The Generalized Spectral Matrices and Radial Matrices on Symmetry Classes of Tensors

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ABSTRACT

Let $A, C$ be $n \times n$ complex matrices. We denote by $\lambda_1, \ldots, \lambda_n; \gamma_1, \ldots, \gamma_n$ the eigenvalues of $A$ and $C$ respectively, and define the following quantities:

$$p^*(A) = \max \left\{ \left\| \text{tr} K \left( \begin{array}{ccc} \gamma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \gamma_n \end{array} \right) P \left( \begin{array}{ccc} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{array} \right) P' \right\| \right\},$$

where $P$ is a permutation matrix,

$$r^*(A) = \max \left\{ |\text{tr} K(CU^*AU)| \mid U \text{ is unitary} \right\},$$

and

$$a^*(A) = \max \left\{ |\text{tr} K(CUAV)| \mid U, V \text{ are unitary} \right\},$$

where $K(A)$ denotes the induced matrix defined by $A$. The relation

$$p^*(A) \leq r^*(A) \leq a^*(A)$$

is obtained. For $C = \text{diag}(1,0,\ldots,0)$ and $K(A) = A$, it reduces to the classical relation

$$\rho(A) \leq r(A) \leq \|A\|.$$  

In this note, we investigate the matrices for which $p^*(A) = a^*(A)$, $r^*(A) = a^*(A)$, or $p^*(A) = r^*(A)$.

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1. INTRODUCTION

Let $n, m$ be positive integers, and denote by $\mathbb{C}^n$ the vector space of complex $n$-tuples; $H$ a subgroup of $S_m$, the group of permutations of the set $\{1, \ldots, m\}$; $\chi : H \to \mathbb{C}$ a character of degree 1 on $H$; and $\otimes^m \mathbb{C}^n$ the $m$th tensor power of $\mathbb{C}^n$. The vector space $\mathbb{C}^m(H) = S(\otimes^m \mathbb{C}^n)$, where $S(\otimes^m \mathbb{C}^n) = (1/|H|)\sum_{\sigma \in H} \chi(\sigma) \otimes \cdots \otimes \otimes \chi(\sigma)$, is a subspace of $\otimes^m \mathbb{C}^n$ called the symmetry class of tensors over $\mathbb{C}^n$ associated with $H$ and $\chi$ [15]. We also denote by $\mathbb{C}_{n \times n}$ the vector space of $n \times n$ complex matrices, and by $\mathbb{U}_n$ the group of $n \times n$ unitary matrices. Throughout the note, we may assume that $A, C \in \mathbb{C}_{n \times n}$ and denote by $\lambda_1, \ldots, \lambda_n (|\lambda_1| \geq \cdots \geq |\lambda_n|), \gamma_1, \ldots, \gamma_n (|\gamma_1| \geq \cdots \geq |\gamma_n|)$ the eigenvalues of $A$ and $C$ respectively, and by $s_1 \geq \cdots \geq s_n, t_1 \geq \cdots \geq t_n$ the singular values of $A$ and $C$ respectively, without any specification. If $A \in \mathbb{C}_{n \times n}$, the induced matrix defined by $A$ is denoted by $K(A)$ and we define the following quantities:

$$\rho^*_\chi(A) = \max \left\{ \left| \operatorname{tr} K \left[ \begin{array}{cc} \gamma_1 & 0 \\ \vdots & \ddots \\ 0 & \gamma_n \end{array} \right] \right| P \in S_n \right\},$$

$$r^*_\chi(A) = \max \left\{ \left| \operatorname{tr} K \left( \begin{array}{cc} \lambda_1 & 0 \\ \vdots & \ddots \\ 0 & \lambda_n \end{array} \right) \right| P \in S_n \right\},$$

and

$$\sigma^*_\chi(A) = \max \left\{ \left| \operatorname{tr} K \left( \begin{array}{cc} \gamma_1 & 0 \\ \vdots & \ddots \\ 0 & \gamma_n \end{array} \right) \right| P \in S_n \right\}.$$

By Schur’s triangularization theorem, we have $\rho^*_\chi(A) \leq r^*_\chi(A)$. The inequality $r^*_\chi(A) \leq \sigma^*_\chi(A)$ is obvious. Then we obtain the relation

$$\rho^*_\chi(A) \leq r^*_\chi(A) \leq \sigma^*_\chi(A).$$

We remark that the three quantities possess the following properties:

$$\rho^*_\chi(U*AU) = \rho^*_\chi(A), \quad \rho^*_\chi(A) = \rho^*_\chi(C),$$

$$r^*_\chi(U*AU) = r^*_\chi(A), \quad r^*_\chi(A) = r^*_\chi(C),$$

$$\sigma^*_\chi(U*AU) = \sigma^*_\chi(A), \quad \sigma^*_\chi(A) = \sigma^*_\chi(C)$$

for all $U \in \mathbb{U}_n, A, C \in \mathbb{C}_{n \times n}$. 

When \( C = \text{diag}(1, 0, \ldots, 0) \) and \( m = 1 \), i.e. \( K(A) = A \) for all \( A \in \mathbb{C}_{n \times n} \), the relation (4) reduces to the classical case

\[
\rho(A) \leq r(A) \leq \|A\|,
\]

where \( \rho(A) \), \( r(A) \), and \( \|A\| \) are the spectral radius, numerical radius, and spectral norm respectively. Furuta, Natamoto, Tokeda, Goldberg, Tadmor, Zwas, Pták, and others [2, 3, 5, 7, 8, 13, 18] have investigated the matrices for which \( \rho(A) = r(A) \), \( r(A) = \|A\| \), or \( \rho(A) = \|A\| \). It turns out that the equality \( \rho(A) = \|A\| \) is equivalent to the equality \( r(A) = \|A\| \), and such \( A \in \mathbb{C}_{n \times n} \) are said to be radial. Those matrices \( A \in \mathbb{C}_{n \times n} \) satisfying \( \rho(A) = r(A) \) are said to be spectral. For \( m = 1 \), the relation (4) reduces to

\[
\rho_c(A) < r_c(A) < \|A\|_c.
\]

Li, Tam, and Tsing [14] have investigated the matrices for which \( \rho_c(A) = r_c(A) \), \( r_c(A) = \|A\|_c \), or \( \rho_c(A) = \|A\|_c \). When \( K(A) = C_n(A) \), the \( m \)th compound of \( A \), for all \( A \in \mathbb{C}_{n \times n} \), and when \( C = \text{diag}(1, \ldots, 1, 0, \ldots, 0) \) with \( \text{tr} C = m \), Andresen, Marcus, and Li [1, 12, 16] have investigated the matrices for which \( \rho^*(A) = r^*(A) \), \( r^*(A) = \sigma_\lambda^*(A) \), or \( \rho^*(A) = \sigma_\lambda^*(A) \). Later the authors investigated the problem when \( C = \text{diag}(1, \ldots, 1, 0, \ldots, 0) \) with \( \text{tr} C = k \), \( k > m \) [20]. In this note, we investigate the problems for the general setting. We say that \( A \in \mathbb{C}_{n \times n} \) is generalized \( C \)-radial if \( \rho_c(A) = \sigma_\lambda^*(A) \), and \( A \in \mathbb{C}_{n \times n} \) is generalized \( C \)-spectral if \( \rho_c(A) = r_c(A) \).

