is* 2*-connected, that is, G is* 3*-connected and* 2*-rigid.*

4. Graph decompositions via connectivity and rigidity

4.1. Connectivity hierarchy

Given a graph $G = (V, E)$, the connected components, edge-2-connected components, and the 2-connected components split $\mathfrak{C}(G)$ into ever finer direct sum decompositions. The 2-connected components correspond to non-separable matroids, and the bridges comprise the trivial components, that is, the singleton direct summands; see Fig. 13. For the graph structure, the 2-connected components, or blocks, are part of a tree structure, the *block-cutpoint tree*. This tree structure is lost in $\mathfrak{C}(G)$.

4.2. Rigid hierarchy

The maximal rigid subgraphs of a graph G partition the edge set into direct summands of the rigidity matroid, which are called the *rigid components* of the graph; see Fig. 14. Similarly one can consider the maximal edge-2-rigid subgraphs of G, that is, the *edge-2-rigid components*; see Fig. 15. These edge-2-rigid subgraphs together with the r-bridges also partition the edge set of G into direct summands of the

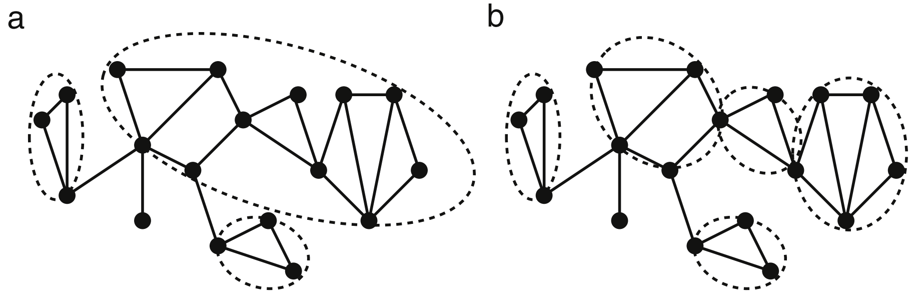

Fig. 13. Non-trivial edge-2-connected components and 2-connected components.

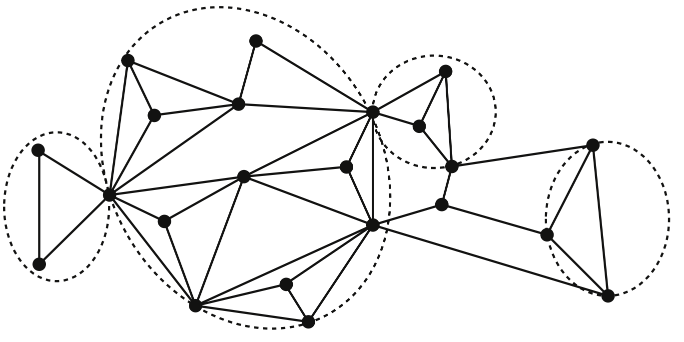

Fig. 14. Non-trivial rigid components.

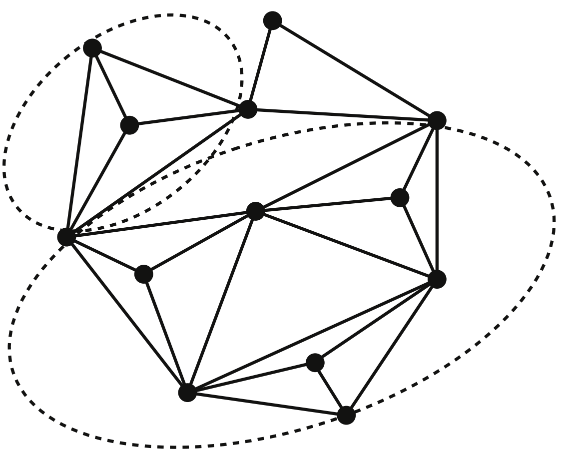

Fig. 15. Non-trivial edge-2-rigid components.

rigidity matroid, necessarily a finer decomposition than that of the rigid components, and are called the *edge-2-rigid components*. Decompositions of $\mathfrak{R}(G)$ into edge-2-rigid components are developed and used in [10,11]. Algorithms to identify edge-2 rigid components and variations are given in [2,12]. We consider r-bridges to be trivial edge-2-rigid components.

