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Note

Ideals with linear quotients ☆

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ABSTRACT

We study basic properties of monomial ideals with linear quotients. It is shown that if the monomial ideal I has linear quotients, then the squarefree part of I and each component of I as well as $\mathfrak{m}I$ have linear quotients, where $\mathfrak{m}$ is the graded maximal ideal of the polynomial ring. As an analogy to the Rearrangement Lemma of Björner and Wachs we also show that for a monomial ideal with linear quotients the admissible order of the generators can be chosen degree increasingly.

0. Introduction

Let K be a field, $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables, and $I \subset S$ a monomial ideal. We denote by $G(I)$ the unique minimal monomial system of generators of I. We say that I has *linear quotients*, if there exists an order $\sigma = u_1, \dots, u_m$ of $G(I)$ such that the colon ideal $(u_1, \dots, u_{i-1}) : u_i$ is generated by a subset of the variables, for $i = 2, \dots, m$. We denote this subset by $q_{u_i,\sigma}(I)$. Any order of the generators for which we have linear quotients will be called an *admissible order*. Ideals with linear quotients were introduced by Herzog and Takayama [10]. If each component of I has linear quotients, then we say I has *componentwise linear quotients*.

The concept of linear quotients, similar to the concept of non-pure shellability, is purely combinatorial. However both concepts have strong algebraic implications. Indeed, an ideal with linear quotients has componentwise linear resolution while shellability of a simplicial complex implies that its Stanley–Reisner ideal is sequentially Cohen–Macaulay. These similarities are not accidental. In fact, let Δ be a simplicial complex and I_Δ its Stanley–Reisner ideal. It is well known, that I_Δ has linear quotients if and only if the Alexander dual of Δ is non-pure shellable, see for example in [8]

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for a short proof of this fact. Thus at least in the squarefree case "linear quotients" and "non-pure shellability" are dual concepts.

In this paper we prove some fundamental properties of monomial ideals with linear quotients. In general, the product of two ideals with linear quotients need not have linear quotients, even if one of them is generated by a subset of the variables, see Example 2.4. However in Lemma 2.5, we show that if $I \subset S$ is a monomial ideal with linear quotients, then $\mathfrak{m}I$ has linear quotients, where $\mathfrak{m} = (x_1, \dots, x_n)$ is the graded maximal ideal of S.

Let I be a monomial ideal with linear quotients and $\sigma = u_1, \dots, u_m$ an admissible order of $G(I)$. It is not hard to see that $\deg u_i \geqslant \min\{\deg u_1, \dots, \deg u_{i-1}\}$, for all $i \in [m] = \{1, \dots, m\}$. But this order need not be a degree increasing order. We show in Lemma 2.1, that there exists a degree increasing admissible order σ' induced by σ. Furthermore, one has $q_{u,\sigma}(I) = q_{u,\sigma'}(I)$ for any $u \in G(I)$, see Proposition 2.2. This implies in particular the "Rearrangement Lemma" of Björner and Wachs [2].

As a main result of this article, we show in Theorem 2.7, that any monomial ideal with linear quotients has componentwise linear quotients, and hence it is componentwise linear. Conversely, assuming that all components of I have linear quotients, we can prove that I has linear quotients only under some extra assumption, see Proposition 2.9. It would be of interest to know whether the converse of Theorem 2.7 is true in general.

Herzog and Hibi showed in [5] that a squarefree monomial ideal I is componentwise linear if and only if the squarefree part of each component has a linear resolution. We would like to remark that the "only if" part of this statement is true more generally. Indeed for *any* componentwise linear monomial ideal, the squarefree part of each component has a linear resolution. Here we prove a slightly different result by showing that if a monomial ideal I has linear quotients, then the squarefree part of I has linear quotients. This together with Theorem 2.7 implies that the squarefree part of each component of I has again linear quotients. As a corollary of the above facts we obtain that if Δ is non-pure shellable, then each facet skeleton (see the definition in Section 2) of Δ is non-pure shellable. Unless Δ is pure, this result differs from the well-known fact that each skeleton of a shellable simplicial complex is again shellable.

1. Preliminaries and background

In this section we fix the terminology, review some notation on simplicial complexes and setup some background.

