Some Ostrowski–Grüss type inequalities and applications

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Abstract

New generalizations of Ostrowski–Grüss type inequalities for functions of Lipschitzian type are established. Applications for cumulative distribution functions are given.

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1. Introduction

In [1], Cheng gave a sharp version of the Ostrowski–Grüss type integral inequality in the following form.

\[ \left| f(x) - \left( x - \frac{a + b}{2} \right) \frac{f(b) - f(a)}{b - a} - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \frac{b - a}{8} (\Gamma - \gamma), \tag{1} \]

for all \( x \in [a, b] \).

In [2], Ujević provided new estimations of the left part of (1) as follows:

\[ \left| f(x) - \left( x - \frac{a + b}{2} \right) \frac{f(b) - f(a)}{b - a} - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \frac{b - a}{2} (S - \gamma), \tag{2} \]

and

\[ \left| f(x) - \left( x - \frac{a + b}{2} \right) \frac{f(b) - f(a)}{b - a} - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \frac{b - a}{2} (\Gamma - S), \tag{3} \]

where \( S = (f(b) - f(a))/(b - a) \).

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All (1)–(3) have been used to get the composite trapezoid and midpoint formulas in numerical integration, respectively.

In this paper, we will generalize Theorems 1 and 2 to functions of some larger classes. For convenience, we define functions of Lipschitzian type as follows:

**Definition 1.** The function \( f : [a, b] \rightarrow \mathbb{R} \) is said to be \( L \)-Lipschitzian on \([a, b]\) if
\[
|f(x) - f(y)| \leq L|x - y| \quad \text{for all } x, y \in [a, b],
\]
where \( L > 0 \) is given.

**Definition 2.** The function \( f : [a, b] \rightarrow \mathbb{R} \) is said to be \((l, L)\)-Lipschitzian on \([a, b]\) if
\[
l(x_2 - x_1) \leq f(x_2) - f(x_1) \leq L(x_2 - x_1) \quad \text{for } a \leq x_1 \leq x_2 \leq b,
\]
where \( l, L \in \mathbb{R} \) with \( l < L \).

We will need the following well-known results.

**Lemma 1.** Let \( h, g : [a, b] \rightarrow \mathbb{R} \) be such that \( h \) is Riemann-integrable on \([a, b]\) and \( g \) is \( L \)-Lipschitzian on \([a, b]\). Then
\[
\left| \int_a^b h(t)g(t) \, dt \right| \leq L \int_a^b |h(t)| \, dt.
\] (4)

**Lemma 2.** Let \( h, g : [a, b] \rightarrow \mathbb{R} \) be such that \( h \) is continuous on \([a, b]\) and \( g \) is of bounded variation on \([a, b]\). Then
\[
\left| \int_a^b h(t)g(t) \, dt \right| \leq \max_{t \in [a,b]} |h(t)| V_a^b(g).
\] (5)

The purpose of this paper is to generalize Theorems 1 and 2 to functions which are \( L \)-Lipschitzian and \((l, L)\)-Lipschitzian respectively. Applications for cumulative distribution functions are given.

2. Main results

**Theorem 3.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be \((l, L)\)-Lipschitzian on \([a, b]\). Then we have
\[
f(x) - \left( x - \frac{a + b}{2} \right) S - \frac{1}{b - a} \int_a^b f(t) \, dt \leq \frac{b - a}{8} (L - l),
\] (6)
\[
f(x) - \left( x - \frac{a + b}{2} \right) S - \frac{1}{b - a} \int_a^b f(t) \, dt \leq \frac{b - a}{2} (S - l)
\] (7)
and
\[
f(x) - \left( x - \frac{a + b}{2} \right) S - \frac{1}{b - a} \int_a^b f(t) \, dt \leq \frac{b - a}{2} (L - S),
\] (8)
for all \( x \in [a, b] \), where \( S = (f(b) - f(a))/(b - a) \).

