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A geometric construction of iterative functions of order three to solve nonlinear equations

Changbum Chun

School of Liberal Arts, Korea University of Technology and Education, Cheonan City, Chungnam 330-708, Republic of Korea

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Abstract

In this paper we consider a geometric construction of iteration functions of order three to develop cubically convergent iterative methods for solving nonlinear equations. This construction can be applied to any iteration function of order two to develop an iteration function of order three. Some examples are given of deriving several third-order iteration methods, and several numerical results follow to illustrate the performance of the derived methods.

Keywords: Newton's method; Iterative methods; Iteration function; Nonlinear equations; Order of convergence; Root-finding

1. Introduction

One of the most important problems in numerical analysis is that of solving nonlinear equations. In this paper, we consider iterative methods to find a simple root α, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, of a nonlinear equation $f(x) = 0$ that uses f and f' but not the higher derivatives of f. The case of multiple roots will not be considered.

Newton's method for the calculation of α is probably the most widely used iterative method, defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

It is well known [1] that this method is quadratically convergent.

In recent years, numerous third-order variants of Newton's method that do not require the computation of second derivatives have been proposed and analyzed for solving nonlinear equations; see [2–9] and the references therein. It is observed that most of these variants are constructed based on considering appropriate quadrature formulas for the computation of the integral involved. It has been shown that these methods are efficient in their performance, and can compete with Newton's method.

In this paper, we are also concerned with developing iterative methods with at least cubic convergence that do not require the computation of second derivatives. Unlike the approaches which were used in deriving existing methods, our approach is based on a simple geometric construction which will be described in detail in the following section,

E-mail address: cbchun@kut.ac.kr.

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and can use any second-order iterative method in deriving a third-order method. Several examples and a numerical experiment result are given to illustrate how our result is employed as a means of deriving cubically convergent methods, and to demonstrate the performance of the derived methods, respectively.

2. Main result

In this section, we construct iteration functions of order three from the given iteration functions of order two based on a geometric observation. In the sequel, whenever we mention that an iteration function ϕ is of order p, it means that the corresponding iterative method defined by

$$x_{n+1} = \phi(x_n), \quad n = 0, 1, 2, \ldots \tag{2}$$

is of convergence order p, that is, the error $|\alpha - x_{n+1}|$ is proportional to $|\alpha - x_n|^p$ as $n \to \infty$. It is well known [7] that if ϕ satisfies

$$\phi(\alpha) = \alpha; \qquad \phi^{(j)}(\alpha) = 0, \quad j = 1, 2, \ldots, p-1; \qquad \phi^{(p)}(\alpha) \neq 0, \tag{3}$$

and

$$M(\epsilon) := \max_{t \in I_\epsilon} |\phi'(t)| < 1, \tag{4}$$

where $I_\epsilon = \{x \in \mathbf{R} : |x - \alpha| \le \epsilon\}$, then the fixed point iteration (2) converges to α for any $x_0 \in I_\epsilon$, and ϕ is of order p. We refer the reader to [8] for further details about the order of an iteration function. For notational convenience, we indicate that ϕ is an iteration function whose order is p by writing $\phi \in I_p$.

Our proposed scheme is constructed geometrically as follows. Let x be a guess for α and let ϕ be an iteration function with $\phi \in I_2$. We consider the function $\bar{\phi}$ satisfying $\frac{\phi(x)+\bar{\phi}(x)}{2} = x$, that is, $\bar{\phi}(x) = 2x - \phi(x)$. Notice that x is the point which bisects that segment which lies between $[\phi(x), 0]$ and $[\bar{\phi}(x), 0]$. As the next guess for α, we consider the intersection $\psi(x)$ with the x-axis of the line through $(\bar{\phi}(x), f[\bar{\phi}(x)])$ which is parallel to the tangent line at $(x, f(x))$. This $\psi(x)$ can be written as follows.

$$\psi(x) = \bar{\phi}(x) - \frac{f[\bar{\phi}(x)]}{f'(x)}.$$

We now prove that the iteration function ψ constructed in this way is of order three.

Theorem 2.1. *Let $\alpha \in J$ be a simple zero of sufficiently differentiable function $f : J \to \mathbf{R}$ for an open interval J, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, and ϕ be an iteration function with $\phi \in I_2$, such that $\phi^{(3)}$ is continuous in a neighborhood of α. Let $\bar{\phi}$ be the function satisfying $\frac{\phi(x)+\bar{\phi}(x)}{2} = x$ and let*

$$\psi(x) = \bar{\phi}(x) - \frac{f[\bar{\phi}(x)]}{f'(x)}. \tag{5}$$

Then $\psi \in I_3$.