A *simplicial complex* Δ on the set of vertices $[n] = \{1, \dots, n\}$ is a collection of subsets of $[n]$ with the property that if $F \in \Delta$ then all the subsets of F are also in Δ (including the empty set). An element of Δ is called a *face* of Δ, and the maximal faces of Δ under inclusion are called *facets*. We denote $\mathscr{F}(\Delta)$ the set of facets of Δ. The simplicial complex with facets $F_1, \dots, F_m$ is denoted by $\langle F_1, \dots, F_m \rangle$. The *dimension* of a face F is defined as $|F| - 1$, where $|F|$ is the number of vertices of F. The dimension of the simplicial complex Δ is the maximal dimension of its facets. A simplicial complex Γ is called a *facet subcomplex* of Δ if $\mathscr{F}(\Gamma) \subset \mathscr{F}(\Delta)$.

A subset C of $[n]$ is called a *vertex cover* of Δ, if $C \cap F \neq \emptyset$ for all facets F of Δ. A vertex cover C is said to be minimal if no proper subset of C is a vertex cover of Δ. Recently, vertex cover algebras were studied in [6] and [7].

We denote by $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables over a field K. To a given simplicial complex Δ on the vertex set $[n]$, the Stanley–Reisner ideal, whose generators correspond to the non-faces of Δ is well studied, see for example in [1,11] and [9] for details. Another squarefree monomial ideal associated to Δ, so-called facet ideal, was first studied by Faridi [4]. The ideal $I(\Delta)$ generated by all monomials $x_{i_1} \cdots x_{i_s}$ where $\{i_1, \dots, i_s\}$ is a facet of Δ, is called the *facet ideal of* Δ. For a simplicial complex of dimension 1, the facet ideal is the *edge ideal*, which was first studied by Villarreal [12].

Recall that the *Alexander dual* $\Delta^\vee$ of a simplicial complex Δ is the simplicial complex whose faces are $\{[n] \setminus F\colon\ F \notin \Delta\}$. Let I be a squarefree monomial ideal in S. We denote by $I^\vee$ the squarefree monomial ideal which is minimally generated by all monomials $x_{i_1} \cdots x_{i_k}$, where $(x_{i_1}, \dots, x_{i_k})$ is a minimal prime ideal of I. It is easy to see that for any simplicial complex Δ, one has $I_{\Delta^\vee} = (I_\Delta)^\vee$. Let $\Delta^c = \langle [n] \setminus F\colon\ F \in \mathscr{F}(\Delta) \rangle$. Then $I_{\Delta^\vee} = I(\Delta^c)$, see [8].

For any set $U \subset [n]$, we denote $u = \prod_{j \in U} x_j$ the squarefree monomial in S whose support is U. In general, for any monomial $u \in S$, the *support* of u is $\operatorname{supp}(u) = \{j\colon\ x_j \mid u\}$.

Remark 1.1. Let Δ be a simplicial complex on $[n]$. Then

$$G\big(I(\Delta)^{\vee}\big) = \Big\{u = \prod_{j \in U} x_j\colon \text{ where } U \text{ is a minimal vertex cover of } \Delta\Big\}.$$

The following notion is important for our later discussion. Let $I = (u_1, \ldots, u_m)$ be a monomial ideal in S. According to [10], the monomial ideal I has linear quotients if one can order the set of minimal generators of I, $G(I) = \{u_1, \ldots, u_m\}$, such that the colon ideal $(u_1, \ldots, u_{i-1}) : u_i$ is generated by a subset of the variables for $i = 2, \ldots, m$. This means for each $j < i$, there exists $k < i$ such that $u_k : u_i = x_t$ and $x_t \mid u_j : u_i$, where $t \in [n]$ and $u_k : u_i = u_k / \gcd(u_k, u_i)$. In the case that I is squarefree, it is enough to show that for each $j < i$, there exists $k < i$ such that $u_k : u_i = x_t$ and $x_t \mid u_j$. Such an order of generators is called an *admissible order* of $G(I)$. Let $\sigma = u_1, \ldots, u_m$ be an admissible order of $G(I)$. We denote by $q_{u_j,\sigma}(I) \subset \{x_1, \ldots, x_n\}$ the set of minimal generators of $(u_1, \ldots, u_{j-1}) : u_j$.

It is known that if I is a monomial ideal with linear quotients and generated in one degree, then I has a linear resolution. See for example in [13] an easy proof.

Remark 1.2. For an ideal which has linear quotients, there might exist several admissible orders. For example, let $I = (x_1x_2, x_1x_3^2x_4, x_2x_4) \subset K[x_1, x_2, x_3, x_4]$. Then $\sigma_1 = x_1x_2, x_1x_3^2x_4, x_2x_4$ and $\sigma_2 = x_1x_2, x_2x_4, x_1x_3^2x_4$ both are admissible orders of $G(I)$.