Section 2 contains the preliminaries and some lemmas which will be used subsequently.

In Section 3 we investigate the matrices for which \( \rho^*(A) = \sigma_\lambda^*(A) \) or \( r^*(A) = \sigma_\lambda^*(A) \). It turns out that they are different concepts in general, and various examples are given.

In Section 4 we study the generalized \( C \)-spectral matrices when the symmetry class of tensors is the exterior space.

2. PRELIMINARY AND SOME LEMMAS

In order to formulate our problems, we summarize some materials in [15].

Let \( \Gamma_{m,n} \) be the set of sequences \( \alpha = (\alpha(1), \ldots, \alpha(m)) \), \( 1 \leq \alpha(i) \leq n \), \( i = 1, \ldots, m \). For \( t = 1, \ldots, n \) and \( \alpha \in \Gamma_{m,n} \), let \( m_t(\alpha) \) be the number of times the integer \( t \) appears in \( \alpha \). Two sequences \( \alpha \) and \( \beta \) in \( \Gamma_{m,n} \) are said to be
equivalent modulo $H$, $\alpha \overset{H}{\sim} \beta$, if there exists a $\sigma \in H$ such that $\alpha = \beta \sigma$. The fact that $H$ is a group immediately implies that the relation $\overset{H}{\sim}$ is an equivalence relation and therefore partitions $\Gamma_{m,n}$ into equivalence classes. Let $\Delta$ denote a system of representations for this equivalence relation, so chosen that each sequence in $\Delta$ is first in lexicographic order in its equivalence class. Define $\tilde{\Delta}$ as the subset of $\Delta$ consisting of those sequences $\omega \in \Delta$ for which $\nu(\omega) = \sum_{\sigma \in H} \chi(\sigma) \neq 0$, where $H_\omega$ is the stabilizer of $\omega$, i.e. $H_\omega = \{ \sigma \in H \mid \omega \sigma = \omega \}$.

The generalized matrix function associated with $H$ and $\chi$ is the scalar valued map defined on $\mathbb{C}_{m \times m}$ by the formula

$$d^H_\chi(X) = \sum_{\theta \in H} \chi(\theta) \prod_{i=1}^{m} x_{i \theta(i)}$$

for $X \in \mathbb{C}_{m \times m}$. Let $A \in \mathbb{C}_{n \times n}$. The induced matrix defined by $A$, denoted by $K(A)$, is the matrix whose $\alpha, \beta$ entry is

$$K(A)_{\alpha, \beta} = \frac{1}{\nu(\alpha) \nu(\beta)} d^H_\chi(A' [\beta | \alpha]), \quad \alpha, \beta \in \tilde{\Delta}.$$

For example,

(a) If $H = \{e\}$ and $\chi$ is the identity map, then $\Delta = \tilde{\Delta} = \Gamma_{m,n}$; $\mathbb{C}^m(H) = \bigotimes_1^{m} \mathbb{C}_1^n$, $d^H_\chi(X) = \prod_{i=1}^{m} x_{i i}$; $K(A) = \bigotimes_1^{m} A$, the $m$th tensor power of $A$.

(b) If $H = S_m$ and $\chi$ is the identity map, then $\Delta = \tilde{\Delta} = \Gamma_{m,n}$, the subset of $\Gamma_{m,n}$ consisting of all nondecreasing sequences; $\mathbb{C}^m(\chi) = \mathbb{C}^{m(m+1)}$, the $m$th completely symmetric space over $\mathbb{C}^n$; $d^H_\chi(X) = \per X$; $K(A) = P_m(A)$, the $m$th induced power of $A$.

(c) If $H = S_m$ and $\chi = \text{sgn}$, then $\Delta = \tilde{\Delta} = Q_{m,n}$, the subset of $G_{m,n}$ consisting of all strictly increasing sequences; $\mathbb{C}^m(\chi) = \wedge_1^{m} \mathbb{C}_1^n$, the $m$th exterior space over $\mathbb{C}^n$; $d^H_\chi(X) = \det X$; $K(A) = C_m(A)$, the $m$th compound of $A$.

We also adopt the notation that $\rho_\chi(A) = \rho_\chi(A)$ $[\rho_\chi(A)]$, $\tau_\chi(A) = \tau_\chi(A)$ $[\tau_\chi(A)]$, and $\sigma_\chi(A) = \sigma_\chi(A)$ $[\sigma_\chi(A)]$ if $H = S_m$ and $\chi = \text{sgn}$ $[H$ is a subgroup of $S_m$ and $\chi = \text{id}]$.

The following lemma is a collection of some well-known results which can be found in [15].
LEMMA 1.

(a) If $\lambda_1, \ldots, \lambda_n$, $s_1, \ldots, s_n$ are the eigenvalues and the singular values of $A \in \mathbb{C}^{n \times n}$ respectively, then $\prod_{i=1}^{n} \lambda_i^{m_i(\alpha)} \prod_{i=1}^{n} s_i^{m_i(\alpha)}$, $\alpha \in \Delta$, are the eigenvalues and the singular values of $K(A)$ respectively.

(b) If $A, B \in \mathbb{C}^{n \times n}$, then $K(AB) = K(A)K(B)$, $K(A^*) = K(A)^*$. 

(c) Let $A \in \mathbb{C}^{n \times n}$, and assume that $\text{rank } A > n$. If $K(A)$ is p.s.d. (positive semidefinite), then there exists $\theta \in \mathbb{R}$ such that $e^{i\theta}A$ is p.s.d.

(d) Assume that $\chi \equiv 1$. If $K(A)$ is p.s.d., then there exists $\theta \in \mathbb{R}$ such that $e^{i\theta}A$ is p.s.d. for $A \in \mathbb{C}^{n \times n}$.

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$, then by Lemma 1(a),

$$\text{tr } K(A) = p_m(\lambda_1, \ldots, \lambda_n),$$

where $p_m(x_1, \ldots, x_n) = \sum_{\alpha \in \Delta} \prod_{i=1}^{n} \lambda_i^{m_i(\alpha)}$ is a polynomial in $x_1, \ldots, x_n$. The polynomial $p_m(x_1, \ldots, x_n)$ is symmetric and homogeneous of degree $m$, i.e.

$$p_m(x_{\tau(1)}, \ldots, x_{\tau(n)}) = p_m(x_1, \ldots, x_n) \quad \text{for all } \tau \in S_n$$

and

$$p_m(cx_1, \ldots, cx_n) = c^m p_m(x_1, \ldots, x_n) \quad \text{for all } c \in \mathbb{C}.$$

We then have another formulation of $\rho^*_c(A)$:

$$\rho^*_c(A) = \max \left\{ \left| p_m(\gamma_1 \lambda_{\sigma(1)}, \ldots, \gamma_n \lambda_{\sigma(n)}) \right| \left| \tau \in S_n \right\}. \quad (8)$$

It can be shown that [19]

$$\sigma^*_c(A) = p_m(s_1 t_1, \ldots, s_n t_n). \quad (9)$$

The following lemma is due to Weyl [21] and Horn [10].