An edge-2-rigid component, however, can be decomposed further if the corresponding restriction of the rigidity matroid is not 2-connected. The direct sum decomposition of the rigidity matroid into its 2-connected components is the finest decomposition we can obtain from the rigidity matroid

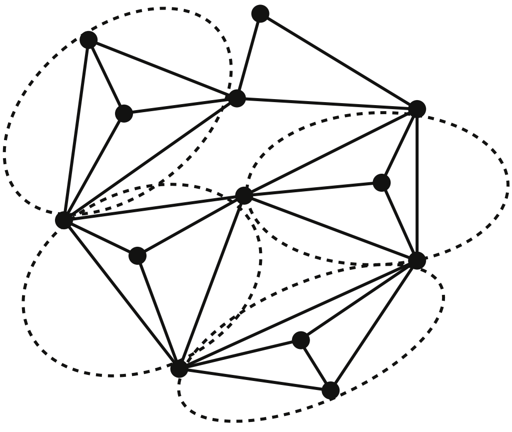

Fig. 16. Non-trivial 2-rigid components.

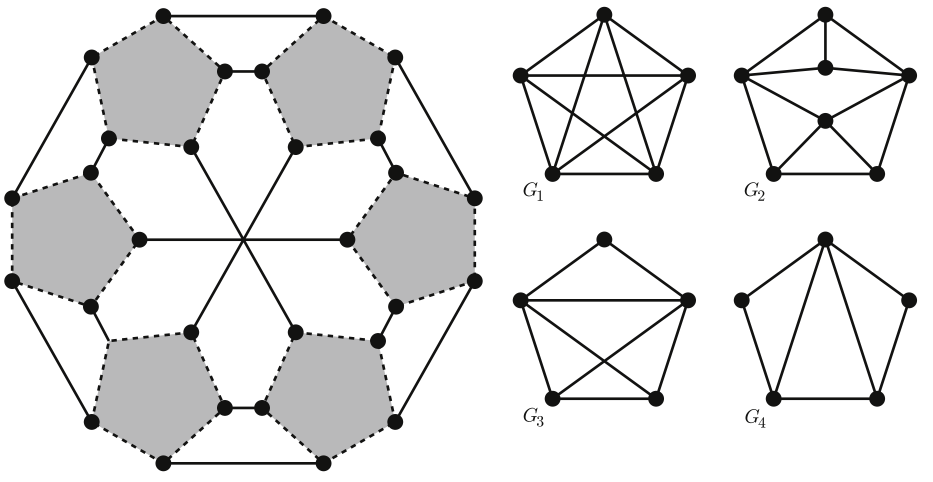

Fig. 17.

information, Fig. 16. Note that the incidence structure of the 2-rigid components can be rather complicated, and there is no obvious analogue of the block-cutpoint tree.

4.3. Decomposing edge-2-rigid graphs which are not 3-connected

In Theorem 5 we saw that if a graph is 2-rigid, then its non-trivial 3-blocks must be globally rigid. If G is only edge-2-rigid, some of the 3-blocks of G need not be globally rigid. In that case, the sequence of 2-sums of graphs as encoded by the 3-block tree need not correspond to a sequence of 2-sums of their rigidity matroids.

Lemma 1. *Let $G = G_1 \bigoplus_{2/e} G_2$ be a rigid graph, with $G_1 - e$ rigid and e an r-bridge of G_2, then $\mathfrak{R}(G) = \mathfrak{R}(G_1 \setminus e) \bigoplus \mathfrak{R}(G_2 \setminus e)$.*

Proof. By assumption, $\rho(G_1 - e) = 2n_1 - 3$ and $\rho(G_2 - e) = 2n_2 - 4$, where n_i denotes the number of vertices of G_i. Since $n = n_1 + n_2 - 2$ is the number of vertices in G, $\rho(G_1 - e) + \rho(G_2 - e) = 2(n_1 + n_2) - 7 = 2n - 3 = r(G)$, as required. $\square$

Lemma 2. *Let $G = (V, E)$ be an edge- 2-rigid graph which is not 3-connected. Let $\{u, v\} \subset V$ be a cutset of G, and $G = G_1 \bigoplus_{2/e} G_2$, where u and v are endpoints of e. If $e = (uv)$ is an r-bridge of G_2, then $G_1 - e$ is edge-2-rigid.*

Proof. Since G is edge-2-rigid, every edge of G is contained in a circuit. By Lemma 1, every circuit of G is either contained in $G_1 - e$ or in $G_2 - e$. Also, we know that $G_1 - e$ is rigid, since $G_2 - e$ is not rigid, but G is rigid. So $G_1 - e$ is rigid and each of its edges is contained in a circuit, so it is edge-2-rigid. □

Note that in Lemma 2, if uv is an edge of G, then G_1 has a doubled edge.