The following result relates squarefree monomial ideals with linear quotients to (non-pure) shellable simplicial complexes. The concept non-pure shellability was first defined by Björner and Wachs [2, Definition 2.1].

Theorem 1.3. *(See [8, Theorem 1.4].) Let Δ be a simplicial complex and $\Delta^{\vee}$ its Alexander dual. Then Δ is (non-pure) shellable if and only if $I_{\Delta^{\vee}}$ has linear quotients.*

2. Monomial ideals with linear quotients

In this section we prove some fundamental properties of ideals with linear quotients.

Let $I \subset S$ be a monomial ideal with linear quotients and $u_1, \ldots, u_m$ an admissible order of $G(I)$. It is easy to see that $\deg u_i \geqslant \min\{\deg u_1, \ldots, \deg u_{i-1}\}$ for $i = 2, \ldots, m$. In particular, $\deg u_1 = \min\{\deg u_1, \ldots, \deg u_m\}$. But in general, this order need not be a degree increasing order. For example, the ideal $I = (x_1x_2, x_1x_3^2x_4, x_2x_4)$ has linear quotients in the given order, but $\deg x_1x_3^2x_4 > \deg x_2x_4$.

In the following lemma we show that for any ideal with linear quotients there exists an admissible order $u_1, \ldots, u_m$ of $G(I)$ such that $\deg u_i \leqslant \deg u_{i+1}$ for $i = 1, \ldots, m-1$. We call such an order a *degree increasing admissible order*.

Lemma 2.1. *Let $I \subset S$ be a monomial ideal with linear quotients. Then there is a degree increasing admissible order of $G(I)$.*

Proof. We use induction on m, the number of generators of I, to prove the statement. If $m = 1$, there is nothing to show.

Assume $m > 1$ and $u_1, \ldots, u_m$ is an admissible order. It is clear that $J = (u_1, \ldots, u_{m-1})$ has linear quotients with the given order. By induction hypothesis, we may assume that $\deg u_i \leqslant \deg u_{i+1}$ for $i = 1, \ldots, m-2$ (if necessary we may reorder the generators $u_1, \ldots, u_{m-1}$ of J, and call them $u_1, \ldots, u_{m-1}$ again). Assume that $\deg u_{m-1} > \deg u_m$. Let $j+1$ be the smallest integer such that $\deg u_{j+1} > \deg u_m$. Since $\deg u_i \geqslant \min\{\deg u_1, \ldots, \deg u_{i-1}\}$ for $i = 2, \ldots, m$, one sees that $j+1 \neq 1$. Now we show that $u_1, \ldots, u_j, u_m, u_{j+1}, \ldots, u_{m-1}$ is an admissible order which is obviously degree increasing.

We need to prove that $(u_1, \ldots, u_j) : u_m$ and $(u_1, \ldots, u_j, u_m, u_{j+1}, \ldots, u_{p-1}) : u_p$ are generated in degree one, for $p = j+1, \ldots, m-1$. Since $\deg u_m < \deg u_q$ for $q = j+1, \ldots, m-1$, we have $\deg(u_q : u_m) > 1$. Since $u_1, \ldots, u_m$ is an admissible order, for any $r \leqslant j$, there exists $k \leqslant j$ such that $\deg(u_k : u_m) = 1$ and $u_k : u_m \mid u_r : u_m$. This shows that $(u_1, \ldots, u_j) : u_m$ is generated in degree one. Now let $j+1 \leqslant p \leqslant m-1$. It is clear that for any $r \leqslant p-1$, there exists $k \leqslant p-1$ such that $\deg(u_k : u_p) = 1$ and $u_k : u_p \mid u_r : u_p$, since the ideal $(u_1, \ldots, u_j, u_{j+1}, \ldots, u_p)$ has linear quotients in this order. It remains to show that there is $h < p$ such that $\deg(u_h : u_p) = 1$ and $u_h : u_p \mid u_m : u_p$. Since $u_1, \ldots, u_j, u_{j+1}, \ldots, u_m$ is an admissible order and $\deg u_m < \deg u_q$ for $q = j+1, \ldots, m-1$, there exists $k \leqslant j$ such that $u_k : u_m = x_d$ and $x_d \mid u_p : u_m$ for some $d \in [n]$. Since $u_1, \ldots, u_j, u_{j+1}, \ldots, u_p$ is an admissible order, there exists $h < p$ such that $u_h : u_p = x_b$ and $x_b \mid u_k : u_p$ for some $b \in [n]$.