LEMMA 2. Let $s_1 \geq \cdots \geq s_n$ be $n$ nonnegative real numbers, and let $\lambda_1, \ldots, \lambda_n$ be $n$ complex numbers with the ordering $|\lambda_1| \geq \cdots \geq |\lambda_n|$. Then there exists $A \in \mathbb{C}^{n \times n}$ such that $A$ has singular values $s_1 \geq \cdots \geq s_n$ and
eigenvalues $\lambda_1, \ldots, \lambda_n$ if and only if

\[
\prod_{i=1}^{k} \lambda_i \leq \prod_{i=1}^{k} s_i, \quad k = 1, \ldots, n-1,
\]

\[
\prod_{i=1}^{n} \lambda_i = \prod_{i=1}^{n} s_i.
\]  

(10)

The next lemma is also due to Horn [9].

**Lemma 3.** Let $A, B \in \mathbb{C}_{n \times n}$ and $C = AB$. Let $s_1 \geq \cdots \geq s_n$, $\beta_1 \geq \cdots \geq \beta_n$, and $t_1 \geq \cdots \geq t_n$ be the singular values of $A$, $B$, and $C$ respectively. Then

\[
\prod_{i=1}^{k} t_i \leq \prod_{i=1}^{k} s_i \beta_i, \quad k = 1, \ldots, n-1,
\]

\[
\prod_{i=1}^{n} t_i = \prod_{i=1}^{n} s_i \beta_i.
\]  

(11)

The following lemma is due to Gohberg and Krein [4].

**Lemma 4.** If $A \in \mathbb{C}_{n \times n}$ and we define $|A| = (A^*A)^{1/2}$, then $|\text{tr} A| = \text{tr} |A|$ if and only if $e^{i\theta}A$ is p.s.d. for some $\theta \in \mathbb{R}$.

The following is the well-known elliptical range theorem [17].

**Lemma 5.** Let $A \in \mathbb{C}_{2 \times 2}$ with eigenvalues $\lambda_1$ and $\lambda_2$. Then $W(A)$, the numerical range of $A$, is an elliptical disk with foci at $\lambda_1$ and $\lambda_2$, and minor axis of length $(\text{tr} A^* A - |\lambda_1|^2 - |\lambda_2|^2)^{1/2}$.

The last lemma is due to Goldberg and Straus [6].

**Lemma 6.** Let $A \in \mathbb{C}_{2 \times 2}$. Then for any $\gamma_1, \gamma_2 \in \mathbb{C}$, the set

\[
\begin{pmatrix} \text{tr} \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} U^* A U \\ U \subset \mathcal{U}_2 \end{pmatrix}
\]

is equal to $(\gamma_1 - \gamma_2)W(A - \frac{1}{2}(\text{tr} A)I_2) + \frac{1}{2}(\gamma_1 + \gamma_2)(\text{tr} A)$. Consequently, it is
an elliptical disk with foci $\gamma_1\lambda_1 + \gamma_2\lambda_2$ and $\gamma_1\lambda_2 + \gamma_2\lambda_1$. Moreover, it is a line segment if and only if $A$ is normal.

3. THE GENERALIZED C-RADIAL MATRICES

The theorem below gives us some information about the generalized C-radial matrices when $\min\{\text{rank } A, \text{rank } C\} > m$.

**Theorem 1.** Let $A, C \in \mathbb{C}_{n \times n}$. Assume that $r = \min\{\text{rank } A, \text{rank } C\} > m$. Then $A$ is generalized $C$-radial if and only if there exist $\theta \in \mathbb{R}$, $\pi, \tau \in S_n$ satisfying $|\gamma_{\pi(1)}| \geq \cdots \geq |\gamma_{\pi(n)}|$, $|\lambda_{\pi(1)}| \geq \cdots \geq |\lambda_{\pi(n)}|$, and $\lambda_{\pi(j)}\gamma_{\pi(j)} = s_j t_j e^{i\theta}$ for all $j = 1, \ldots, n$ and $A, C$ are unitarily similar to

$$
\tilde{A} = \text{diag}(\lambda_{\pi(1)}, \ldots, \lambda_{\pi(r)}) \Theta A_1, \quad \tilde{C} = \text{diag}(\gamma_{\pi(1)}, \ldots, \gamma_{\pi(r)}) \Theta C_1,
$$

respectively, where $A_1, C_1 \in \mathbb{C}_{(n-r) \times (n-r)}$.

**Proof.** $\Leftarrow$: Since $p_m$ is symmetric and is homogeneous of degree $m$, we have

$$
\rho^*_C(A) = \max \left\{ |p_m(\lambda_{\pi(1)}, \ldots, \lambda_{\pi(n)})| \bigg| \pi \in S_n \right\} \quad \text{[by (8)]}
$$

$$
> \left| p_m(\lambda_{\pi(1)}\gamma_{\pi(1)}, \ldots, \lambda_{\pi(n)}\gamma_{\pi(n)}) \right|
$$

$$
= \left| p_m(s_1 t_1 e^{i\theta}, \ldots, s_n t_n e^{i\theta}) \right|
$$

$$
= p_m(s_1 t_1, \ldots, s_n t_n)
$$

$$
= \sigma^*_C(A). \quad \text{[by (9)]}
$$

Since $\rho^*_C(A) \leq \sigma^*_C(A)$ is always true, we have $\rho^*_C(A) = \sigma^*_C(A)$.

$\Rightarrow$: There exists $\omega \in S_n$ such that $A, C$ are unitarily similar to the upper triangular matrices

$$
\tilde{A} = \begin{pmatrix}
\lambda_1 & \ast \\
& \ddots & \ast \\
0 & \cdots & \lambda_n
\end{pmatrix}, \quad \tilde{C} = \begin{pmatrix}
\gamma_{\omega(1)} & \ast \\
& \ddots & \ast \\
0 & \cdots & \lambda_{\omega(n)}
\end{pmatrix}
$$
respectively, by Schur's triangularization theorem and $\rho_\star^\star(A) = |\text{tr } K(\tilde{A}\tilde{C})| = \sigma_\star^\star(A)$. Let $\eta_1 \geq \cdots \geq \eta_n$ be the singular values of $\tilde{A}\tilde{C}$. Since $p_m(\eta_1, \ldots, \eta_n) = \text{sum of singular values of } K(\tilde{A}\tilde{C})$, we have

$$|\text{tr } K(\tilde{A}\tilde{C})| \leq p_m(\eta_1, \ldots, \eta_n).$$

(12)

Because $\eta_1 \geq \cdots \geq \eta_n$ are the singular values of the product of $\tilde{A}$ and $\tilde{C}$, the numbers $\eta_1 \geq \cdots \geq \eta_n$, $\sigma_1 \geq \cdots \geq \sigma_n$, and $t_1 \geq \cdots \geq t_n$ then satisfy the condition (11). Hence the numbers $\eta_1 \geq \cdots \geq \eta_n$ and $s_1t_1 \geq \cdots \geq s_nt_n$ also satisfy the condition (10). By Lemma 2, there exists $\Lambda \in \mathbb{C}_{n \times n}$ which has eigenvalues $\eta_1 \geq \cdots \geq \eta_n$ and singular values $s_1t_1 \geq \cdots \geq s_nt_n$. We have

$$p_m(\eta_1, \ldots, \eta_n) = \text{tr } K(\Lambda)$$

$$\leq \text{sum of singular values of } K(\Lambda)$$

$$= p_m(s_1t_1, \ldots, s_nt_n)$$

$$= \sigma_\star^\star(A).$$

(13)

