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k-Subdomination in graphs

Gerard J. Chang[a,*,1], Sheng-Chyang Liaw[b], Hong-Gwa Yeh[c,2]

[a] *Department of Mathematics, National Taiwan University, Taipei 106, Taiwan*
[b] *Department of Mathematics, National Central University, Chungli 32054, Taiwan*
[c] *Department of Applied Mathematics, National University of Kaohsiung, Kaohsiung 811, Taiwan*

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Abstract

For a positive integer k, a k-subdominating function of a graph $G=(V,E)$ is a function $f:V\to\{-1,1\}$ such that $\sum_{u\in N_G[v]} f(u)\geqslant 1$ for at least k vertices v of G. The k-subdomination number of G, denoted by $\gamma_{ks}(G)$, is the minimum of $\sum_{v\in V} f(v)$ taken over all k-subdominating functions f of G. In this article, we prove a conjecture for k-subdomination on trees proposed by Cockayne and Mynhardt. We also give a lower bound for $\gamma_{ks}(G)$ in terms of the degree sequence of G. This generalizes some known results on the k-subdomination number $\gamma_{ks}(G)$, the signed domination number $\gamma_s(G)$ and the majority domination number $\gamma_{\mathrm{maj}}(G)$.

Keywords: Domination; k-subdomination; Majority domination; Signed domination; Tree; Leaf

1. Introduction

The concept of domination is a good model for many location problems in operations research. In a graph $G=(V,E)$, the (*open*) *neighborhood* of a vertex v is the set $N_G(v)$ consisting of all vertices adjacent to v; and the *closed neighborhood* $N_G[v]=N_G(v)\cup\{v\}$. The *degree* of a vertex v is $\deg(v)=|N_G(v)|$. A *leaf* is a vertex of degree 1. A *leaf neighbor* is a neighbor that is a leaf. A *dominating set* of G is a subset D of V for which every vertex in V–D is adjacent to some vertex of D; or equivalently, $|N_G[v]\cap D|\geqslant 1$. The *domination number* $\gamma(G)$ is the smallest cardinality of a dominating set. Alternatively, we can view a dominating set as a *dominating function* which is a function $g:V\to\{0,1\}$ such that $g(N_G[v])\geqslant 1$ for all vertices $v\in V$,

* Corresponding author. Tel.: +886-3-573-1945; fax: +886-3-542-2682.
E-mail addresses: gjchang@math.ntu.edu.tw (G.J. Chang), scliaw@math.ncu.edu.tw (S.-C. Liaw), hgyeh@math.nuk.edu.tw (H.-G. Yeh).
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where $g(S)=\sum_{x\in S} g(x)$ for any $S \subseteq V$. In this case, $\gamma(G)$ is the minimum of $g(V)$ taken over all dominating functions of G.

Variations of domination have been defined by replacing $\{0,1\}$ in the above definition by $\{-1,1\}$ or $\{-1,0,1\}$, and requiring the condition $g(N_G[v]) \geqslant 1$ for different number of vertices. For a positive integer k, a *k-subdominating function* of $G=(V,E)$ is defined in [3] as a function $g: V \to \{-1,1\}$ such that $g(N_G[v]) \geqslant 1$ for at least k vertices v of G. The *k-subdomination number* of G is

$$\gamma_{ks}(G)=\min\{g(V)\colon g \text{ is a } k\text{-subdominating function of } G\}.$$

In the special cases where $k=|V|$ and $k=\lceil |V|/2 \rceil$, $\gamma_{ks}(G)$ is, respectively, the *signed domination number* $\gamma_s(G)$ defined in [4] and the *majority domination number* $\gamma_{maj}(G)$ defined in [2] of G. For more study on signed domination and majority domination, see [1,5–13].