Proof. We will show that $\psi(\alpha) = \alpha, \psi'(\alpha) = \psi''(\alpha) = 0$. Observe that $\bar{\phi}(\alpha) = \alpha, \bar{\phi}'(\alpha) = 2; \bar{\phi}^{(j)}(\alpha) = -\phi^{(j)}(\alpha)$, $j \ge 2$. It is easy to verify that $\psi(\alpha) = \alpha$, $\psi'(\alpha) = 0$. Rewrite the definition of $\psi(x)$ as

$$f'(x)\psi(x) = f'(x)\bar{\phi}(x) - f[\bar{\phi}(x)]. \tag{6}$$

Let l be any integer. Differentiating (6) l times with respect to x yields

$$\sum_{j=0}^{l} C[l, j] f^{(l-j+1)}(x)\psi^{(j)}(x) = \sum_{j=0}^{l} C[l, j] f^{(l-j+1)}(x)\bar{\phi}^{(j)}(x) - \frac{\mathrm{d}^l f[\bar{\phi}(x)]}{\mathrm{d}x^l}, \tag{7}$$

where $C[l, j]$ is a binomial coefficient. Set $x = \alpha$ and $l = 2$ in (7). An elementary calculation shows that

$$f'(\alpha)\psi''(\alpha) = 4f''(\alpha) + f'(\alpha)\bar{\phi}''(\alpha) - \left.\frac{\mathrm{d}^2 f[\bar{\phi}(x)]}{\mathrm{d}x^l}\right|_{x=\alpha},$$

and

$$\left.\frac{\mathrm{d}^2 f[\bar{\phi}(x)]}{\mathrm{d}x^l}\right|_{x=\alpha} = 4f''(\alpha) + f'(\alpha)\bar{\phi}''(\alpha).$$

Hence

$$f'(\alpha)\psi''(\alpha) = 0.$$

Since $f'(\alpha) \neq 0$, we conclude that $\psi''(\alpha) = 0$. □

An elementary computation for $\psi^{(3)}(\alpha)$ yields

$$\psi^{(3)}(\alpha) = \frac{3f''(\alpha)\phi''(\alpha) - 2f^{(3)}(\alpha)}{f'(\alpha)}, \tag{8}$$

and, in general, $\psi^{(3)}(\alpha) \neq 0$. So, in most cases the iteration function defined by (5) is of order three.

3. Examples and numerical results

It should be noted that the result of Theorem 2.1 is independent of the structure of the iteration function of order two involved, so any iteration function of order two gives rise to an iteration function of order three. In this section several iteration functions of order two are considered to illustrate how Theorem 2.1 can be employed as a means of deriving cubically convergent methods. Some of the obtained methods are then compared with some of the existing third-order methods.

Example 3.1. Consider Newton's iteration function defined by $\phi(x) = x - f(x)/f'(x)$. In this case, $\bar{\phi}(x) = x + f(x)/f'(x)$. By Theorem 2.1 the iteration function ψ defined by

$$\psi(x) = x - \frac{f(x + f(x)/f'(x)) - f(x)}{f'(x)}$$

is of order three. Therefore we obtain the iterative method with third-order convergence given by

$$x_{n+1} = x_n - \frac{f(x_n + f(x_n)/f'(x_n)) - f(x_n)}{f'(x_n)}. \tag{9}$$

We remark that this method has also been derived by Kou et al. [9] by using Newton's theorem for the function f on a new interval of integration.

Example 3.2. Let $\phi(x) = x - f(x)/(f(x) + f'(x))$. This iteration function was derived in [10] and it was proved that it is of order two. Theorem 2.1 with $\bar{\phi}(x) = x + f(x)/(f(x) + f'(x))$ implies that the iteration function ψ defined by

$$\psi(x) = x + \frac{f(x)}{f(x) + f'(x)} - \frac{f(x + f(x)/(f(x) + f'(x)))}{f'(x)}$$

is of order three. Hence we obtain the new iterative method with third-order convergence given by

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}, \tag{10}$$

where $z_n = x_n + \frac{f(x_n)}{f(x_n) + f'(x_n)}$.