In order that $\rho_\star^\star(A) = \sigma_\star^\star(A)$, (12) and (13) must be equalities. If (13) is an equality, i.e. $\text{tr } K(\Lambda) = \text{tr } |K(\Lambda)|$, then $K(\Lambda)$ is p.s.d. by Lemma 4 and hence $\eta_j = s_jt_j$ for all $j = 1, \ldots, n$. The inequality (12) being an equality implies that there exists $\phi \in \mathbb{R}$ such that $e^{i\phi} K(\tilde{A}\tilde{C}) = K(e^{i\phi/m} A\tilde{C})$ is p.s.d., by Lemma 4. Then $\rho_\star^\star(A) = \sigma_\star^\star(A)$ implies that there exists $\theta \in \mathbb{R}$ such that $e^{-i\theta} \tilde{A}\tilde{C}$ is positive semidefinite, by Lemma 1(c) (this is because rank $\tilde{A}\tilde{C} > m$, as $\eta_j = s_jt_j$, $j = 1, \ldots, n$, and $r = \min\{\text{rank } A, \text{rank } C\} > m$). Immediately $\tilde{A}\tilde{C} = \text{diag}(\lambda_1 \gamma_{\omega(1)}, \ldots, \lambda_n \gamma_{\omega(n)})$, for $\tilde{A}\tilde{C}$ is upper triangular, and there exists $\beta \in S_n$ such that $\lambda_{\beta(j)} \gamma_{\omega(\beta(j))} = s_jt_j e^{i\theta}$ for all $j = 1, \ldots, n$. Let $\tau, \sigma \in S_n$ such that $\tau(j) = \beta(j)$, $\sigma(j) = \omega \circ \beta(j)$, $j = 1, \ldots, r$, and $|\lambda_{\tau(r+1)}| \geq \cdots \geq |\lambda_{\tau(n)}|$, $|\gamma_{\sigma(r+1)}| \geq \cdots \geq |\gamma_{\sigma(n)}|$. By (10) and (11), we have $s_j = |\lambda_{\tau(j)}|$, $t_j = |\gamma_{\sigma(j)}|$, $j = 1, \ldots, r$. Then $|\lambda_{\tau(1)}| \geq \cdots \geq |\lambda_{\tau(n)}|$ and $|\gamma_{\sigma(1)}| \geq \cdots \geq |\gamma_{\sigma(n)}|$ and $A, C$ are unitarily similar to

$$\hat{A} = \begin{pmatrix} \lambda_{\tau(1)} & \cdots & a_{ij} \\ 0 & \ddots & \ddots \\ 0 & \cdots & \lambda_{\tau(n)} \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} \gamma_{\sigma(1)} & \cdots & c_{ij} \\ 0 & \ddots & \ddots \\ 0 & \cdots & \gamma_{\sigma(n)} \end{pmatrix}$$

respectively.

It remains to show that $a_{ij} = c_{ij} = 0$ for $i = 1, \ldots, r$, $i < j$. Let $\xi_1 \geq \cdots \geq \xi_r$ be the singular values of the submatrix $\hat{A}[1, \ldots, r][1, \ldots, n]$. Then by the
interlacing inequalities, we have \( s_j \geq \xi_j, \ j = 1, \ldots, r \). Moreover, we have

\[
\sum_{j=1}^{r} \xi_j^2 = \sum_{j=1}^{r} |\lambda_{r(j)}|^2 + \sum_{i=1}^{r} \sum_{i<j} |a_{ij}|^2
\]

\[
= \sum_{j=1}^{r} s_j^2 + \sum_{i=1}^{r} \sum_{i<j} |a_{ij}|^2.
\]

We then obtain \( a_{ij} = 0, \ i = 1, \ldots, r, \ i < j \). Similarly, \( c_{ij} = 0, \ i = 1, \ldots, r, \ i < j \).

By putting \( m = 1 \), we have the following corollary.

**Corollary 1** (See [14, Theorem 5.15]). Let \( A, C \in \mathbb{C}_{n \times n} \) be nonsingular. Then \( \rho_c(A) = \|A\|_C \) if and only if \( A, C \) are normal and there exist \( \theta \in \mathbb{R}, \pi \in S_n \), satisfying \( |\gamma_{\pi(1)}| \geq \cdots \geq |\gamma_{\pi(n)}| \) and for all \( j = 1, \ldots, n, \lambda_j \gamma_{\pi(j)} = s_j t_j e^{i\theta} \).

**Remark 1.** The nonsingularity of \( A \) and \( C \) is necessary. For if we take \( C \) to be the zero matrix, then we have \( \rho_c(A) = \|A\|_C \) for all \( A \in \mathbb{C}_{n \times n} \).

**Remark 2.** The theorem below indicates the essentiality of the condition \( r = \min\{\text{rank } A, \text{rank } C\} > m \) in Theorem 1.

**Theorem 2.** Let \( A, C \in \mathbb{C}_{n \times n} \) and \( r = \min\{\text{rank } A, \text{rank } C\} \).

(a) If \( r > m \), then \( \rho_c(A) = \sigma_c(A) \) if and only if there exist \( \theta \in \mathbb{R}, \pi \in S_n \), satisfying \( |\gamma_{\pi(1)}| \geq \cdots \geq |\gamma_{\pi(n)}| \) and for all \( j = 1, \ldots, n \) and \( A, C \) are unitarily similar to

\[
\hat{A} = \text{diag}(\lambda_{\pi(1)}, \ldots, \lambda_{\pi(r)}) \oplus A_1, \quad \hat{C} = \text{diag}(\gamma_{\pi(1)}, \ldots, \gamma_{\pi(r)}) \oplus C_1
\]

respectively, where \( A_1, C_1 \in \mathbb{C}_{(n-r) \times (n-r)} \).

(b) If \( r < m \), then \( \rho_c(A) = \sigma_c(A) = 0 \).

(c) If \( r = m \), then \( \rho_c(A) = \sigma_c(A) \) if and only if \( A, C \) are unitarily similar to

\[
\hat{A} = \begin{pmatrix}
\lambda_1 & * \\
0 & \ddots & \ddots \\
0 & & \lambda_m
\end{pmatrix} \oplus A_1, \quad \hat{C} = \begin{pmatrix}
\gamma_1 & * \\
0 & \ddots & \ddots \\
0 & & \gamma_m
\end{pmatrix} \oplus C_1
\]

where \( A_1, C_1 \in \mathbb{C}_{(n-m) \times (n-m)} \) and \( |\prod_{i=1}^{m-1} \lambda_i| = \prod_{i=1}^{m} s_i, \ |\prod_{i=1}^{m} \gamma_i| = \prod_{i=1}^{m} t_i \).
Proof. (a) follows immediately from Theorem 1.