Cockayne and Mynhardt [3] proved that for any tree T of n vertices, $\gamma_{ks}(T) \leqslant 2k+2-n$. This upper bound is sharp for $k \leqslant n/2$ as shown by the example $K_{1,n-1}$. They then gave the following conjecture:

Conjecture. If T is a tree of n vertices and $n/2 < k \leqslant n$, then $\gamma_{ks}(T) \leqslant 2k-n$.

Note that the upper bound in the conjecture is sharp as shown by the same example $K_{1,n-1}$. They gave some partial results which support the conjecture.

Theorem 1 (Cockayne and Mynhardt [3]). *Suppose T is an n-vertex tree rooted at v, where $\deg(v)=s$ and v has exactly t leaf neighbors; say $N(v)=\{w_1,\ldots,w_t, u_1,\ldots,u_{s-t}\}$ such that $w_1,\ldots,w_t$ are leaves and $2 \leqslant |V(T(u_1))| \leqslant \cdots \leqslant |V(T(u_{s-t}))|$, where $T(u)$ is the subtree of T induced by u and its descendants. If $r=\lceil s/2+1 \rceil \leqslant s-t$ and $n \geqslant k \geqslant |V(T(u_1))|+\cdots+|V(T(u_r))|$, then $\gamma_{ks}(T) \leqslant 2k-n$.*

Theorem 2 (Cockayne and Mynhardt [3]). *For any full m-ary tree of n vertices, $\gamma_{ks}(T) \leqslant 2k-n$ whenever $2\lceil (m+3)/2 \rceil \leqslant k \leqslant n$.*

The main result of this paper is to settle the conjecture. We also give a lower bound for $\gamma_{ks}(G)$ in terms of the degree sequence of G. This generalizes some previous results on the k-subdomination number $\gamma_{ks}(G)$, the signed domination number $\gamma_s(G)$ and the majority domination number $\gamma_{maj}(G)$.

2. Upper bound conjecture

We first establish the conjecture given by Cockayne and Mynhardt [3].

Theorem 3. *If T is a tree of n vertices and $n/2 < k \leqslant n$, then $\gamma_{ks}(T) \leqslant 2k-n$.*

Proof. We actually prove the stronger assertion that T has a *good* k-subdominating function g, which is one such that $g(V(T)) = 2k - n$ and there are exactly k *good vertices* that are vertices v with $g(v) = 1$ and $g(N_G[v]) \geqslant 1$. Suppose to the contrary that the assertion is not true. Choose a tree T with a minimum number of vertices having no good k-subdominating function. It is obvious that $k \leqslant n - 1$.

Claim 1. *The only neighbor of a leaf in* T *is of odd degree.*

Proof. Assume x is a leaf whose only neighbor y is of even degree. Let tree $T' = T - x$. Since $(n-1)/2 < k \leqslant n - 1$, the tree T' has a good k-subdominating function g' by the choice of T. Extend g' to $g: V(T) \to \{-1, 1\}$ by $g(v) = g'(v)$ for all $v \in V(T) - \{x\}$ and $g(x) = -1$. Then $g(N_T[v]) = g'(N_{T'}[v])$ for all $v \in V(T) - \{x, y\}$ and $g(N_T[y]) = g'(N_{T'}[y]) - 1$. Since y is of even degree in T, we have that $|N_{T'}[y]|$ is even. Consequently, $g'(N_{T'}[y]) \geqslant 1$ implies $g(N_T[y]) \geqslant 1$. Therefore, g is a good k-subdominating function of T, a contradiction. □

Choose a longest path P: $v_1 v_2 \ldots v_m$ in T. Note that $m \geqslant 4$, for otherwise T is a star which certainly has a good k-subdominating function as $n/2 < k$. Note that v_2 has exactly one non-leaf neighbor v_3 and $2a \geqslant 2$ leaf neighbors by Claim 1. Also, v_{m-1} has exactly one non-leaf neighbor v_{m-2} and $2b \geqslant 2$ leaf neighbors. We may assume $a \geqslant b$, otherwise reverse the path P. Now $m \geqslant 5$, for otherwise $m = 4$ which implies that $n = 2a + 2b + 2$ and $k > n/2 = a + b + 1$. Choose $S_a \subseteq N(v_2) - \{v_3\}$ and $S_b \subseteq N(v_3) - \{v_2\}$ with $|S_a| \geqslant a$, $|S_b| \geqslant b$ and $|S_a| + |S_b| = k - 2$. Then there exists a good k-subdominating function g of T such that $g(v) = 1$ for $v \in S_a \cup S_b \cup \{v_2, v_3\}$ and $g(v) = -1$ for all other vertices.