**Proof.** Let us define the function
\[
K(x, t) := \begin{cases} t - a, & t \in [a, x], \\ t - b, & t \in (x, b]. \end{cases}
\]
Put
\[
g(t) := f(t) - \frac{L + l}{2} t.
\] (9)
It is easy to show that the function \( g : [a, b] \rightarrow \mathbb{R} \) is \( M \)-Lipschitzian on \([a, b]\) with \( M = \frac{L - l}{2} \). So, the Riemann–Stieltjes integral \( \int_a^b K(x, t)dg(t) \) exists. Using the integration by parts formula for Riemann–Stieltjes integrals, we have

\[
\frac{1}{b-a} \int_a^b K(x, t)dg(t) = g(x) - \frac{1}{b-a} \int_a^b g(t)dt. \tag{10}
\]

We also have

\[
\frac{1}{b-a} \int_a^b K(x, t)dt = x - \frac{a + b}{2} \tag{11}
\]

and

\[
\int_a^b dg(t) = g(b) - g(a). \tag{12}
\]

From (10)–(12), it follows that

\[
g(x) - \left( x - \frac{a + b}{2} \right) \frac{g(b) - g(a)}{b-a} - \frac{1}{b-a} \int_a^b g(t)dt
\]

\[
= \frac{1}{b-a} \int_a^b K(x, t)dg(t) - \frac{1}{(b-a)^2} \int_a^b dg(t) \int_a^b K(x, t)dt
\]

\[
= \frac{1}{b-a} \int_a^b \left[ K(x, t) - \frac{1}{b-a} \int_a^b K(x, s)ds \right] dg(t). \tag{13}
\]

From (4) of Lemma 1 we have

\[
\left| \int_a^b \left[ K(x, t) - \frac{1}{b-a} \int_a^b K(x, s)ds \right] dg(t) \right| \leq \frac{L - l}{2} \int_a^b \left| K(x, t) - \frac{1}{b-a} \int_a^b K(x, s)ds \right| dt. \tag{14}
\]

It is not difficult to find that

\[
\int_a^b \left| K(x, t) - \frac{1}{b-a} \int_a^b K(x, s)ds \right| dt = \frac{1}{4} (b-a)^2, \tag{15}
\]

and so from (13)–(15) we get

\[
\left| g(x) - \left( x - \frac{a + b}{2} \right) \frac{g(b) - g(a)}{b-a} - \frac{1}{b-a} \int_a^b g(t)dt \right| \leq \frac{b-a}{8} (L - l). \tag{16}
\]

Consequently, the inequality (6) follows from substituting (9) in the left hand side of the inequality (16).

Now we proceed to prove the inequalities (7) and (8).

Put \( g_1(t) := f(t) - lt \) and \( g_2(t) := f(t) - Lt \).

It is easy to show that both \( g_1, g_2 : [a, b] \rightarrow \mathbb{R} \) are functions of bounded variation on \([a, b]\) with

\[
V_a^b(g_1) = f(b) - f(a) - l(b-a) \quad \text{and} \quad V_a^b(g_2) = L(b-a) - [f(b) - f(a)]. \tag{18}
\]

So, the Riemann–Stieltjes integrals \( \int_a^b K(x, t)dg_1(t) \) and \( \int_a^b K(x, t)dg_2(t) \) exist. Using the integration by parts formula for Riemann–Stieltjes integrals, we have

\[
\frac{1}{b-a} \int_a^b K(x, t)dg_1(t) = g_1(x) - \frac{1}{b-a} \int_a^b g_1(t)dt \tag{19}
\]

and

\[
\frac{1}{b-a} \int_a^b K(x, t)dg_2(t) = g_2(x) - \frac{1}{b-a} \int_a^b g_2(t)dt. \tag{20}
\]
From (19), (20) and (11) we can easily derive that

\[ g_1(x) = \left( x - \frac{a + b}{2} \right) \frac{g_1(b) - g_1(a)}{b - a} - \frac{1}{b - a} \int_a^b g_1(t) \, dt \]

\[ = \frac{1}{b - a} \int_a^b \left[ K(x, t) - \frac{1}{b - a} \int_a^b K(x, s) \, ds \right] \, dg_1(t) \]

and

\[ g_2(x) = \left( x - \frac{a + b}{2} \right) \frac{g_2(b) - g_2(a)}{b - a} - \frac{1}{b - a} \int_a^b g_2(t) \, dt \]

\[ = \frac{1}{b - a} \int_a^b \left[ K(x, t) - \frac{1}{b - a} \int_a^b K(x, s) \, ds \right] \, dg_2(t). \]