(b): Since \( s_m t_m = 0 \) because \( r < m \), we have \( E_m(s_1 t_1, \ldots, s_n t_n) = 0 \). Hence \( \rho_c(A) = \sigma_c(A) = 0 \).

(c): =:

\[
\rho_c(A) = \max \left\{ \left| E_m(\lambda_1 \gamma_{\pi(1)}, \ldots, \lambda_n \gamma_{\pi(n)}) \right| \left| \pi \in S_n \right\}
\]

\[
\geq \left| E_m(\lambda_1 \gamma_1, \ldots, \lambda_n \gamma_n) \right|
\]

\[
= \prod_{i=1}^{m} \lambda_i \gamma_i
\]

\[
= \prod_{i=1}^{m} s_i t_i
\]

\[
= E_m(s_1 t_1, \ldots, s_n t_n)
\]

\[
= \sigma_c(A),
\]

under the assumption that \( r = m \). By (4), we get \( \rho_c(A) = \sigma_c(A) \).

\( \Rightarrow \): Without loss of generality, we may assume that \( \text{rank } A = m \). Then \( \rho_c(A) = \sigma_c(A) \) implies that \( |\prod_{i=1}^{m} \lambda_i \gamma_i| = \rho_c(A) = \sigma_c(A) = \prod_{i=1}^{m} s_i t_i \). Then \( |\prod_{i=1}^{m} \lambda_i| = \prod_{i=1}^{m} s_i \), and \( |\prod_{i=1}^{m} \lambda_i| = \prod_{i=1}^{m} t_i \) by (10). Moreover, \( A \) and \( C \) are unitarily similar to

\[
\hat{A} = \begin{pmatrix}
\lambda_1 & & a_{12} & & \\
& \ddots & \ddots & \ddots & \\
& & \lambda_m & & a_{mj} \\
0 & & & \ddots & \\
0 & & & & 0
\end{pmatrix}, \quad \hat{C} = \begin{pmatrix}
\gamma_1 & & c_{1j} \\
& \ddots & \\
0 & & \ddots & \\
0 & & & \gamma_n
\end{pmatrix}
\]

respectively. It remains to show that \( a_{ij} = c_{ij} = 0 \) for \( i = 1, \ldots, m, j = m + 1, \ldots, n \). Write

\[
\hat{A} = \begin{pmatrix}
D & E \\
O & F
\end{pmatrix} \in \mathbb{C}_{n \times n},
\]

where \( D = (d_{ij}) \in \mathbb{C}_{m \times m}, \ E = (e_{ij}) \in \mathbb{C}_{m \times (n-m)}, \ F = (f_{ij}) \in \mathbb{C}_{(n-m) \times (n-m)} \).

Let \( \xi_1 \geq \cdots \geq \xi_m \) be the singular values of \( D \). By interlacing inequalities, we have

\[
\xi_j \leq s_j, \quad j = 1, \ldots, m
\] (14)
Moreover,
\[
\sum_{j=1}^{m} s_j^2 = \text{tr}(\hat{A}^*\hat{A})
\]
\[
= \sum_{i,j} |d_{ij}|^2 + \sum_{i,j} |e_{ij}|^2 + \sum_{i,j} |f_{ij}|^2
\]
\[
= \sum_{j=1}^{m} \xi_j^2 + \sum_{i,j} |e_{ij}|^2 + \sum_{i,j} |f_{ij}|^2
\]  
(15)

If \( e_{ij} \neq 0 \) for some \( i, j \), then
\[
s_j > \xi_j \quad \text{for some } 1 \leq j \leq m \text{ by (14) and (15). However}
\]
\[
\left| \prod_{i=1}^{m} \lambda_i \right| = \prod_{i=1}^{m} \xi_i
\]
\[
< \prod_{i=1}^{m} s_i
\]
by (14) and (16), which contradicts the equality \( |\prod_{i=1}^{m} \lambda_i| = |\prod_{i=1}^{m} s_i| \). Hence, we conclude that \( E = 0 \), i.e. \( a_{ij} = 0 \) for \( i = 1, \ldots, m, \ j = m+1, \ldots, n \). Similarly, we have \( c_{ij} = 0 \) for \( i = 1, \ldots, m, \ j = m+1, \ldots, n \). The proof is completed.

However, the condition \( r = \min\{\text{rank } A, \text{rank } C\} > m \) can be omitted for some particular symmetry classes of tensors. The following theorem suggests an example.

**Theorem 3.** Let \( A, C \in \mathbb{C}_{n \times n} \). Then \( \rho_\tau^r(A) = \sigma_\tau^r(A) \) if and only if there exist \( \theta \in \mathbb{R}, \tau, \sigma \in S_n \) satisfying \( |\gamma_{(1)}^\tau| \geq \cdots \geq |\gamma_{(n)}^\tau| \), \( |\lambda_{(1)}^\sigma| \geq \cdots \geq |\lambda_{(n)}^\sigma| \), \( \lambda_{(j)}^\sigma \gamma_{(j)}^\tau = s_j f_j e^{i\theta} \) for all \( j = 1, \ldots, n \), and \( A, C \) are unitarily similar to
\[
\hat{A} = \text{diag}(\lambda_{(1)}^\sigma, \ldots, \lambda_{(r)}^\sigma) \oplus A_1, \quad \hat{C} = \text{diag}(\gamma_{(1)}^\tau, \ldots, \gamma_{(r)}^\tau) \oplus C_1
\]
respectively, where \( r = \min\{\text{rank } A, \text{rank } C\} \) and \( A_1, C_1 \in \mathbb{C}_{(n-r) \times (n-r)} \).

**Proof.** \( \Leftarrow \): Same as the proof of one implication in Theorem 1.

\( \Rightarrow \): By Lemma 1(d) and following the technical details in the proof of Theorem 1, we can obtain the result.
From the relation (4), if $\rho_{\mathcal{C}}(A) = \sigma_{\mathcal{C}}(A)$, then $r_{\mathcal{C}}(A) = \sigma_{\mathcal{C}}(A)$. The following theorem shows that the converse of the statement is also true under certain conditions.

**Theorem 4.** Let $A, C \in \mathbb{C}_{n \times n}$. Suppose

(i) $C$ is normal, i.e. $C$ is unitarily similar to $\text{diag}(\gamma_1, \ldots, \gamma_n)$,
(ii) $r = \min\{\text{rank } A, \text{rank } C\} > m$,
(iii) if $\gamma_i \neq \gamma_j$, then $|\gamma_i| \neq |\gamma_j|$.

Then the following are equivalent:

(a) $A$ is generalized $C$-radial, i.e. $\rho_{\mathcal{C}}(A) = \sigma_{\mathcal{C}}(A)$.
(b) $r_{\mathcal{C}}(A) = \sigma_{\mathcal{C}}(A)$.
(c) There exist $\theta \in \mathbb{R}$, $\tau, \pi \in S_n$ such that $|\gamma_{\pi(1)}| > \cdots > |\gamma_{\pi(n)}|$, $|\lambda_{\pi(j)}| > \cdots > |\lambda_{\pi(n)}|$, $\lambda_{\pi(j)} \gamma_{\pi(j)} = s^j e^{i\theta}$ for all $j = 1, \ldots, n$, and $A, C$ are unitarily similar to

$$
\hat{A} = \text{diag}(\lambda_{\pi(1)}, \ldots, \lambda_{\pi(r)}), \quad \hat{C} = \text{diag}(\gamma_{\pi(1)}, \ldots, \gamma_{\pi(r)}) \oplus C_1
$$

respectively.

