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The sharpness of some results on stable solutions of $-\Delta u = f(u)$ in $\mathbb{R}^N$

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ABSTRACT

In this note we give a complete answer to a question raised by Dupaigne and Farina (2009) [8] related to the existence of nonconstant stable solutions of the equation $-\Delta u = f(u)$ in $\mathbb{R}^N$, where $N \le 9$ and f is a very general non-negative, non-decreasing and convex nonlinearity.

1. Introduction

This paper deals with the stability of nonconstant solutions of

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $f \in C^1(\mathbb{R})$. We consider classical solutions $u \in C^2(\mathbb{R}^N)$.

A solution u of (1.1) is called stable if

$$\int_{\mathbb{R}^N} \left(|\nabla v|^2 - f'(u)v^2\right) dx \ge 0$$

for every $v \in C^\infty(\mathbb{R}^N)$ with compact support in $\mathbb{R}^N$. Note that the above expression is nothing but the second variation of the energy functional associated with (1.1) in a bounded domain Ω: $E_\Omega(u) = \int_\Omega \left(|\nabla u|^2/2 - F(u)\right) dx$, where $F' = f$. Thus, if $u \in C^1(\mathbb{R}^N)$ is a local minimizer of E_Ω for every bounded smooth domain $\Omega \subset \mathbb{R}^N$ (i.e., a minimizer under every small enough $C^1(\Omega)$ perturbation vanishing on $\partial\Omega$), then u is a stable solution of (1.1).

Stable radial solutions of (1.1) are well-understood: by the work of Cabré and Capella [1], refined by the second author [2], every bounded radial solution of (1.1) must be constant if $N \le 10$. Also, in these works there are examples of nonconstant bounded radial stable solutions for when $N \ge 11$. For dimensions $N \le 4$, Dupaigne and Farina [3] have obtained that every bounded stable solution of (1.1) must be constant if $f \ge 0$. For the case $N = 2$, Farina et al. [4] proved that any stable solution of (1.1) with bounded gradient is one-dimensional (i.e. up to a rotation of the space, u depends on only one variable). For every dimension N of the space, for the case of the nonlinearities $f(u) = |u|^{p-1}u$, $p > 1$, and $f(u) = e^u$, classification results have been obtained by Farina [5–7]. On the other hand Dupaigne and Farina [8] considered, in any dimension, the case of very general non-negative, non-decreasing and convex nonlinearities. Specifically they obtained:

Theorem 1.1 (*Dupaigne and Farina [8]*). *Let $I = (a, b) \subset \mathbb{R}$ be a maximal open interval, possibly unbounded, such that $0 \not\equiv f \in C^2(I; \mathbb{R}) \cap C^0(\bar{I}; \overline{\mathbb{R}})$ is non-negative, non-decreasing, convex in I and vanishes at some point of $\bar{I}$. Define*

$$q(u) := \frac{f'^2}{ff''}(u); \qquad \overline{q_0} = \limsup_{u \to z^+} q(u); \qquad \underline{q_0} = \liminf_{u \to z^+} q(u); \qquad \overline{q_\infty} = \limsup_{u \to b^-} q(u),$$

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where $z = \sup\left\{u \in \bar{I} = [a, b] \subset \overline{\mathbb{R}} : f(u) = 0\right\}$ *and* $b = \sup I$. *Assume that* $u \in C^2(\mathbb{R}^N)$ *is a stable solution of* (1.1). *Then, u is constant if* $N \le 2$ *and*

$$0 < \underline{q_0} \le \overline{q_0} < +\infty \quad \text{and} \quad 0 < \overline{q_\infty} < +\infty \tag{1.2}$$

or if $N \ge 3$ *and the following conditions hold:*

$$\overline{q_0} < +\infty \quad \text{and} \quad \frac{4}{N-2}\left(1 + 1/\sqrt{\overline{q_0}}\right) > 1/\underline{q_0}. \tag{1.3}$$

$$\overline{q_\infty} < +\infty \quad \text{and} \quad \frac{4}{N-2}\left(1 + 1/\sqrt{\overline{q_\infty}}\right) > 1/\overline{q_\infty}. \tag{1.4}$$

In [8, Remark 1.3] the authors showed that conditions (1.3) and (1.4) are sharp if $N \ge 10$; a counterexample is given by the nonlinearity $f(u) = e^u$ if $N = 10$ and by $f(u) = u^p$ (for certain $p > 1$) if $N \ge 11$. In this remark the authors also raised the question of whether conditions (1.2)–(1.4) are sharp in dimensions $1 \le N \le 9$.

In this paper we respond to this question. We show that, in Theorem 1.1, condition (1.2) is not sharp if $1 \le N \le 2$, while conditions (1.3) and (1.4) are sharp if $3 \le N \le 9$. More precisely we obtained the following results:

Theorem 1.2. *Let* $N \le 2$, $f \in C^1(\mathbb{R})$ *a non-decreasing function and u a stable solution of* (1.1). *Then f is constant in the interval* $J := u(\mathbb{R}^N)$.