We claim that $x_b \mid u_m : u_p$. In order to prove this we first show that $b \neq d$. Suppose $b = d$. Then we have $x_d = u_k : u_m$ and $x_d = x_b \mid u_k : u_p$. Hence $\deg_{x_d} u_k = \deg_{x_d} u_m + 1$ and $\deg_{x_d} u_k \geqslant \deg_{x_d} u_p + 1$, where by $\deg_{x_d} u$ we mean the degree of x_d in u. Therefore $\deg_{x_d} u_m \geqslant \deg_{x_d} u_p$, which is a contradiction, since $x_d \mid u_p : u_m$.

Now since $x_b = u_h : u_p$ and $x_b \mid u_k : u_p$, we have $\deg_{x_b} u_h = \deg_{x_b} u_p + 1$ and $\deg_{x_b} u_k \geqslant \deg_{x_b} u_p + 1$. On the other hand, since $x_d = u_k : u_m$ and $b \neq d$, we have $\deg_{x_b} u_m \geqslant \deg_{x_b} u_k \geqslant \deg_{x_b} u_p + 1 > \deg_{x_b} u_p$. This implies that $x_b \mid u_m : u_p$. □

If $\sigma = u_1, \ldots, u_m$ is any admissible order of $G(I)$, we denote by $\sigma' = u_{i_1}, \ldots, u_{i_m}$ the degree increasing admissible order derived from σ as given in Lemma 2.1. The order σ' is called the degree increasing admissible order induced by σ. Attached to an admissible order σ are the sets $q_{u,\sigma}(I)$ as defined in the previous section. We have the following result.

Proposition 2.2. *Let I be a monomial ideal with linear quotients with respect to the admissible order σ of the generators. Then for all $u \in G(I)$ we have*

$$q_{u,\sigma}(I) = q_{u,\sigma'}(I).$$

Proof. Let $\sigma = u_1, \ldots, u_m$ and $\sigma' = u_{i_1}, \ldots, u_{i_m}$. Suppose $u = u_k$ in σ and $u = u_{i_t}$ in σ'. Let $x_d \in q_{u,\sigma}(I)$, for some $d \in [n]$, then there exists $j < k$ such that $u_j : u_k = x_d$. In particular, $\deg u_j \leqslant \deg u_k$. According to the definition of σ', u_j comes before u_{i_t} and hence $x_d \in q_{u,\sigma'}(I)$.

Conversely, let $x_d \in q_{u,\sigma'}(I)$ for some $d \in [n]$. Then there exists an i_j with $j < t$, such that $u_{i_j} : u_{i_t} = x_d$. We may assume that j is the smallest integer with this property and $u_{i_j} = u_r$ in σ.

Suppose $x_d \notin q_{u,\sigma}(I)$. Then $r > k$ and $\deg u_r < \deg u_k$ according to the definition of σ'. Therefore $u_r = x_d v$ and $u_k = wv$ where v and w are monomials with $\deg w \geqslant 2$ and $x_d \nmid w$. Since $u_1, \ldots, u_r$ is an admissible order and $k < r$, there exists $s < r$ such that $u_s : u_r = x_b$ and $x_b \mid u_k : u_r = w$ $(b \neq d)$. Hence $\deg u_s \leqslant \deg u_r = \deg u_{i_j}$. Therefore $u_s = u_{i_l}$ with $l < j$.

It follows that $\deg_{x_b} u_s = \deg_{x_b} u_r + 1 \leqslant \deg_{x_b} u_k$, $\deg_{x_c} u_s \leqslant \deg_{x_c} u_r \leqslant \deg_{x_c} u_k$ for any $c \neq d, b$, and $\deg_{x_d} u_s \leqslant \deg_{x_d} u_r = \deg_{x_d} u_k + 1$. If $\deg_{x_d} u_s < \deg_{x_d} u_k + 1$, then we have $u_s \mid u_k$, a contradiction. Therefore $\deg_{x_d} u_s = \deg_{x_d} u_k + 1$, and hence $x_d = u_s : u_k = u_{i_l} : u_{i_t}$, contradicting the choice of j. □