Then by (5) of the Lemma 2 we can deduce that

\[ \left| g_1(x) - \left( x - \frac{a + b}{2} \right) \frac{g_1(b) - g_1(a)}{b - a} - \frac{1}{b - a} \int_a^b g_1(t) \, dt \right| \leq \frac{1}{b - a} \max_{t \in [a, b]} \left| K(x, t) - \left( x - \frac{a + b}{2} \right) \right| V_a^b (g_1) \]

and

\[ \left| g_2(x) - \left( x - \frac{a + b}{2} \right) \frac{g_2(b) - g_2(a)}{b - a} - \frac{1}{b - a} \int_a^b g_2(t) \, dt \right| \leq \frac{1}{b - a} \max_{t \in [a, b]} \left| K(x, t) - \left( x - \frac{a + b}{2} \right) \right| V_a^b (g_2). \]

Notice that

\[ \max_{t \in [a, b]} \left| K(x, t) - \left( x - \frac{a + b}{2} \right) \right| = \frac{b - a}{2} \]

and from (18), we get

\[ \left| g_1(x) - \left( x - \frac{a + b}{2} \right) \frac{g_1(b) - g_1(a)}{b - a} - \frac{1}{b - a} \int_a^b g_1(t) \, dt \right| \leq \frac{b - a}{2} (S - l) \tag{21} \]

and

\[ \left| g_2(x) - \left( x - \frac{a + b}{2} \right) \frac{g_2(b) - g_2(a)}{b - a} - \frac{1}{b - a} \int_a^b g_2(t) \, dt \right| \leq \frac{b - a}{2} (L - S). \tag{22} \]

Consequently, inequalities (7) and (8) follow from substituting (17) in the left hand sides of (21) and (22), respectively. □

**Corollary 1.** Under the assumptions of Theorem 3, we get midpoint inequalities

\[ \left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \frac{b - a}{8} (L - l), \]

\[ \left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \frac{b - a}{2} (S - l) \]

and

\[ \left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \frac{b - a}{2} (L - S). \]

**Proof.** We set \( x = (a + b)/2 \) in the above theorem. □
Corollary 2. Under the assumptions of Theorem 3, we get trapezoid inequalities

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{b-a}{8} (L - l),
\]
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{b-a}{2} (S - l)
\]

and

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{b-a}{2} (L - S).
\]

Proof. We set \( x = a \) or \( x = b \) in the above theorem. \(\square\)

Remark 1. It is clear that Theorem 3 can be regarded as a generalization of Theorems 1 and 2.

Theorem 4. Let \( f : [a, b] \to \mathbb{R} \) be \( L \)-Lipschitzian on \( [a, b] \). Then we have

\[
\left| f(x) - \left( x - \frac{a+b}{2} \right) S - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{b-a}{4} L
\]

where \( S = (f(b) - f(a))/(b-a) \).

Proof. We get inequality (23) immediately with

\[
\left| f(x) - \left( x - \frac{a+b}{2} \right) S - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{b-a}{2} (S + L)
\]

and

\[
\left| f(x) - \left( x - \frac{a+b}{2} \right) S - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{b-a}{2} (L - S)
\]

by taking \( l = -L \) in Theorem 3.

Consequently, the inequality (24) follows from (25) and (26) by considering the fact that \( \min\{S + L, L - S\} = L - |S| \). \(\square\)

3. Applications for cumulative distribution functions

Now we consider some applications for cumulative distribution functions.

Let \( X \) be a random variable having the probability density function \( f : [a, b] \to \mathbb{R}^+ \) and the cumulative distribution function \( F(x) = \Pr(X \leq x) \), i.e.,

\[
F(x) = \int_a^x f(t) \, dt, \quad x \in [a, b].
\]

\( E(X) \) is the expectation of \( X \). Then we have the following inequalities.