Theorem 7. *If G is an edge-2-rigid graph, the leaves of its 3-block tree are globally rigid.*

Proof. Since every edge of G is contained in a circuit, the leaves of the 3-block tree must be 3-connected graphs on at least 4 vertices. They are either 2-rigid or, by Theorem 6, contain more than one r-bridge. Pruning a leaf of the 3-block tree yields, using Lemma 2, either two edge-2-rigid 2-summands, or at most one summand with an r-bridge, which cannot be the summand corresponding to the leaf by Theorem 6 and Lemma 1. □

After pruning the leaves of the 3-block tree of an edge-2-rigid graph we can look at the number of r-bridges of the pruned graph. If there are none, the pruned graph is still edge-2-rigid and its rigidity matroid has the same number of direct summands as the original graph had. If the pruned tree contains r-bridges, then the number of r-bridges equals the number of indecomposable direct summands of the rigidity matroid of the original graph which correspond to the leaves of the 3-block tree by Lemma 1.

4.4. Decomposing 3-connected graphs which are not 2-rigid.

If a 3-connected graph is not globally rigid, then it contains r-bridges. We now first decompose $\mathfrak{R}(G)$ into its 2-connected components. These components are either r-bridges or edge-2-rigid graphs on at least four vertices whose rigidity matroid is 2-connected. If they are 3-connected, they are globally rigid. If not, we compute their 3-block tree to write them as the 2-sum over globally rigid blocks and multilinks.

For example, we can replace each of the shaded pentagonal regions in Fig. 17 by one of $G_1, \ldots, G_4$ to produce a 3-connected graph, in fact 5-connected with the shaded pentagonal regions all replaced with G_1's. Moreover, G is also rigid, so there is a single rigid component.

$\mathfrak{R}(G)$ is not 2-connected however. Each of the edges connecting pentagonal regions is a singleton direct summand of $\mathfrak{R}(G)$. In fact, if all pentagonal regions are replaced by G_4, then G is isostatic and $\mathfrak{R}(G)$ decomposes to the sum over singleton edges, and all the edge-2-rigid components are trivial.

If the pentagonal regions are replaced by G_3, then the two edges incident to the vertex of valence two in G_3 are also singleton summands of $\mathfrak{R}(G)$, so that there are a total of 27 r-bridges, as well as six summands corresponding to the six K_4's which are edge-2-rigid components.

If the pentagonal regions are replaced by G_2, which is only 2-connected then further decomposition is necessary to determine the globally rigid pieces.

If all pentagonal regions are replaced by G_1, which is 3-connected, the edge-2-rigid components are single edges, r-bridges, and the K_5's which are 3-connected and 2-rigid and hence all are globally rigid.

In general, the procedures of the previous two sections can be combined to give an algorithm for decomposing any graph into globally rigid pieces. If the graph is not 2-rigid, determine the 2-rigid components. The non-trivial 2-rigid components are 2-connected. Now an application of the Hopcroft and Tarjan algorithm [9] yields globally rigid blocks. This procedure reveals some non-trivial structure unless the individual edges of the graph are the globally rigid pieces, which happens exactly when G is 3-connected and isostatic.

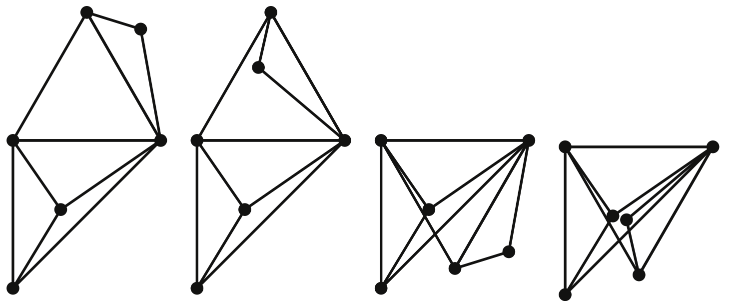

Fig. 18. The four congruence classes of embeddings.

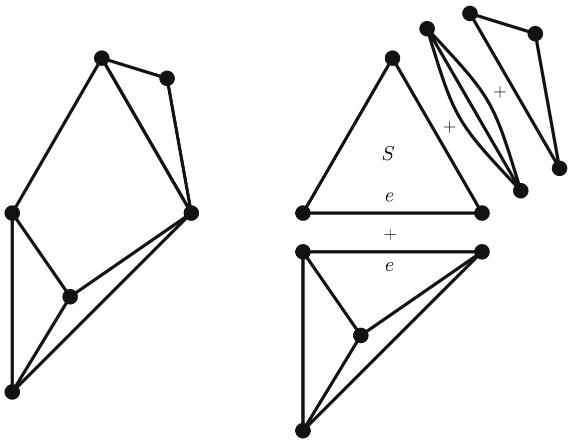

Fig. 19. G and its 3-block decomposition.