As a corollary of this theorem we obtain the following result, which proves that condition (1.2) of Theorem 1.1 is not sharp in dimensions $1 \le N \le 2$.

Corollary 1.3. *Let* $N \le 2$, *and* $0 \not\equiv f \in C^1(\mathbb{R})$ *be a non-decreasing function vanishing at some point of* $\overline{\mathbb{R}}$. *Assume* $u \in C^2(\mathbb{R}^N)$ *is a stable solution of* (1.1). *Then u is constant.*

For dimensions $3 \le N \le 9$, the following result shows that conditions (1.3) and (1.4) of Theorem 1.1 are sharp, at least for the case $z = -\infty$. It would be interesting to find counterexamples for the case $z \in \mathbb{R}$.

Proposition 1.4. *Let* $3 \le N \le 9$ *and* $q > 0$, *satisfying*

$$\frac{4}{N-2}\left(1 + \frac{1}{\sqrt{q}}\right) \le \frac{1}{q}. \tag{1.5}$$

Then there exists $u_q \in C^\infty(\mathbb{R}^N)$ *with* $u_q(\mathbb{R}^N) = (-\infty, -1]$ *and* $f_q \in C^\infty(\mathbb{R})$ *such that* u_q *is a stable solution of* (1.1) *with* $f = f_q$, *and* f_q *satisfies* $f_q, f_q', f_q'' > 0$ *in* $\mathbb{R}$, $\lim_{u\to-\infty} f_q(u) = 0$ *and* $\overline{q_0} = \underline{q_0} = \lim_{u\to-\infty} \frac{f_q'^2}{f_q f_q''}(u) = q$.

Remark 1. Note that the number $q_\infty = \lim_{u\to+\infty} q(u)$ is not relevant, since $u_q(\mathbb{R}^N) = (-\infty, -1]$. In fact, it is a simple matter to obtain any value $q_\infty \in [1, +\infty]$ modifying appropriately the function f_q in $(1, +\infty)$.

2. Proof of the main results

To prove Theorem 1.2 we will need the lemma below. It has not appeared anywhere but it is essentially known. In fact, a similar result, using the same ideas as this lemma (a capacity test function), has been written in the case of the biharmonic operator (see e.g. [9, Theorem 6]).

Lemma 2.1. *Let* $N \le 2$ *and* $h \in L^1_{\text{loc}}\left(\mathbb{R}^N\right)$ *with* $h \ge 0$. *If*

$$\int_{\mathbb{R}^N} |\nabla w|^2\, dx \ge \int_{\mathbb{R}^N} h\, w^2\, dx, \quad \forall w \in C_c^\infty\left(\mathbb{R}^N\right), \tag{2.1}$$

then $h \equiv 0$.

Remark 2. Lemma 2.1 is optimal for dimensions $N = 1, 2$, but not for dimensions $N \ge 3$ due to the Hardy inequality: $\int_{\mathbb{R}^N} |\nabla w|^2\, dx \ge \int_{\mathbb{R}^N} \left((N-2)^2/(4\|x\|^2)\right) w^2\, dx$, for every $w \in C_c^\infty\left(\mathbb{R}^N\right)$.

Proof of Lemma 2.1. Let us first note that (2.1) remains true if we consider functions $w \in W_0^{1,p}(B(0, R))$ where $2 < p < \infty$, and $R > 0$.

Since $p > 2$ and $p > N$, we have that $W_0^{1,p}(B(0, R)) \subset \left(W_0^{1,2} \cap L^\infty\right)(B(0, R))$. Therefore the functional $w \mapsto \int_{B(0,R)} \left(|\nabla w|^2 - h w^2\right)$ is continuous in $W_0^{1,p}(B(0, R))$. The density of $C_c^\infty(B(0, R))$ in $W_0^{1,p}(B(0, R))$, ensures that (2.1) holds for any $w \in W_0^{1,p}(B(0, R))$.