Theorem 5. With the above assumptions and if there exist constants \( M, m \) such that \( 0 \leq m \leq f(t) \leq M \) for all \( t \in [a, b] \), then we have the inequalities

\[
\left| \Pr(X \leq x) - \frac{1}{b-a} \left( x - \frac{a+b}{2} \right) - \frac{b - E(X)}{b-a} \right| \leq \frac{b-a}{8} (M - m),
\]
\[
\left| \Pr(X \leq x) - \frac{1}{b-a} \left( x - \frac{a+b}{2} \right) - \frac{b - E(X)}{b-a} \right| \leq \frac{b-a}{2} \left( \frac{1}{b-a} - m \right)
\]
and

\[
\left| P_r\left( X \leq x \right) - \frac{1}{b-a} \left( x - \frac{a+b}{2} \right) - \frac{b-E(X)}{b-a} \right| \leq \frac{b-a}{2} \left( M - \frac{1}{b-a} \right). \tag{29}
\]

**Proof.** It is easy to show that the function \( F(x) = \int_a^x f(t) \, dt \) is \((m, M)\)-Lipschitzian on \([a, b]\). So, by Theorem 3 we get

\[
\left| F(x) - \left( x - \frac{a+b}{2} \right) \frac{F(b) - F(a)}{b-a} - \int_a^b F(t) \, dt \right| \leq \frac{b-a}{8} (M-m),
\]

and

\[
\left| F(x) - \left( x - \frac{a+b}{2} \right) \frac{F(b) - F(a)}{b-a} - \int_a^b F(t) \, dt \right| \leq \frac{b-a}{2} (S-m),
\]

where \( S = \frac{F(b) - F(a)}{b-a} \).

As \( F(a) = 0, F(b) = 1 \), and

\[
\int_a^b F(t) \, dt = b - E(X),
\]

then we can easily deduce inequalities (27)–(29). \( \square \)

**Corollary 3.** Under the assumptions of Theorem 5, we have

\[
\left| E(X) - \frac{a+b}{2} \right| \leq \frac{(b-a)^2}{8} (M-m), \quad \tag{30}
\]

\[
\left| E(X) - \frac{a+b}{2} \right| \leq \frac{(b-a)^2}{2} \left( \frac{1}{b-a} - m \right) \quad \tag{31}
\]

and

\[
\left| E(X) - \frac{a+b}{2} \right| \leq \frac{(b-a)^2}{2} \left( M - \frac{1}{b-a} \right). \quad \tag{32}
\]

**Proof.** We set \( x = a \) or \( x = b \) in (27)–(29) to get (30)–(32). \( \square \)

**Remark 2.** It should be noted that the inequality (30) improves the inequality (5.4) in [3].

**Corollary 4.** Under the assumptions of Theorem 5, we have

\[
\left| P_r\left( X \leq \frac{a+b}{2} \right) - \frac{b-E(X)}{b-a} \right| \leq \frac{b-a}{8} (M-m), \quad \tag{33}
\]

\[
\left| P_r\left( X \leq \frac{a+b}{2} \right) - \frac{b-E(X)}{b-a} \right| \leq \frac{b-a}{2} \left( \frac{1}{b-a} - m \right) \quad \tag{34}
\]

and

\[
\left| P_r\left( X \leq \frac{a+b}{2} \right) - \frac{b-E(X)}{b-a} \right| \leq \frac{b-a}{2} \left( M - \frac{1}{b-a} \right). \quad \tag{35}
\]

**Proof.** We set \( x = \frac{a+b}{2} \) in (27)–(29) to get (33)–(35).
Corollary 5. Under the assumptions of Theorem 5, we have

\[ |Pr\left(X \leq \frac{a + b}{2}\right) - \frac{1}{2}| \leq \frac{b - a}{4} (M - m). \] (36)

\[ |Pr\left(X \leq \frac{a + b}{2}\right) - \frac{1}{2}| \leq \lfloor 1 - m(b - a) \rfloor \] (37)

and

\[ |Pr\left(X \leq \frac{a + b}{2}\right) - \frac{1}{2}| \leq \lfloor M(b - a) - 1 \rfloor. \] (38)

Proof. Using the triangle inequality, we get

\[ |Pr\left(X \leq \frac{a + b}{2}\right) - \frac{1}{2}| = |Pr\left(X \leq \frac{a + b}{2}\right) - \frac{1}{2} + \frac{1}{b - a} \left( E(X) - \frac{a + b}{2} \right) - \frac{1}{b - a} \left( E(X) - \frac{a + b}{2} \right) | \]

\[ \leq |Pr\left(X \leq \frac{a + b}{2}\right) - \left( \frac{b - E(X)}{b - a} \right) \frac{1}{b - a} | \bigg| E(X) - \frac{a + b}{2} \bigg|, \]

and then inequalities (36)–(38) follow from (30)–(32) and (33)–(35). \( \square \)

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