By putting $m = 1$, we have the following corollaries.

**Corollary 2** (See [14, Theorem 5.5]). Let $A, C \in \mathbb{C}_{n \times n}$. If $C = \text{diag}(\gamma_1, \ldots, \gamma_r, 0, \ldots, 0)$ is such that $|\gamma_1| > \cdots > |\gamma_r| > 0$, then the following are equivalent:

(a) $\rho_{\mathcal{C}}(A) = \|A\|_{\mathcal{C}}$.
(b) $r_{\mathcal{C}}(A) = \|A\|_{\mathcal{C}}$.
(c) There exist $\theta \in \mathbb{R}$, $\pi \in S_n$ such that $|\lambda_{\pi(1)}| > \cdots > |\lambda_{\pi(n)}|$, $\lambda_{\pi(j)} \gamma_{\pi(j)} = s^j e^{i\theta}$ for all $j = 1, \ldots, n$ and $A$ is unitarily similar to

$$
\hat{A} = \text{diag}(\lambda_{\pi(1)}, \ldots, \lambda_{\pi(r)}), \quad A_1 \in \mathbb{C}_{(n-r) \times (n-r)}.
$$

**Corollary 3** (See [14, Theorem 5.11]). Let $A, C \in \mathbb{C}_{n \times n}$. If $C = \text{diag}(h_1, \ldots, h_r, 0, \ldots, 0)$ where $h_1 > \cdots > h_r > 0$, then the following are equivalent:

(a) $\rho_{\mathcal{C}}(A) = \|A\|_{\mathcal{C}}$.
(b) $r_{\mathcal{C}}(A) = \|A\|_{\mathcal{C}}$.
(c) There exist $\theta \in \mathbb{R}$, $\pi \in S_n$ such that $|\lambda_{\pi(1)}| > \cdots > |\lambda_{\pi(n)}|$, $\lambda_{\pi(j)} = s^j e^{i\theta}$ for all $j = 1, \ldots, n$ and $A$ is unitarily similar to

$$
\hat{A} = \text{diag}(\lambda_1, \ldots, \lambda_r), \quad A_1 \in \mathbb{C}_{(n-r) \times (n-r)}.
$$
Remark 3. The following example shows that assumption (i) in Theorem 4 is essential.

**Example.**

\[
A = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 0 & 1 \\
  0 & 0 & 0
\end{pmatrix}, \quad C = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 1 & 0
\end{pmatrix}, \quad m = 1.
\]

Then \( \rho_C(A) = 1 \), \( r_C(A) = \|A\|_C = 2 \).

Remark 4. The following example shows that assumption (iii) in Theorem 4 is essential.

**Example.**

\[
C = \text{diag}(1, 1, -1), \quad \|C\|_C = 1.
\]

Then \( \rho_C(A) = 1 \), \( r_C(A) = \|A\|_C = 2 \).

Remark 5. The following theorem shows that assumption (ii) in Theorem 4 is essential.

**Theorem 5.** Let \( A, C \in \mathbb{C}_{n \times n} \), and let \( r = \min\{\text{rank} A, \text{rank} C\} \). Suppose that

(i) \( C \) is normal,

(ii) if \( \gamma_i \neq \gamma_j \), then \( |\gamma_i| \neq |\gamma_j| \).

(A) If \( r > m \), then the following are equivalent:

(a) \( \rho_C^*(A) = \sigma_C^*(A) \).

(b) \( r_C^*(A) = \sigma_C^*(A) \).

(c) There exist \( \theta \in \mathbb{R}, \tau, \pi \in S_n \) such that \( |\gamma_{\pi(1)}| > \cdots > |\gamma_{\pi(n)}| \), \( |\lambda_{\tau(1)}| \geq \cdots \geq |\lambda_{\tau(n)}| \), and \( A, C \) are unitarily similar to

\[
\hat{A} = \text{diag}(\lambda_{\tau(1)}, \ldots, \lambda_{\tau(r)}) \oplus A_1,
\]

\[
\hat{C} = \text{diag}(\gamma_{\pi(1)}, \ldots, \gamma_{\pi(r)}) \oplus C_1,
\]

with \( A_1, C_1 \in \mathbb{C}_{(n-r) \times (n-r)} \), \( \lambda_{\tau(j)} \gamma_{\pi(j)} = s_j t_j e^{i\theta} \) for all \( j = 1, \ldots, n \).
(B) If $r < m$, then $\rho_\alpha(A) = r_\alpha(A) = \sigma_\alpha(A) = 0$.

(C) If $r = m$, then the following are equivalent:

   (a) $\rho_\alpha(A) = \sigma_\alpha(A)$.
   (b) $r_\alpha(A) = \sigma_\alpha(A)$.
   (c) $A$ is unitarily similar to

\[
A = \begin{pmatrix}
\lambda_1 & & * \\
& \ddots & \\
0 & & \lambda_m
\end{pmatrix} \oplus A_1,
\]

where $A_1 \in \mathbb{C}_{(n-m)\times(n-m)}$ and $|\prod_{i=1}^m \lambda_i| = \prod_{i=1}^m s_i$.

Immediately we have the following corollaries.

**Corollary 4** (See [20, Theorem 1]). Let $C = \text{diag}(1, \ldots, 1, 0, \ldots, 0)$ with $\text{tr} \, C = k, k > m$. If $A \in \mathbb{C}_{n \times n}$ and $\text{rank} \, A > m$, then the following are equivalent:

   (a) $\rho_\alpha(A) = \sigma_\alpha(A)$.
   (b) $r_\alpha(A) = \sigma_\alpha(A)$.
   (c) There exists $\theta \in \mathbb{R}$ such that $A$ is unitarily similar to $\text{diag}(s_1 e^{i\theta}, \ldots, s_k e^{i\theta}) \oplus \Lambda_1$, where $\Lambda_1 \in \mathbb{C}_{(n-k)\times(n-k)}$.

**Corollary 5** (See [20, Theorem 2]). Let $C = \text{diag}(1, \ldots, 1, 0, \ldots, 0)$ with $\text{tr} \, C = k, k > m$. If $A \in \mathbb{C}_{n \times n}$ and $\text{rank} \, A = m$, then the following are equivalent:

   (a) $\rho_\alpha(A) = \sigma_\alpha(A)$.
   (b) $r_\alpha(A) = \sigma_\alpha(A)$.
   (c) $A$ is unitarily similar to

\[
A = \begin{pmatrix}
\lambda_1 & & * \\
& \ddots & \\
0 & & \lambda_m
\end{pmatrix} \oplus \mathbf{0}_{m \times m},
\]

and

\[
|\prod_{j=1}^m \lambda_j| = \prod_{j=1}^m s_j.
\]
COROLLARY 6 (See [12, Theorem 5]). Let $A \in \mathbb{C}_{n \times n}$ with rank $A > m$. Assume that $C = (1, \ldots, 1, 0, \ldots, 0)$ with $\text{tr} C = m$. Then the following are equivalent:

(a) $\rho_C^*(A) = \sigma_C^*(A)$.
(b) $r_C^*(A) = \sigma_C^*(A)$.
(c) $A$ is unitarily similar to $A_1 \oplus A_2$, where $A_1 \in \mathbb{C}_{m \times m}$ and $|\det A_1| = \prod_{j=1}^{m} s_j$.