Let Δ be a simplicial complex with $\mathscr{F}(\Delta) = \{F_1, \ldots, F_m\}$. Then $I_\Delta = \bigcap_{i=1}^m P_{F_i}$ where $P_{F_i} = (x_j\colon\ j \notin F_i)$, see [1, Theorem 5.4.1]. It follows from [8, Lemma 1.2] that $I_{\Delta^\vee} = (u_1, \ldots, u_m)$, where $u_i = \prod_{j \notin F_i} x_j$. We follow the notation in [2]: if $\delta = F_1, \ldots, F_m$ is any order of facets of Δ, then we set $\Delta_k = \langle F_1, \ldots, F_k \rangle$ and $R_\delta(F_k) = \{i \in F_k\colon\ F_k - \{i\} \in \Delta_{k-1}\}$ for any $k \in [m]$.

We observe the following simple but important fact: Δ is shellable with shelling $\delta = F_1, \ldots, F_m$ if and only if $I_{\Delta^\vee}$ has linear quotients with the admissible order $\sigma = u_1, \ldots, u_m$. Moreover, if the equivalent conditions hold, then $R_\delta(F_k) = q_{u_k,\sigma}(I_{\Delta^\vee})$.

As an immediate consequence of Lemma 2.1, Proposition 2.2 and the observation above we rediscover the following well-known "Rearrangement Lemma" of Björner and Wachs [2, Lemma 2.6].

Corollary 2.3. *Let $\delta = F_1, \ldots, F_m$ be a shelling of the simplicial complex Δ. There exists a shelling $\delta' = F_{i_1}, \ldots, F_{i_m}$ of Δ induced by δ such that* $\dim F_{i_k} \geqslant \dim F_{i_{k+1}}$ *for $k = 1, \ldots, m-1$. Furthermore we have $R_\delta(F) = R_{\delta'}(F)$ for any facet F of Δ.*

It is known that the product of two ideals with linear quotients need not to have again linear quotients, even if one of them is generated by linear forms. Such an example was given by Conca and Herzog [3].

Example 2.4. Let $R = K[a,b,c,d]$, $I = (b,c)$ and $J = (a^2b, abc, bcd, cd^2)$. Then J has linear quotients, and I is generated by a subset of the variables. But the product IJ has no linear quotients (not even a linear resolution).

However, we have the following:

Lemma 2.5. *Let $I \subset S$ be a monomial ideal. If I has linear quotients, then $\mathfrak{m}I$ has linear quotients, where $\mathfrak{m} = (x_1,\dots,x_n)$ is the graded maximal ideal of S.*

Proof. We may assume $G(I) = \{u_1,\dots,u_m\}$ and $u_1,\dots,u_m$ is a degree increasing admissible order. We prove the assertion by using induction on m.

The case $m = 1$ is trivial. Let $m > 1$. Consider the multi-set

$$T = \{u_1x_1,\dots,u_1x_n, u_2x_1,\dots,u_2x_n,\dots,u_mx_1,\dots,u_mx_n\}.$$

It is a system of generator of $\mathfrak{m}I$. If $u_ix_j \mid u_rx_s$ for some $i < r$, then we remove u_rx_s from T. In this way, we get the minimal set

$$T' = \{u_ix_j\colon\ i = 1,\dots,m,\ \ j \in A_i\}$$

of monomial generators of $\mathfrak{m}I$, where $A_1 = [n]$ and $A_i \subset [n]$ for $i = 2,\dots,m$. We shall order $G(\mathfrak{m}I)$ in the following way: u_kx_l comes before u_tx_s if $k < t$ or $k = t$ and $l < s$. Now we show that the above order σ of $G(\mathfrak{m}I)$ is an admissible order. We define the order of the generators of $\mathfrak{m}(u_1,\dots,u_{m-1})$ in the same way as we did for $\mathfrak{m}I$. Then the ordered sequence τ of the generators of $\mathfrak{m}(u_1\dots,u_{m-1})$ is an initial sequence of σ. Moreover, by induction hypothesis, τ is an admissible order of $G(\mathfrak{m}(u_1,\dots,u_{m-1}))$.

For a given $j \in A_m$ let J be the ideal generated by all monomials in T' which come before u_mx_j with respect to σ. It remains to be shown that $J : u_mx_j$ is generated by monomials of degree 1.

