Abstract

Define for integer \( m \geq 1 \) and \( z \) a complex number with \( z^2, z^m \neq 1 \), the polynomial of degree \( m \)
\[
w_m(z; z) = (z + z)^m - (1 + z)^m,
\]
whose (simple) zeros can be seen as the Möbius transforms of the \( m \)th roots of unity.

In this paper it will be shown that \((0, m)\) Pál-type interpolation on the zeros of \( \{ w_{n+m}(z; z), w_n(z; z) \} \) is regular for all integers \( n, m \geq 1 \) and complex numbers \( z \) with \( z^2, z^m, z^n, z^{n+m} \neq 1 \).

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1. Introduction

Let \( \Pi_N \) denote the linear space of all polynomials with complex coefficients of degree at most \( N \). Given a pair of polynomials \( \{ A(z), B(z) \} \) with \( A(z) \in \Pi_s \), \( B(z) \in \Pi_n \), and having simple zeros, we shall study here the \((0, m)\) Pál-type problem of interpolation by polynomials \( P(z) \in \Pi_{s+n-1} \), which are required to take prescribed values at the zeros of \( A(z) \) and prescribed values of the \( m \)th derivative of \( P(z) \) at the zeros of \( B(z) \).

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We shall say that the problem of interpolation is regular if a unique polynomial \( P(z) \in \Pi_{s+n-1} \) can be determined to satisfy the foregoing conditions of interpolation. It can easily be seen that the Pál-type problem of \((0, m)\) interpolation on the zeros of the pair \(\{A(z), B(z)\}\) is regular, if and only if, for any \( P(z) \in \Pi_{s+n-1} \), the following interpolatory conditions

\[
P(z_k) = 0 \text{ for the zeros } z_k \text{ (}1 \leq k \leq s\text{) of } A(z),
\]

\[
P^{(m)}(w_j) = 0 \text{ for the zeros } w_j \text{ (}1 \leq j \leq n\text{) of } B(z)
\]

imply that \( P(z) \equiv 0 \).

For \( A(z) \equiv B(z) \), we have so called Hermite–Birkhoff interpolation (cf. the excellent book [8]; there are, however, authors that use this name for problems that do not have the same nodes for the prescribed values and of derivatives).

Study of Birkhoff interpolation problems on nonuniformly distributed nodes is of a more recent origin [2,4,5]. There are, however, few examples where lacunary problems (the orders of the derivatives for which data are given, are nonconsecutive) are regular (see [3]; for an overview [6]).

Along with the continuing interest in interpolation in general, a number of papers on Pál-type interpolation have appeared, cf. [1,7]: the case when the nodes for the values differ from those for the derivatives. This type of interpolation problems started with paper [9] by L.G. Pál in 1975.

After making a special choice for the node-generating polynomials, in Section 2 the new result will be stated, followed by the proof in Section 3. Finally a (short) list of references is given.

2. Main result

Consider for integer \( m \geq 1 \) and \( z \) a complex number the polynomial

\[
w_m(z; \alpha) = (z + \alpha)^m - (1 + \alpha z)^m.
\]

(3)

It is obvious that under the conditions \( \alpha^2, \alpha^m \neq 1 \) this polynomial has simple zeros and is of degree \( m \); the zeros can be seen as the Möbius transforms of the \( m \)th roots of unity.

Then the following theorem will be proved:

**Theorem 2.1.** Let \( n, m \geq 1 \) be integers and let \( z \) be a complex number with

\[
\alpha^2, \alpha^m, \alpha^n, \alpha^{n+m} \neq 1.
\]

Then the \((0, m)\) Pál-type interpolation problem on the zeros of the pair

\[
\{w_{n+m}(z; \alpha), w_n(z; \alpha)\}
\]

is regular.

**Remark.** It is obvious that \( z = 1 \) is a common zero of the two node-generating polynomials \( w_{n+m}(z; \alpha) \) and \( w_n(z; \alpha) \) for all values of \( n, m \geq 1 \). For \( m \) divisible by \( n \) the second polynomial is a divisor of the first and when \( n \) and \( m \) have a common divisor \( k \geq 2 \), there are also several common zeros.