Condition (ii) in Theorem 4 can be omitted for some particular symmetry classes of tensors. The following theorem gives an example.

**THEOREM 6.** Let $A, C \in \mathbb{C}_{n \times n}$. Suppose $r = \min\{\text{rank} A, \text{rank} C\}$ and

(i) $C$ is normal, i.e., $C$ is unitarily similar to $\text{diag}(\gamma_1, \ldots, \gamma_n)$,
(ii) if $\gamma_i \neq \gamma_j$, then $|\gamma_i| \neq |\gamma_j|$.

Then the following are equivalent:

(a) $\rho^*_C(A) = \sigma_C^*(A)$.
(b) $r^*_C(A) = \sigma_C^*(A)$.
(c) There exist $\theta \in \mathbb{R}$, $\tau, \pi \in S_n$ such that $|\gamma_{\pi(1)}| \geq \cdots \geq |\gamma_{\pi(n)}|$, $|\lambda_{\tau(1)}| \geq \cdots \geq |\lambda_{\tau(n)}|$,

$\lambda_{\tau(j)} \gamma_{\pi(j)} = s_j t_j e^{i\theta}$ for all $j = 1, \ldots, n$, and $A, C$ are unitarily similar to

$$\hat{A} = \text{diag}(\lambda_{\tau(1)}, \ldots, \lambda_{\tau(r)}) \oplus A_1$$

$$\hat{C} = \text{diag}(\gamma_{\pi(1)}, \ldots, \gamma_{\pi(r)}) \oplus C_1$$

respectively.

**THEOREM 7.** Let $A, C \in \mathbb{C}_{n \times n}$. Suppose

(i) $C$ is normal, i.e., $C$ is unitarily similar to $\text{diag}(\gamma_1, \ldots, \gamma_n)$,
(ii) $\text{rank} A > \text{rank} C = r > m$ and $s_1 \geq \cdots \geq s_p > s_{p+1} \geq \cdots \geq s_n$, where $p \geq r$.

Then the following are equivalent:

(a) $\rho_C^*(A) = \sigma_C^*(A)$.
(b) $r_C^*(A) = \sigma_C^*(A)$.
(c) There exist $\theta \in \mathbb{R}$, $\tau, \pi \in S_n$ such that $|\gamma_{\pi(1)}| \geq \cdots \geq |\gamma_{\pi(n)}|$, $|\lambda_{\tau(1)}| \geq \cdots \geq |\lambda_{\tau(n)}|$,

$\lambda_{\tau(j)} \gamma_{\pi(j)} = s_j t_j e^{i\theta}$ for all $j = 1, \ldots, n$, and $A, C$ are unitarily similar to

$$\hat{A} = \text{diag}(\lambda_{\tau(1)}, \ldots, \lambda_{\tau(r)}) \oplus A_1$$

$$\hat{C} = \text{diag}(\gamma_{\pi(1)}, \ldots, \gamma_{\pi(r)}, 0, \ldots, 0)$$

respectively.
**Corollary 7 (See [13, Theorem 3.6]).** Let \( A \in \mathbb{C}^{n \times n} \). Assume that \( C \) is normal, rank \( C = r \), and \( A \) has singular values \( s_1 = \cdots = s_p > s_{p+1} \geq \cdots \geq s_n \) with \( r \leq p \). Then the following are equivalent:

(a) \( \rho_C(A) = \|A\|_C \).

(b) \( r_C(A) = \|A\|_C \).

(c) There exist \( \theta \in \mathbb{R} \), \( \tau, \pi \in S_n \) such that \( |\gamma_{\pi(1)}| \geq \cdots \geq |\gamma_{\pi(n)}| \), \( |\lambda_{\tau(1)}| \geq \cdots \geq |\lambda_{\tau(n)}| \), \( \lambda_{\tau(j)}\gamma_{\pi(j)} = s_j t_j e^{i\theta} \) for all \( j = 1, \ldots, n \), and \( A, C \) are unitarily similar to

\[
\hat{A} = \text{diag}(\lambda_{\tau(1)}, \ldots, \lambda_{\tau(r)}) \oplus A_1, \\
\hat{C} = \text{diag}(\gamma_{\pi(1)}, \ldots, \gamma_{\pi(r)}, 0, \ldots, 0)
\]

respectively.

The following example shows that assumption (ii) in Theorem 7 is essential.

**Example.**

\[
A = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}, \quad C = \text{diag}(1, 1, -1), \quad m = 1.
\]

Then \( \rho_C(A) = 1 \) but \( r_C(A) = \sigma_C(A) = 2 \) and \( A \) is not normal.

The proofs of Theorem 4 to Theorem 7 are quite technical. So we omit them.

4. **The Generalized C-Spectral Matrices on the Exterior Space**

**Theorem 8.** Let \( A, C \in \mathbb{C}^{n \times n} \).

(a) If \( m = n \), then \( \rho_C(A) = r_C(A) = |\det A \det C| \).

(b) If \( m < n \) and under the assumptions

(i) \( C \) is normal and has distinct eigenvalues, i.e. \( C = \text{diag}(\gamma_1, \ldots, \gamma_n) \) and \( \gamma_i \neq \gamma_j \) for \( i \neq j \),

(ii) There exists \( \pi \in S_n \) such that \( \rho_C(A) = |F_m(\gamma_1\lambda_{\pi(1)}, \ldots, \gamma_n\lambda_{\pi(n)})| \) and \( E_{m-1}(\gamma_1\lambda_{\pi(1)}, \ldots, \gamma_i\lambda_{\pi(i)}, \gamma_{i+1}\lambda_{\pi(i+1)}, \ldots, \gamma_n\lambda_{\pi(n)}) \neq 0 \) (denotes deletion, \( E_0 = 1 \)) for arbitrary \( i, j \),

then \( \rho_C(A) = r_C(A) \) if and only if \( A \) is normal.
Proof. (a): Obvious.