Let $u_kx_l \in G(J)$. If $k = m$, then $u_kx_l : u_mx_j = x_l$. If $k < m$, then we shall find an element $u_rx_s \in G(J)$ and $t \in [n]$ such that $u_rx_s : u_mx_j = x_t$ and $x_t \mid u_kx_l : u_mx_j$. Indeed since $u_1,\dots,u_m$ is an admissible order of $G(I)$, there exists $q < m$ such that $u_q : u_m = x_t$ and $x_t \mid u_k : u_m$. This implies that $u_qx_j : u_mx_j = u_q : u_m = x_t$. Since $u_qx_j \in \mathfrak{m}I$, there exists, by the definition of σ, a monomial $u_rx_s \in G(J)$ such that $u_rx_s \mid u_qx_j$.

We claim that $u_rx_s : u_mx_j = x_t$ and $x_t \mid u_kx_l : u_mx_j$. Notice that $u_rx_s : u_mx_j \mid u_qx_j : u_mx_j = x_t$. If $u_rx_s : u_mx_j \neq x_t$, then $u_rx_s : u_mx_j = 1$, that is, $u_rx_s \mid u_mx_j$ which contradicts the fact that $j \in A_m$. This shows that $u_rx_s : u_mx_j = x_t$.

Since $x_t \mid u_k : u_m$, it is enough to show that $x_t \neq x_j$ in order to prove that $x_t \mid u_kx_l : u_mx_j$. Assume that $x_t = x_j$. Since $u_q : u_m = x_t$, we have $u_q = x_tu$ for some monomial u such that $u \mid u_m$. Since $\deg u_q \leqslant \deg u_m$, it follows that $u_m = uw$ for some monomial w with $\deg w \geqslant 1$ and $x_t \nmid w$. Hence there exists some variable x_d with $d \neq t$ such that $x_d \mid w$. But then $x_du_q = x_dux_t \mid wux_t = u_mx_j$, contradicting again the fact that $j \in A_m$. $\square$

Remark 2.6. The converse of the above lemma is not true. For example, let $I = (ab, cd) \subset K[a,b,c,d]$. Then $\mathfrak{m}I = (a^2b, ab^2, abc, abd, acd, bcd, c^2d, cd^2)$ has linear quotients in the given order, but I has no linear quotients.

Now we present the main theorem of this section.

Theorem 2.7. *Let $I \subset S$ be a monomial ideal. If I has linear quotients, then I has componentwise linear quotients.*

Proof. By Lemmas 2.5 and 2.1, we may assume that I is generated by monomials of two different degrees a and $a+1$. We denote by $I_{\langle a\rangle}$ the ideal generated by the ath graded component of the ideal I. Let $G(I)=\{u_1,\dots,u_s,v_1,\dots,v_t\}$, where $\deg u_i=a$ for $i=1,\dots,s$ and $\deg v_j=a+1$ for $j=1,\dots,t$. By Lemma 2.1, we may assume that $u_1,\dots,u_s,v_1,\dots,v_t$ is an admissible order, hence $I_{\langle a\rangle}$ has linear quotients. Now we show that $I_{\langle a+1\rangle}$ has also linear quotients.

We have $I_{\langle a+1\rangle}=\mathfrak{m}(u_1,\dots,u_s)+(v_1,\dots,v_t)$. Let $G(I_{\langle a+1\rangle})=\{w_1,\dots,w_l,v_1,\dots,v_t\}$, where $w_1,\dots,w_l$ is ordered as in Lemma 2.5. In particular, $w_1,\dots,w_l$ is an admissible order. We only need to show that $(w_1,\dots,w_l,v_1,\dots,v_{p-1}):v_p$ is generated by a subset of the variables, for $1\leqslant p\leqslant t$.

First we consider $v_j:v_p$ where $j<p$. Since $u_1,\dots,u_s,v_1,\dots,v_t$ is an admissible order of $G(I)$, there exist some $u\in\{u_1,\dots,u_s,v_1,\dots,v_{p-1}\}$ and $d\in[n]$ such that $u:v_p=x_d$ and $x_d\mid v_j:v_p$. If $u\in\{v_1,\dots,v_t\}$ we are done. So we may assume $u\in\{u_1,\dots,u_s\}$. Therefore, $\deg u=\deg v_p-1$. Since $u:v_p=x_d$, we have $\deg_{x_d}u=\deg_{x_d}v_p+1$ and $\deg_{x_b}u\leqslant\deg_{x_b}v_p$ for any $b\neq d$. Since $\deg u<\deg v_p$, there exists a variable x_c with $c\neq d$ such that $\deg_{x_c}u\leqslant\deg_{x_c}v_p-1$. Since $x_cu\in\mathfrak{m}I_{\langle a\rangle}$, one has $x_cu=w_k$ for some $k\leqslant l$. All this implies that $\deg_{x_d}w_k=\deg_{x_d}u=\deg_{x_d}v_p+1$ and $\deg_{x_b}w_k\leqslant\deg_{x_b}v_p$ for any $b\neq d$. Therefore $w_k:v_p=x_d$ and $x_d\mid v_j:v_p$.