Furthermore, the conditions \( \alpha^2, \alpha^n, \alpha^{n+m} \neq 1 \) ensure that the polynomials have degree equal to their index and that all zeros are simple; from the proof it will become clear that the condition \( \alpha^m \neq 1 \) finally ensures regularity.
3. Proof

As the proof for \( z = 0 \) is easier than the general situation, that value of \( z \) will be treated separately.

3.1. The case \( z \neq 0 \).

Let \( N = 2n + m - 1 \) and put \( P_N \) in the following form in order to satisfy (1):
\[
P_N(z) = w_{n+m}(z)Q(z)
\]  
with \( Q(z) \in \Pi_{n-1} \).

Introduce for nonnegative integers \( m, k \) the descending factorial by
\[
[m]_0 = 1;
[m]_k = m(m - 1)(m - 2) \cdots (m - k + 1), \quad k \geq 1.
\]  
(5)

Thus \( [m]_k = 0 \) for \( m < k \). Furthermore, in the sequel, empty sums will be taken 0 and empty products 1.

In order to derive a condition for the unknown polynomial \( Q(z) \) using (2), we have to calculate the \( m \)th derivative with Leibniz’ formula:
\[
P^{(m)}(z) = \sum_{k=0}^{m} \binom{m}{k} [n + m]_k \{(z + \alpha)^{n+m-k} - \alpha^k (1 + \alpha z)^{n+m-k}\}Q^{(m-k)}(z).
\]  
(6)

Now according to (2) the value of the left-hand side of (6) is zero for \( z = z_j \), where \( z_j \) runs through the \( n \) zeros of the polynomial \( w_n(\alpha; z) \). For these zeros we have
\[
(1 + \alpha z_j)^{n+m-k} = (z_j + \alpha)^n (1 + \alpha z_j)^{m-k}, \quad 0 \leq k \leq m, \quad 1 \leq j \leq n,
\]
and we find for \( j = 1, 2, \ldots, n \), after division by \( z_j + \alpha \neq 0 \):
\[
\sum_{k=0}^{m} \binom{m}{k} [n + m]_k \{(z_j + \alpha)^{m-k} - \alpha^k (1 + \alpha z_j)^{m-k}\}Q^{(m-k)}(z_j) = 0.
\]  
(7)

Dropping the index \( j \) on \( z \) on the left-hand side, we see that Eq. (7) implies that a polynomial of degree \( n - 1 \) has at least \( n \) zeros, and thus has to be identically zero. This leads to a differential equation for \( Q(z) \):
\[
\sum_{k=0}^{m} \binom{m}{k} [n + m]_k \{(z + \alpha)^{m-k} - \alpha^k (1 + \alpha z)^{m-k}\}Q^{(m-k)}(z) = 0.
\]  
(8)

Put
\[
Q(z) = \sum_{j=0}^{n-1} c_j z^j,
\]  
(9)

then
\[
Q^{(k)}(z) = \sum_{j=0}^{n-1} [j]_k c_j z^{j-k}, \quad 0 \leq k \leq m.
\]  
(10)
Remark. Because of (5), in this form (10) the summation automatically starts with \( j = k \) and it holds even for \( k > n \) without leading to negative powers of \( z \).

Now replace \( k \) by \( m - k \) in (8) and write the form between brackets as a sum

\[
(z + x)^k - x^{m-k}(1 + x z)^k = \sum_{i=0}^{k} a_i^{[k]} z^i
\]

with

\[
a_i^{[k]} = \binom{k}{i} (x^{k-i} - x^{m-k+i}), \quad 0 \leq i \leq k \leq m. (12)
\]

Put (10), (11) and (12) into (8) to obtain a condition in the form of a vanishing triple sum:

\[
\sum_{j=0}^{n-1} \sum_{k=0}^{m-1} \sum_{i=0}^{k} \binom{m}{k} [n + m]_{m-k} j_k a_i^{[k]} c_j z^{j+i-k} \equiv 0. (13)
\]

First write in this triple sum \( r = k - i \):

\[
\sum_{j=0}^{n-1} \sum_{k=0}^{m-1} \sum_{r=0}^{k} \binom{m}{k} [n + m]_{m-k} j_k a_i^{[k]} c_j z^{j-r} \equiv 0, (14)
\]

and then interchange the two inner summations:

\[
\sum_{j=0}^{n-1} \sum_{k=0}^{m-1} \sum_{r=0}^{k} \binom{m}{k} [n + m]_{m-k} j_k a_i^{[k]} c_j z^{j-i} \equiv 0. (15)
\]