Claim 2. *The neighbors of* v_3 *not in* P *are leaves, and* $m \geqslant 6$.

Proof. Assume v_3 has a non-leaf neighbor x not in P or $m = 5$, in which case we set $x = v_4$. Since P is a longest path in T or $x = v_4$ (for $m = 5$), all neighbors of x are leaves except v_3. By Claim 1, assume that x has $2c \geqslant 2$ leaf neighbors. Moreover, we may assume $a \geqslant c$, for otherwise we just interchange the role of v_2 and x. Let tree $T' = T - (N_T[v_2] \cup N_T[x] - \{v_3\})$ have n' vertices and $k' = k - a - c - 1$. Then $n' = n - 2a - 2c - 2$ and $k' > n'/2$. If $k' > n'$, then $k \geqslant n - a - c$ and so T has a good k-subdominating function g such that $g(v) = 1$ for all vertices v except $g(v) = -1$ for at most a leaf neighbors v of v_2 and at most c leaf neighbors v of x, a contradiction. Now $n'/2 < k' \leqslant n'$. Then T' has a good k'-subdominating function g' by the choice of T. Let S be the vertex set containing v_2 and $a + c$ of its leaf neighbors. Extend g' to $g: V(T) \to \{-1, 1\}$ by $g(v) = g'(v)$ for all $v \in V(T')$, $g(v) = 1$ for $v \in S$ and $g(v) = -1$ for $v \in N_T[x] \cup N_T[v_2] - (S \cup \{v_3\})$. Then $g(N_T[v]) = 2$ for $v \in S - \{v_2\}$ and $g(N_T[v_2]) = a + c + 1 - (a - c) + g(v_3) \geqslant 2$. Also, since $g(v_2) = 1$ and $g(x) = -1$, we have $g(N_T[v_3]) = g'(N_{T'}[v_3])$ and so $g(N_T[v]) = g'(N_{T'}[v])$ for all $v \in V(T')$. Therefore, g has $k' + a + c + 1 = k$ good vertices, a contradiction. □

Claim 3. *The vertex* v_3 *(respectively,* v_{m-2}*) has at most one leaf neighbor.*

Proof. If v_3 has at least one leaf neighbor, then the number of such leaves is odd by Claim 1. Assume there are three leaves x, y and z in $N_T(v_3) - P$. Let tree $T' = T - (\{x, y, z\} \cup N_T[v_2] - \{v_3\})$ have n' vertices and $k' = k - a - 2$. Then $n' = n - 2a - 4$ and $k' > n'/2$. If $k' > n'$, then $k \geqslant n - a - 1$ and so T has a good k-subdominating function g such that $g(v) = 1$ for all vertices v except $g(x) = -1$ and $g(v) = -1$ for at most a leaf neighbors of v_2, a contradiction. Now $n'/2 < k' \leqslant n'$. Then T' has a good k'-subdominating function g' by the choice of T. Let S be the vertex set containing v_2 and $a + 1 - (g'(v_3) + 1)/2$ of its leaves. Extend g' to $g : V(T) \to \{-1, 1\}$ by $g(v) = g'(v)$ for $v \in V(T')$, $g(x) = g'(v_3)$, $g(v) = 1$ for $v \in S$, and $g(v) = -1$ for $v \in \{y, z\} \cup N[v_2] - (S \cup \{v_3\})$. Then $g(N_T[v]) = 2$ for $v \in S$. Since $g(N_T[v_3]) = g'(N_{T'}[v_3]) + g(v_2) + g(x) + g(y) + g(z) = g'(N_{T'}[v_3]) + g'(v_3) - 1$, we have $g(N_T[v_3]) = g'(N_{T'}[v_3]) \geqslant 1$ and $g(N_T[x]) = 2$ whenever $g'(v_3) = 1$. Therefore, g has $k' + |S| + (g'(v_3) + 1)/2 = k$ good vertices, where $(g'(v_3) + 1)/2$ is for vertex x, a contradiction. □