It remains to consider $w_j:v_p$. In this case $w_j=x_bu_i$ for some $i\in[s]$ and some $b\in[n]$. Since $u_1,\dots,u_s,v_1,\dots,v_t$ is an admissible order, there exist some $u\in\{u_1,\dots,u_s,v_1,\dots,v_t\}$ and $d\in[n]$ such that $u:v_p=x_d$ and $x_d\mid u_i:v_p$. Therefore $x_d\mid w_j:v_p$, since $u_i:v_p\mid w_j:v_p$. If $u\in\{v_1,\dots,v_t\}$, then we are done. So we may assume $u\in\{u_1,\dots,u_s\}$. Then, as before, there exists a variable x_c with $c\neq d$ such that $x_cu\in\mathfrak{m}I_{\langle a\rangle}$, $\deg_{x_d}x_cu=\deg_{x_d}u=\deg_{x_d}v_p+1$ and $\deg_{x_b}x_cu\leqslant\deg_{x_b}v_p$ for any $b\neq d$. This implies that $x_cu:v_p=x_d$ and $x_d\mid w_j:v_p$. $\square$

Corollary 2.8. *If $I\subset S$ is a monomial ideal with linear quotients, then I is componentwise linear.*

We do not know if the converse of Theorem 2.7 is true in general. However we could prove the following:

Proposition 2.9. *Let I be a monomial ideal with componentwise linear quotients. Suppose for each component $I_{\langle a\rangle}$ there exists an admissible order σ_a of $G(I_{\langle a\rangle})$ with the property that the elements of $G(\mathfrak{m}I_{\langle a-1\rangle})$ form the initial part of σ_a. Then I has linear quotients.*

Proof. We choose the order $\sigma=u_1,\dots,u_s$ of $G(I)$ such that $i<j$ if $\deg u_i<\deg u_j$ or $\deg u_i=\deg u_j=a$ and u_i comes before u_j in σ_a.

We show that $(u_1,\dots,u_{p-1}):u_p$ is generated by linear forms. If $\deg u_1=\deg u_p$, then there is nothing to prove.

Now assume that $\deg u_1<\deg u_p=b$. Let $l<p$ be the largest number such that $\deg u_l<b$. Then, by our assumption, there exists an admissible order $w_1,\dots,w_t,u_{l+1},\dots,u_p$ (which is an initial part of an admissible order of $I_{\langle b\rangle}$), where $w_1,\dots,w_t\in G(\mathfrak{m}I_{\langle b-1\rangle})$.

Let $j<p$ and suppose that $\deg(u_j:u_p)\geqslant 2$. Let m be a monomial such that $\deg(mu_j)=\deg u_p$ and $mu_j:u_p=u_j:u_p$. Since $mu_j\in\{w_1,\dots,w_t,u_{l+1},\dots,u_{p-1}\}$ there exist $w\in\{w_1,\dots,w_t,u_{l+1},\dots,u_{p-1}\}$ and some $d\in[n]$ such that $w:u_p=x_d$ and $x_d\mid u_j:u_p$ because $mu_j:u_p=u_j:u_p$.

If $w\in\{u_{l+1},\dots,u_{p-1}\}$, then we are done. On the other hand, if $w\in\{w_1,\dots,w_t\}$, then $w=m'u_i$ for some $i\leqslant l$ and some monomial m'. Since $w:u_p=x_d$, one has $\deg_{x_b}w\leqslant\deg_{x_b}u_p$ for all $b\neq d$. Hence x_d does not divide m', otherwise $u_i\mid u_p$ which contradicts the fact that $u_i,u_p\in G(I)$. Therefore $x_d=u_i:u_p$ and $x_d\mid u_j:u_p$. $\square$

Let $I\subset S$ be a monomial ideal. We denote by I_* the monomial ideal generated by the squarefree monomials in I and call it the squarefree part of I. Indeed $I_*=(u\colon u\in G(I)$ and u is squarefree$)$. We follow [5] and denote by $I_{[a]}$ the squarefree part of $I_{\langle a\rangle}$. In [5, Proposition 1.5], the authors proved that if I is squarefree, then $I_{\langle a\rangle}$ has a linear resolution if and only if $I_{[a]}$ has a linear resolution. Indeed for the "only if" part one does not need the assumption that I is squarefree. We have the following slightly different result.