(b), \(\sigma(A)\), if A and C are normal, then

\[
\sigma^*_C(A) = \max \left\{ \| \text{tr} C_m(CU^*AU) \| \mid U \in \mathbb{U}_n \right\}
\]

\[
= \max \left\{ \| \text{tr} C_m \left( \begin{array}{ccc}
\gamma_1 & 0 & \\
0 & \ddots & 0 \\
0 & \ddots & \gamma_n
\end{array} \right) U^* \left( \begin{array}{ccc}
\lambda_1 & 0 & \\
0 & \ddots & 0 \\
0 & \ddots & \lambda_n
\end{array} \right) U \| \mid U \in \mathbb{U}_n \right\}
\]

\[
= \max \left\{ \| \text{tr} C_m \left( \begin{array}{ccc}
\gamma_1 & 0 & \\
0 & \ddots & 0 \\
0 & \ddots & \gamma_n
\end{array} \right) \left( \begin{array}{ccc}
\lambda_1 & 0 & \\
0 & \ddots & 0 \\
0 & \ddots & \lambda_n
\end{array} \right) \| \mid U \in \mathbb{U}_n \right\}
\]

\[
\leq \max \left\{ \| \text{tr} C_m \left( \begin{array}{ccc}
\gamma_1 & 0 & \\
0 & \ddots & 0 \\
0 & \ddots & \gamma_n
\end{array} \right) \right\} U^* \left( \begin{array}{ccc}
\lambda_1 & 0 & \\
0 & \ddots & 0 \\
0 & \ddots & \lambda_n
\end{array} \right) \left( \begin{array}{ccc}
\lambda_1 & 0 & \\
0 & \ddots & 0 \\
0 & \ddots & \lambda_n
\end{array} \right) U \mid U \in \mathbb{U}_n \right\}
\]

\[
\leq \rho^*_C(A).
\]

(see [11, Theorem 7]). Hence \(\sigma^*_C(A) = \rho^*_C(A)\) by (4).

\(\Rightarrow\): Let \(\pi \in S_n\) such that \(\rho^*_C(A) = |E_m(\gamma_1\lambda_{\pi(1)}, \ldots, \gamma_n\lambda_{\pi(n)})|\) satisfying assumption (ii). By Schur's triangularization theorem, A is unitarily similar to

\[
\begin{pmatrix}
\lambda_{\pi(1)} & a_{ij} \\
& \ddots \\
& & \lambda_{\pi(n)}
\end{pmatrix}
\]
We are going to show \( a_{ij} = 0 \) for \( i < j \). If \( a_{ij} \neq 0 \) for \( j = i + 1, i = 1, \ldots, n - 1 \), by Lemma 6 there exists \( V \in \mathbb{R}_2 \) such that

\[
\text{tr} \left( \begin{pmatrix} \gamma_i & 0 \\ 0 & \gamma_j \end{pmatrix} V^* \begin{pmatrix} \lambda_{\pi(i)} & a_{ij} \\ 0 & \lambda_{\pi(j)} \end{pmatrix} V \right) = \xi_i + \xi_j,
\]

where \( \xi_i, \xi_j \) are the eigenvalues of

\[
\begin{pmatrix} \gamma_i & 0 \\ 0 & \gamma_j \end{pmatrix} V^* \begin{pmatrix} \lambda_{\pi(i)} & a_{ij} \\ 0 & \lambda_{\pi(j)} \end{pmatrix} V
\]

satisfying \( \xi_i, \xi_j = \gamma_i \gamma_j \lambda_{\pi(i)} \lambda_{\pi(j)} \) and

\[
\left| E_m(\gamma_1 \lambda_{\pi(1)}, \ldots, \xi_i, \ldots, \xi_j, \ldots, \gamma_n \lambda_{\pi(n)}) \right|
\]

\[
= \left| \xi_i, \xi_j E_{m-2}(\gamma_1 \lambda_{\pi(1)}, \ldots, \tilde{\xi}_i, \ldots, \tilde{\xi}_j, \ldots, \gamma_n \lambda_{\pi(n)}) \right|
\]

\[
+ (\xi_i + \xi_j) E_{m-1}(\gamma_1 \lambda_{\pi(1)}, \ldots, \tilde{\xi}_i, \ldots, \tilde{\xi}_j, \ldots, \gamma_n \lambda_{\pi(n)})
\]

\[
+ E_m(\gamma_1 \lambda_{\pi(1)}, \ldots, \tilde{\xi}_i, \ldots, \tilde{\xi}_j, \ldots, \gamma_n \lambda_{\pi(n)})
\]

\[
> \left| E_{m-2}(\gamma_1 \lambda_{\pi(1)}, \ldots, \gamma_1 \tilde{\lambda}_{\pi(i)}, \ldots, \gamma_j \tilde{\lambda}_{\pi(j)}, \ldots, \gamma_n \lambda_{\pi(n)}) \right|
\]

\[
+ (\gamma_i \lambda_{\pi(i)} + \gamma_j \lambda_{\pi(j)}) E_{m-1}(\gamma_1 \lambda_{\pi(1)}, \ldots, \gamma_i \tilde{\lambda}_{\pi(i)}, \ldots, \gamma_j \tilde{\lambda}_{\pi(j)}, \ldots, \gamma_n \lambda_{\pi(n)})
\]

\[
+ E_m(\gamma_1 \lambda_{\pi(1)}, \ldots, \gamma_i \tilde{\lambda}_{\pi(i)}, \ldots, \gamma_j \tilde{\lambda}_{\pi(j)}, \ldots, \gamma_n \lambda_{\pi(n)})
\]

\[
= \left| E_m(\gamma_1 \lambda_{\pi(1)}, \ldots, \gamma_n \lambda_{\pi(n)}) \right| = \rho^C(A).
\]

If \( U \subset \mathbb{R}_n \) such that \( U[i, j|i, j] - V, U(i, j|i, j) = I_{n-2} \), we have

\[
\left| \text{tr} C_m \begin{pmatrix} \gamma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \gamma_n \end{pmatrix} U^* \begin{pmatrix} \lambda_{\pi(1)} & \cdots & a_{ij} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{\pi(n)} \end{pmatrix} U \right|
\]

\[
= E_m(\gamma_1 \lambda_{\pi(1)}, \ldots, \xi_i, \ldots, \xi_j, \ldots, \gamma_n \lambda_{\pi(n)})
\]

\[
> \rho^C(A),
\]
which contradicts the hypothesis $\rho_c^*(A) = r_c^*(A)$. Hence $A$ is unitarily similar to

$$
\begin{pmatrix}
\lambda_{\pi(1)} & 0 & a_{ij} \\
0 & \ddots & 0 \\
0 & \ddots & \lambda_{\pi(n)}
\end{pmatrix}
$$

Using the previous argument, we can conclude that $a_{ij} = 0$ for $j = i + 2$, $i = 1, \ldots, n - 1$ (the argument works because $a_{ij} = 0$ for $j = i + 1$, $i = 1, \ldots, n - 1$). Inductively, we have $a_{ij} = 0$ for $i < j$, $i = 1, \ldots, n - 1$. Hence $A$ is normal.

**Remark 6.** The following example shows that assumption (i) in Theorem 8 is essential.

**Example.**

$$
A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad m = 1.
$$

Then $\rho_c(A) = r_c(A) = 2$, but $A$ is not normal.

The following example shows that assumption (ii) in Theorem 8 is essential.

**Example.**

$$
A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad m = 2.
$$

Then $\rho_c^*(A) = r_c^*(A) = 2$, but $A$ is not normal.

**Corollary 8** (See [14, Theorem 3.1]). Let $A \in \mathbb{C}_{n \times n}$. Suppose $C = \text{diag}(\gamma_1, \ldots, \gamma_n)$ and the $\gamma_i$’s are distinct. Then $\rho_c^*(A) = r_c^*(A)$ if and only if $A$ is normal.
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