In the remainder, we shall give a good k-subdominating function of T to complete the proof. If v_3 has a unique leaf neighbor x not in P, then set $x' = v_3$; otherwise $N_T(v_3) = \{v_2, v_4\}$, in this case set $x = v_3$ and $x' = v_4$. If v_{m-2} has a unique leaf neighbor y not in P, then set $y' = v_{m-2}$; otherwise $N_T(v_{m-2}) = \{v_{m-1}, v_{m-3}\}$, in this case set $y = v_{m-2}$ and $y' = v_{m-3}$. Let tree $T' = T - ((N_T[v_2] \cup N_T[v_{m-1}] - \{v_3, v_{m-2}\}) \cup \{x, y\})$ have n' vertices and $k' = k - a - b - 2$. Then $n' = n - 2a - 2b - 4$ and $k' > n'/2$.

If $k' > n'$, then $k \geqslant n - a - b - 1$. For the case of $k \geqslant n - a - b$, T has a good k-subdominating function g such that $g(v) = 1$ for all vertices v except $g(v) = -1$ for at most a leaf neighbors v of v_2 and at most b leaf neighbors v of v_{m-1}. For the case of $k = n - a - b - 1$, T has a good k-subdominating function g such that $g(v) = 1$ for all vertices except $g(v) = -1$ for exactly $a - b$ leaf neighbors v of v_2 and all vertices v in $N[v_{m-1}] - \{v_{m-2}\}$.

Now consider the case when $n'/2 < k' \leqslant n'$. Then T' has a good k'-subdominating function g'. We construct a function g on $V(T)$ as follows. Let $g(v) = g'(v)$ for all $v \in V(T')$. If $g'(y') = 1$, then set $g(y) = i = 1$, otherwise set $g(y) = -1$ and $i = 0$. Let $g(v) = -1$ for all $v \in N_T[v_{m-1}] - \{v_{m-2}\}$. If $g'(x') = 1$, then set $g(x) = j = 1$, otherwise set $g(x) = -1$ and $j = 0$. If $i = j = 0$ and $v_3 \neq x$, then reset $g(v_3) = g(x) = i = j = 1$. If $i = j = 0$ and $v_3 = x$, then reset $g(v_3) = i = 1$. Let the g value of v_2 and $a + b + 1 - i - j \leqslant a + b$ leaves of $N_T(v_2)$ be 1 and the other leaves be -1. Since g preserves the property of g' that $g'(N_{T'}[v]) \geqslant 1$ for those $v \in V(T')$ with $g'(v) = 1$, and there are $a + b + 2$ vertices for which v is not in T' or $g'(v) = -1$, $g(v) = 1$ and $g(N_T[v]) \geqslant 1$, g is a good k-subdominating function of T. □

We have recently learned from the referee that Cockayne and Mynhardt's conjecture has independently been settled by Kang Li-ying, Shan Er-fang, and Cai Mao-cheng using different techniques.

3. Lower bound

This section establishes a lower bound for $\gamma_{ks}(G)$ in terms of the degree sequence by a simple argument. This generalizes some known results on $\gamma_{ks}(G)$, $\gamma_{\mathrm{maj}}(G)$ and $\gamma_{\mathrm{s}}(G)$, whose proofs were more involved.