Proposition 2.10. *Let I be a monomial ideal in S. If I has linear quotients, then I_* has linear quotients.*

Proof. Let $u_1, \dots, u_m$ be an admissible order of $G(I)$. Assume $I_* = (u_{i_1}, \dots, u_{i_t})$, where $1 \leqslant i_1 < i_2 < \dots < i_t \leqslant m$. We shall show $u_{i_1}, \dots, u_{i_t}$ is an admissible order of $G(I_*)$ by using induction on m.

The case $m = 1$ is trivial. Now assume $m > 1$. It is clear that $(u_{i_1}, \dots, u_{i_{t-1}})$ is the squarefree part of the monomial ideal $(u_1, \dots, u_{i_{t-1}})$, where $u_1, \dots, u_{i_{t-1}}$ is an admissible order. By induction hypothesis $u_{i_1}, \dots, u_{i_{t-1}}$ is an admissible order of $G((u_{i_1}, \dots, u_{i_{t-1}}))$. Consider $u_{i_j} : u_{i_t}$ with $j < t$. Since $u_1, \dots, u_m$ is an admissible order of $G(I)$, there exists $k < i_t$ and some $d \in [n]$ such that $u_k : u_{i_t} = x_d$ and $x_d \mid u_{i_j} : u_{i_t}$. Since u_{i_j} and u_{i_t} are squarefree, we have $x_d \nmid u_{i_t}$. On the other hand, since $u_k : u_{i_t} = x_d$, one has $\deg_{x_d} u_k = 1$ and $\deg_{x_b} u_k \leqslant \deg_{x_b} u_{i_t} \leqslant 1$ for any $b \neq d$. Hence $u_k \in G(I_*)$. $\square$

Combining Proposition 2.10 with Theorem 2.7, we obtain:

Corollary 2.11. *Let $I \subset S$ be a monomial ideal with linear quotients. Then $I_{[a]}$ has linear quotients for all a.*

Let Δ be a d-dimensional simplicial complex. We define the 1-*facet skeleton* of Δ to be the simplicial complex

$$\Delta^{[1]} = \big\langle G\colon\ G \subset F \in \mathscr{F}(\Delta) \text{ and } |G| = |F| - 1\big\rangle.$$

Recursively, the *i-facet skeleton* is defined to be the 1-facet skeleton of $\Delta^{[i-1]}$, for $i = 1, \dots, d$. For example if $\Delta = \langle\{1,2,3\}, \{2,3,4\}, \{4,5\}\rangle$, then

$$\Delta^{[1]} = \big\langle\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}, \{5\}\big\rangle \quad \text{and} \quad \Delta^{[2]} = \big\langle\{1\}, \{2\}, \{3\}, \{4\}\big\rangle.$$

If Δ is pure of dimension d, then the i-facet skeleton of Δ is just the $(d-i)$-skeleton of Δ. Now let Γ be a shellable simplicial complex with facets $F_1, \dots, F_m$. It is known that any skeleton of Γ is shellable, see [2, Theorem 2.9]. Since $I_\Gamma = \bigcap_{i=1}^m P_{F_i}$ where $P_{F_i} = (x_j\colon\ j \notin F_i)$, we have $(I_\Gamma)^\vee = (u_1, \dots, u_m)$, where $u_i = \prod_{j \notin F_i} x_j$. By Theorem 1.3 $(I_\Gamma)^\vee$ has linear quotients. Hence $\mathfrak{m}(I_\Gamma)^\vee$ and the squarefree part of $\mathfrak{m}(I_\Gamma)^\vee$ have linear quotients by Lemma 2.5 and Proposition 2.10. It is not hard to see that the squarefree part of $\mathfrak{m}(I_\Gamma)^\vee$ is the Alexander dual of $I_{\Gamma^{[1]}}$. Hence our discussions yield the following:

Corollary 2.12. *If Γ is a shellable simplicial complex of dimension d, then $\Gamma^{[i]}$ is shellable, for $i \leqslant d$. In particular, if Γ is pure, then any skeleton of Γ is again shellable.*

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