Consider the following sequence of functions:

$$w_n(x) = \begin{cases} 1, & |x| \le n, \\ 2 - \dfrac{\ln|x|}{\ln n}, & n < |x| < n^2, \\ 0, & |x| \ge n^2. \end{cases}$$

It follows immediately that

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 \, dx \longrightarrow 0.$$

Hence, from (2.1), we deduce that

$$\int_{\mathbb{R}^N} h w_n^2 \, dx \longrightarrow 0.$$

Finally, since $\int_{\mathbb{R}^N} h w_n^2 \ge \int_{B(0,n)} h$ and $h \ge 0$ in $\mathbb{R}^N$, we conclude that $h \equiv 0$ in $\mathbb{R}^N$. □

Proof of Theorem 1.2. Applying the previous lemma with $h(x) = f'(u(x))$ we deduce that $f'(u(x)) = 0$ for every $x \in \mathbb{R}^N$. Thus $f'(s) = 0$ for every $s \in J$ and the theorem follows. □

Proof of Corollary 1.3. Applying Theorem 1.2 we can assert that $-\Delta u = C$ in $\mathbb{R}^N$ for some constant $C \in \mathbb{R}$. Let us consider the function

$$w(x) = u(x) + \frac{C}{2} x_1^2.$$

- Case $C > 0$.
  We claim that u is not bounded from below. This is obvious if w is constant. Otherwise, since w is harmonic it follows that w is not bounded from below. Therefore $u \le w$ is not bounded from below. Hence the interval $J \subset I$ is not bounded from below and $f(s) = C > 0$ in J. This contradicts our assumptions on f, which is non-decreasing and vanishing at some point of $\bar{I}$.
- Case $C < 0$.
  Like for the previous case we deduce that u is not bounded from above. Hence the interval $J \subset I$ is not bounded from above and $f(s) = C < 0$ in J. This contradicts again our assumptions on f, which is non-decreasing and vanishing at some point of $\bar{I}$.
- Case $C = 0$.
  In this case u is an harmonic function in $\mathbb{R}^N$. If u is not constant then u is neither bounded above nor bounded below. Thus $J = \mathbb{R}$ and $f \equiv 0$, contradicting our assumptions. We conclude that u must be constant. □

Proof of Proposition 1.4. First of all, it is easily seen that $3 \le N \le 9$, $q > 0$ and (1.5) imply that

$$0 < q \le \frac{N}{4} - \frac{\sqrt{N-1}}{2} < 1. \tag{2.2}$$

Define the radial function

$$u_q(x) = -\left(1 + |x|^2\right)^{1-q}$$

and

$$f_q(s) = \begin{cases} 4q(1-q)(-s)^{\frac{q+1}{q-1}} + 2(1-q)(N-2q)(-s)^{\frac{q}{q-1}}, & s \le -1, \\ g_q(s), & s > -1, \end{cases}$$

where $g_q(s)$ is chosen such that $f_q \in C^\infty(\mathbb{R})$ and $f, f_q, f_q'' > 0$ in $\mathbb{R}$. Since $q \in (0, 1)$, it is easy to check that $\lim_{u \to -\infty} f_q(u) = 0$ and $\overline{q_0} = \underline{q_0} = \lim_{u \to -\infty} \frac{f_q'^2}{f_q f_q''}(u) = q$.

It remains to prove that u_q is stable. For this purpose, taking into account (2.2), an easy computation shows that

$$f_q'\left(u_q(x)\right) = \frac{2q\left((N-2q)|x|^2 + (N+2)\right)}{\left(1+|x|^2\right)^2} < \frac{2q(N-2q)}{|x|^2} \le \frac{(N-2)^2}{4|x|^2} \quad \forall x \in \mathbb{R}^N$$

and, by the Hardy inequality, we conclude that u_q is stable. □

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References

[1] X. Cabré, A. Capella, On the stability of radial solutions of semilinear elliptic equations in all of $\mathbb{R}^n$, C. R. Math. Acad. Sci. Paris 338 (2004) 769–774.
[2] S. Villegas, Asymptotic behavior of stable radial solutions of semilinear elliptic equations in $\mathbb{R}^N$, J. Math. Pures Appl. 88 (2007) 241–250.
[3] L. Dupaigne, A. Farina, Stable solutions of $-\Delta u = f(u)$ in $\mathbb{R}^N$, J. Eur. Math. Soc. (JEMS) 12 (2010) 855–882.
[4] A. Farina, B. Sciunzi, E. Valdinoci, Bernstein and De Giorgi type problems: new results via a geometric approach, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 7 (2008) 741–791.
[5] A. Farina, Liouville-type results for solutions of $-\Delta u = |u|^{p-1}u$ on unbounded domains of $\mathbb{R}^N$, C. R. Math. Acad. Sci. Paris 341 (2005) 415–418.
[6] A. Farina, On the classification of solutions of the Lane–Emden equation on unbounded domains of $\mathbb{R}^N$, J. Math. Pures Appl. 87 (2007) 537–561.
[7] A. Farina, Stable solutions of $-\Delta u = e^u$ on $\mathbb{R}^N$, C. R. Math. Acad. Sci. Paris 345 (2007) 63–66.
[8] L. Dupaigne, A. Farina, Liouville theorems for stable solutions of semilinear elliptic equations with convex nonlinearities, Nonlinear Anal. 70 (2009) 2882–2888.
[9] G. Warnault, Liouville theorems for stable radial solutions for the biharmonic operator, Asymptot. Anal. 69 (2010) 87–98.