Theorem 4. *If $G=(V,E)$ is a graph of order n with degree sequence $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$, then*

$$\gamma_{ks}(G) \geqslant -n + \frac{2}{d_n+1}\sum_{j=1}^{k}\left\lceil \frac{d_j+2}{2} \right\rceil.$$

Proof. Suppose g is an optimal k-subdominating function for G, say, $g(N_G[v]) \geqslant 1$ for k distinct vertices v in $\{v_{j_1}, v_{j_2}, \ldots, v_{j_k}\}$. Let $f(x)=(g(x)+1)/2$ for all vertices $x \in V$. Then f is a 0–1 valued function. First,

$$\sum_{i=1}^{k} f(N_G[v_{j_i}]) = \sum_{i=1}^{k}\left\lceil \frac{g(N_G[v_{j_i}])+d_{j_i}+1}{2} \right\rceil \geqslant \sum_{i=1}^{k}\left\lceil \frac{d_{j_i}+2}{2} \right\rceil \geqslant \sum_{j=1}^{k}\left\lceil \frac{d_j+2}{2} \right\rceil.$$

On the other hand,

$$\sum_{i=1}^{k} f(N_G[v_{j_i}]) \leqslant \sum_{j=1}^{n} f(N_G[v_j]) = \sum_{i=1}^{n} (d_i+1) f(v_i) \leqslant (d_n+1) f(V).$$

Therefore, $f(V) \geqslant 1/(d_n+1)\sum_{j=1}^{k}\lceil (d_j+2)/2 \rceil$ and so

$$\gamma_{ks}(G) = g(V) = 2f(V) - n \geqslant -n + \frac{2}{d_n+1}\sum_{j=1}^{k}\left\lceil \frac{d_j+2}{2} \right\rceil. \qquad \square$$

By setting $d_1 = d_2 = \cdots = d_n = r$ in Theorem 4, we have

Theorem 5 (Hattingh et al. [8]). *For every r-regular ($r \geqslant 2$) graph G of order n,*

$$\gamma_{ks}(G) \geqslant \begin{cases} k\frac{r+3}{r+1} - n & \text{if } r \text{ odd}, \\ k\frac{r+2}{r+1} - n & \text{if } r \text{ is even}. \end{cases}$$

Moreover, taking $k=n$ and $k=\lceil n/2 \rceil$, respectively, we have the following two theorems.

Theorem 6 (Dunbar et al. [4] and Henning et al. [11]). *For every r-regular ($r \geqslant 2$) graph G of order n,*

$$\gamma_{\mathrm{s}}(G) \geqslant \begin{cases} \frac{2n}{r+1} & \text{if } r \text{ odd}, \\ \frac{n}{r+1} & \text{if } r \text{ is even}. \end{cases}$$

Theorem 7 (Henning [9]). *For every r-regular* ($r \geqslant 2$) *graph G of order n*,

$$\gamma_{\mathrm{maj}}(G) \geqslant \begin{cases} \frac{(1-r)n}{2(r+1)} & \text{if } r \text{ odd}, \\ \frac{-rn}{2(r+1)} & \text{if } r \text{ is even}. \end{cases}$$

Corollary 8. *If G is a graph with n vertices, m edges and maximum degree* $\triangle$, *then*

$$\gamma_{ks}(G) \geqslant k - 2n + \frac{2m+n+k}{\Delta+1}.$$

Proof. According to Theorem 4, we have

$$\begin{aligned} \gamma_{ks}(G) &\geqslant -n + \frac{2}{d_n+1}\sum_{j=1}^{k}\left\lceil \frac{d_j+2}{2} \right\rceil \geqslant -n + \frac{2k + \sum_{j=1}^{k} d_j}{\Delta+1} \\ &= -n + \frac{2k + 2m - \sum_{j=k+1}^{n} d_j}{\Delta+1} \geqslant -n + \frac{2k+2m-(n-k)\Delta}{\Delta+1} \\ &= k - 2n + \frac{2m+n+k}{\Delta+1}. \qquad \square \end{aligned}$$

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