Example 3.3. Let $\phi(x) = x - f(x)f'(x)/(f^2(x) + f'^2(x))$. This is the iteration function due to Mamta et al. which converges quadratically [11]. In this case, $\bar{\phi}(x) = x + f(x)f'(x)/(f^2(x) + f'^2(x))$. By Theorem 2.1 the iteration function ψ defined by

$$\psi(x) = x + \frac{f(x)f'(x)}{f^2(x) + f'^2(x)} - \frac{f(x + f(x)f'(x)/(f^2(x) + f'^2(x)))}{f'(x)}$$

Table 1
Comparison of various third-order iterative methods and Newton's method

$f(x)$	IT						NFE					
	NM	WF	MM	HM	KM	CM	NM	WF	MM	HM	KM	CM
$f_1, x_0 = 0.8$	8	5	5	5	6	13	16	15	15	15	18	39
$f_2, x_0 = 2.3$	8	5	6	5	6	6	16	15	18	15	18	18
$f_3, x_0 = 1.5$	7	5	5	5	6	5	14	15	15	15	18	15
$f_4, x_0 = 1.7$	7	5	5	5	5	5	14	15	15	15	15	15
$f_5, x_0 = -2$	10	7	7	7	6	6	20	21	21	21	18	18
$f_6, x_0 = 4.7$	11	10	7	6	8	6	22	30	21	18	24	18
$f_7, x_0 = 3$	105	66	67	52	76	76	210	198	201	156	228	228

is of order three. Thus we obtain the new iterative method with third-order convergence given by

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}, \tag{11}$$

where $z_n = x_n + \frac{f(x_n)f'(x_n)}{f^2(x_n)+f'^2(x_n)}$.

It should be pointed out that per iteration the methods (9)–(11) require two function and one first-derivative evaluations, but they converge cubically. In a similar fashion to that in the above examples, one can continue to derive iterative methods with cubic convergence by making use of existing second-order methods as long as they are provided.

Finally, we present some numerical test results for various cubically convergent iterative schemes and the Newton method in Table 1. Compared were the Newton method (NM), the method of Weerakoon and Fernando (WF) defined by

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n - f(x_n)/f'(x_n))}, \tag{12}$$

the mid-point rule (MM) defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - f(x_n)/(2f'(x_n)))}, \tag{13}$$

the method of Homeier (HM) defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(x_n - f(x_n)/f'(x_n))}\right), \tag{14}$$

the methods of (9) (KM), and (10) (CM) newly obtained in the present contribution. We note that these methods do not require the computation of second derivatives to carry out iterations.

All computations were done using the MAPLE using 128-digit floating point arithmetic (Digits := 128). We accept an approximate solution rather than the exact root, depending on the precision (ϵ) of the computer. We use the following stopping criteria for computer programs: (i) $|x_{n+1} - x_n| < \epsilon$, (ii) $|f(x_{n+1})| < \epsilon$, and so, when the stopping criterion is satisfied, x_{n+1} is taken as the exact root α computed. We used $\epsilon = 10^{-32}$ and the same test functions as in Weerakoon and Fernando [2], and display the approximate zeros x_* found up to the 28th decimal places.

$$f_1(x) = x^3 + 4x^2 - 10, \qquad x_* = 1.3652300134140968457608068290,$$
$$f_2(x) = \sin^2 x - x^2 + 1, \qquad x_* = 1.4044916482153412260350868178,$$
$$f_3(x) = x^2 - \mathrm{e}^x - 3x + 2, \qquad x_* = 0.2575302854398607604553673049 4,$$
$$f_4(x) = \cos x - x, \qquad x_* = 0.7390851332151606416553120876 7,$$
$$f_5(x) = x\mathrm{e}^{x^2} - \sin^2 x + 3\cos x + 5, \qquad x_* = -1.2076478271309189270094167584,$$
$$f_6(x) = x^2\sin^2(x) + \mathrm{e}^{x^2\cos x \sin x} - 28, \qquad x_* = 4.6221041635528383439278532516,$$
$$f_7(x) = (x^3 + 4x^2 - 10)^2, \qquad x_* = 1.3652300134140968457608068290.$$

Displayed are the number of iterations to approximate the zero (IT), and the number of function evaluations (NFE) counted as the sum of the number of evaluations of the function itself plus the number of evaluations of the derivative. The results presented in Table 1 show that for most of the functions we tested, the methods introduced in the present presentation demonstrate at least equal performance compared to the other methods of order three in consideration, and converge more rapidly than Newton's method. It is clear that the function f_7 has a repeated zero, and in this case, it is noteworthy that all of the third-order methods under consideration also show linear convergence just as in Newton's method, which is well known.

4. Conclusion

In this work we considered a geometric construction of iteration functions of order three to develop cubically convergent iterative methods for solving nonlinear equations where in most cases the derived methods do not require the computation of second derivatives. The presented methods are compared in their performance with various cubically convergent iteration methods, and it is observed that they have at least equal performance. The result presented in this work can be continuously applied to developing the other cubically convergent iterative schemes.

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