5. Configuration index

5.1. Definition and examples

The *configuration index* $\iota(G, \mathbf{p})$ of a graph $G = (V, E)$ whose vertices are embedded in the plane by $\mathbf{p} : V \to \mathbb{R}^2$ is the cardinality of the set of congruence classes of embeddings of G with the same edge lengths as in $(G, \mathbf{p})$. We call $\mathbf{p}$ *generic* if the coordinates of $\mathbf{p}(V)$ as point in $\mathbb{R}^{2|V|}$ are algebraically independent over $\mathbb{Q}$. If $\mathbf{p}$ is generic, $\iota(G, \mathbf{p}) = 1$ exactly when G is globally rigid.

The graph in Fig. 18 is not globally rigid since it is neither 3-connected nor edge-2-rigid, but its 3-block decomposition consists of two K_3's and one K_4 (plus two 3-links), all globally rigid, so $\iota(G) = 4$ for all generic embeddings of G in the plane. If we remove the edge shared by K_4 and K_3 in G, the situation becomes more delicate; see Fig. 19. The 3-blocks are still globally rigid, but now there is only one 3-link; see Fig. 19. However, the 2-sum joins an edge-2-rigid graph with an r-independent graph, and the edge e is an r-bridge of one of the summands, S. Removing e from S yields a framework of degree of freedom 1. On the other hand, even though $K_4 - e$ is still rigid, removal of e destroys 3-connectivity as well as edge-2-rigidity and $\iota(K_4 - e) = 2$. For both possible realizations we have to see which of them are compatible with the range of distance allowed to the endpoints of e by the motion of $S - e$. We conclude that the configuration index for G depends on the embedding, even if the embedding is generic. This was already pointed out in [11]. For the embedding $\mathbf{p}$ of Fig. 18, $\iota(G, \mathbf{p}) = 8$; see Fig. 20.

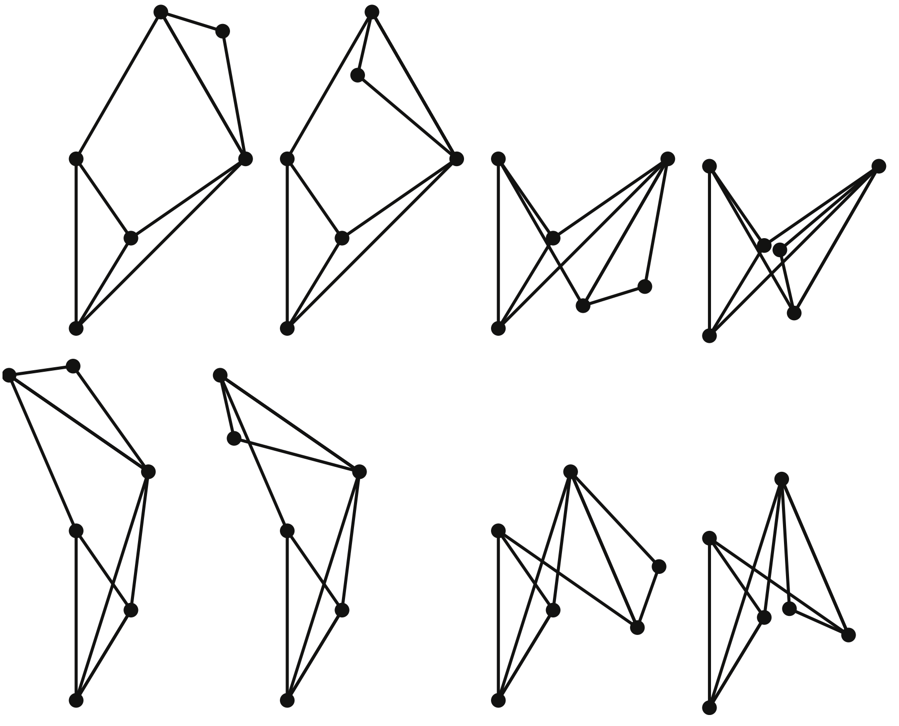

Fig. 20.

5.2. *The configuration index of a 2-rigid graph*

Let G be rigid and let $\mathfrak{R}(G)$ be 2-connected. From Theorem 5 we know that its 3-blocks are globally rigid or multilinks, which makes it easy to compute their configuration index; see also [11].

Theorem 8 ([11]). *Let $G = (V, E)$ be rigid, $|V| \geq 4$, and let $\mathfrak{R}(G)$ be 2-connected. If k is the number of globally rigid 3-blocks of $\mathfrak{R}(G)$ (which are not multilinks), then $\iota(G, \mathbf{p}) = 2^{k-1}$ for any generic embedding $\mathbf{p}$.*

In order to compute the configuration index of an edge-2-rigid graph which is not 3-connected, the 3-block tree may also be used to inductively calculate the configuration index, provided that one can compute the configuration index of the 3-blocks. If the configuration index of a 3-block B is not equal to one, then we can use the direct sum decomposition of $\mathfrak{R}(B)$ to bound the configuration index.

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