Because \([j]_k = 0\) for \( j < k \) and the sum over \( k \) starts with \( k = r \), the sum over \( j \) has to start with at least \( j = r \). Then take the summation over \( r \) as outer sum and write \( s = j - r \):

\[
\sum_{r=0}^{m} \sum_{s=0}^{n-1-r} \left[ \sum_{k=r}^{m} \binom{m}{k} [n + m]_{m-k} [r + s] k a_i^{[k]} \right] c_{r+s} z^s \equiv 0. (16)
\]

To make this more manageable (as system of linear, homogenous equations for the unknowns \( c_j \)) the inner sum, denote it by \( S_{r,s} \), will be calculated explicitly, using the \( a_i^{[k]} \) from (12):

\[
S_{r,s} = \sum_{k=r}^{m} \binom{m}{k} [n + m]_{m-k} [r + s] k a_i^{[k]} = \sum_{k=r}^{m} \binom{m}{k} [n + m]_{m-k} [r + s] k \binom{k}{k-r} (x^{k-r} - x^{m-k+(k-r)}).
\]

Therefore, the condition becomes:

\[
\sum_{r=0}^{m} \sum_{s=0}^{n-1-r} \left[ \sum_{k=r}^{m} \binom{m}{k} [n + m]_{m-k} [r + s] k a_i^{[k]} \right] c_{r+s} z^s \equiv 0. (17)
\]
As the inner sum is empty for \( n \), we find the ascending factorial \((a)_k\) given by \((a)_0 = 1\), \((a)_k = a(a + 1) \cdots (a + k - 1)\), \(k \geq 1\), a form containing a terminating hypergeometric series of \( _2F_1 \)-type:

\[
S_{r,s} = \binom{m}{m-r}[n + m]_{n-r}[s + r][x^r - x^{n-r}] \sum_{t=0}^{m-r} \frac{(-s)_t(-m+r)_t}{(n+r+1)_t t!}.
\]  

(18)

According to the Chu–Vandermonde theorem

\[
_2F_1 \left( \begin{array}{c}
-m + r, -s \\
n + r + 1
\end{array} \right) 1 = \frac{((n + r + 1) - (-s))_{m-r}}{(n + r + 1)_{m-r}} = \frac{[n + m + s]_{m-r}}{[n + m]_{m-r}},
\]

and inserting this with (18) into (16) the final result is

\[
\sum_{r=0}^{m} \sum_{s=0}^{n-1-r} \binom{m}{r}[s + r][n + m + s]_{m-r}(x^r - x^{n-r})c_{s+r}z_s \equiv 0.
\]  

(19)

As the inner sum is empty for \( n - 1 - r < 0 \), we see that we have to distinguish between \( m \geq n \) and \( m < n \).

(A) The case \( m < n \): In this case always \( r \leq n - 1 \) and \( r \) can run through its full range in Eq. (19).

Interchange the order of summation to find

\[
\sum_{s=0}^{n-1} \sum_{r=0}^{m} \binom{m}{r}[s + r][n + m + s]_{m-r}(x^r - x^{n-r})c_{s+r}z_s \equiv 0.
\]

(20)

Equating the coefficients of \( z^s \) to 0, we arrive at the following set of equations for the unknown \( c \)'s:

\[
\sum_{r=0}^{m} \binom{m}{r}[s + r][n + m + s]_{m-r}(x^r - x^{n-r})c_{s+r} = 0, \quad \text{for } s = 0, 1, \ldots, n - m - 1,
\]  

(21)

\[
\sum_{r=0}^{n-1-s} \binom{m}{r}[s + r][n + m + s]_{m-r}(x^r - x^{n-r})c_{s+r} = 0, \quad \text{for } s = n-m, n-m+1, \ldots, n-1.
\]  

(22)

Writing Eq. (22) for the values \( s=n, n-1, \ldots, n-m \), we have a triangular system of homogeneous linear equations for \( c_{n-1}, c_{n-2}, \ldots, c_{n-m} \), where the “new” unknown appearing at each step has coefficient following from \( r = 0 \):

\[
[n + m + s]_m(1 - x^m) \neq 0,
\]

showing

\[
c_{n-1} = c_{n-2} = \cdots = c_{n-m} = 0.
\]  

(23)
In Eq. (21) the $c$'s appearing are
\[
s = n - m - 1 : \quad c_{n-m-1}, c_{n-m}, \ldots, c_{n-1},
\]
\[
s = n - m - 2 : \quad c_{n-m-2}, c_{n-m-1}, \ldots, c_{n-2},
\]
\[
\vdots
\]
\[
s = 1 : \quad c_1, c_2, \ldots, c_{m+1},
\]
\[
s = 0 : \quad c_0, c_1, \ldots, c_m,
\]
and the coefficient of the “new” $c$ is $[n + 1 + s]m(1 - z^m) \neq 0$ for each step, while the “old” $c$'s already are equal to 0, thus
\[
c_{n-m-1} = c_{n-m-2} = \cdots = c_0 = 0.
\] (24)

Eqs. (24) and (23) together imply that the interpolation problem is regular.

(B) The case $m \geq n$: In this case the sum over $r$ in Eq. (19) has to run from 0 to $n - 1$ as the values $r \geq n$ lead to an empty inner sum. Interchanging the order of summation and equating the coefficients of $z^s$ to zero, we find
\[
\sum_{r=0}^{n-1-s} \binom{m}{s+r} [n + m + s]_{m-r}(z^r - z^{m-r})c_{s+r} = 0, \quad \text{for } s = 0, 1, \ldots, n - 1.
\] (25)

We can then proceed as for Eq. (21) and we find a triangular system for the $c$'s with diagonal coefficients $[n + m + s]_m(1 - z^m) \neq 0$ for $s = 0, 1, \ldots, n - 1$, leading to
\[
c_{n-1} = c_{n-2} = \cdots = c_0.
\] (26)

But this shows that the interpolation problem is regular in this case too.

3.2. The case $z = 0$.

The node generating polynomials now simplify to polynomials generating roots of unity:
\[
w_{n+m}(z) = z^{n+m} - 1, \quad w_n(z) = z^n - 1.
\]

The interpolating polynomial, on account of (1), takes the form
\[
P(z) = (z^{n+m} - 1) \sum_{j=0}^{n-1} c_j z^j.
\] (27)

On account of (2) the coefficients $c_j$ follows from
\[
\sum_{j=0}^{n-1} [n + m + j]_m c_j z_i^{n+j} - \sum_{j=m}^{n-1} [j]_m c_j z_i^{j-m} = 0 \quad (1 \leq i \leq n),
\] (28)

where $z_i$ are the $n$th roots of unity: $z_i^n = 1$.

Now when $m \geq n$, the second sum in (28) has to be omitted and the remaining expression exhibits a polynomial of degree $n - 1$ (drop the index $i$ on $z$) having $n$ zeros. This immediately implies that all $c_j$ are zero or, equivalently, that the interpolation problem is regular.
In the case $m < n$, conditions (28) can be written as (shift the index in the second sum)

$$\sum_{j=0}^{n-m-1} ([n + m + j]_m c_j - [m + j]_{m+j} z_i^j) + \sum_{j=n-m}^{n-1} [n + m + j]_m c_j z_i^j = 0 \quad (1 \leq i \leq n).$$  \(29\)

As in the case $m \geq n$, this implies

$$[n + m + j]_m c_j - [m + j]_{m+j} = 0, \quad 0 \leq j \leq n - m - 1$$  \(30\)

and

$$[n + m + j]_m c_j = 0, \quad n - m \leq j \leq n - 1.$$  \(31\)

Eq. (31) immediately given rise to

$$c_j = 0, \quad n - m \leq j \leq n - 1.$$  \(32\)

Then Eq. (30) shows that (32) implies

$$c_j = 0, \quad n - 2m \leq j \leq n - m - 1.$$  \(33\)

Thus another $k$-fold application of this lowering method, with $k$ the integer part of $n/m$, shows that all coefficients $c_j$ are zero, leading to regularity.

References