Homotopy minimal periods for maps of three-dimensional solvmanifolds

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Abstract

A natural number \( m \) is called a homotopy minimal period of a map \( f : X \rightarrow X \) if every map \( g \) homotopic to \( f \) has periodic points of minimal period \( m \). In this paper we give a description for the sets of homotopy minimal periods of maps of all compact solvmanifolds of dimension three. Techniques based on the notion of a model solvmanifold are different than those previously used to study tori, compact nilmanifolds, and special \( NR \)-solvmanifolds.

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1. Introduction

Let \( f : M \rightarrow M \), be a self-map of a topological space, in particular a connected polyhedron or compact connected smooth manifold. \( \text{Per}(f) \subset \mathbb{N} \) denotes those \( n \) for which \( f \) has periodic points of minimal period \( n \) (i.e. \( \exists \ x \in M, f^n(x) = x \) but \( f^m(x) \neq x \) for all \( m < n \)).

Definition 1.1. Define the set of homotopy minimal periods of \( f \) to be the set

\[
\text{HPer}(f) := \bigcap_{g \simeq f} \text{Per}(g),
\]

i.e. \( m \in \mathbb{N} \) is a homotopy minimal period of \( f \) iff it is a minimal period for every \( g \) homotopic to \( f \).

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By definition $\text{HPer}(f) \subset \text{Per}(f)$ with a proper inclusion in general (cf. [22, Chapter VI]). The set $\text{HPer}(f)$ describes a rigid part of the dynamics of $f$, i.e. $\text{HPer}(f)$ does not depend on a perturbation of $f$, and $\text{HPer}(f)$ also has some relations to the topological entropy of $f$ (cf. [22, Chapter VI]). This was first studied and described in [4] for self-maps of the circle in order to derive a variant of the Šarkovskii theorem (cf. [30]). The set $\text{HPer}(f)$ is described for self-maps of the two torus $\mathbb{T}^2$ in [1], the torus $\mathbb{T}^d$ of any dimension in [24], a nilmanifold $N$ in [20], any $NR$-solvmanifold of dimension $\geq 4$ in [19], and any $NR$-solvmanifold of dimension $\geq 3$ in [22]. Such descriptions are based on a combinatorial and algebraic scheme developed in [24], properties of the full and prime Nielsen–Jiang periodic points given in [9] (see [22] and [26] for a definition of these numbers), and the Wecken theorem for periodic points of [15,17]. This combinatorial scheme does not work for maps between arbitrary solvmanifolds where the Anosov theorem $N(f) = |L(f)|$ may not hold (cf. [25]).

In addition to having a general theorem about the possible form of the sets $\text{HPer}(f)$ it is also of interest to give a complete description of all these sets as a list in tabular format when the dimension of $M$ is small. This was done for the 3-torus in [24], 3-nilmanifolds in [21], and for the remaining $NR$-solvmanifolds of dimension 3 in [19] (see also [22]). A complete description of the possibilities for $\text{HPer}(f)$ for a map of a real projective space using a direct argument based on the Wecken theorem for periodic points of [15,17] is given in [16].

In this work we describe the set $\text{HPer}(f)$ for a map of any solvmanifold of dimension 3. There are two aspects to the calculations. To establish that the Nielsen periodic numbers are indeed accurate for determining $\text{HPer}(f)$ we use the Wecken results of [14,15]. We can then complete the problem with a direct calculation of the Reidemeister classes for fibre maps of fibrations $S^1 \hookrightarrow M \rightarrow \mathbb{T}^2$ and $\mathbb{T}^2 \hookrightarrow M \rightarrow S^1$ respectively. In the case $\mathbb{T}^2 \hookrightarrow M \rightarrow S^1$ we represent homotopy classes by “linear maps” as in [10] where the notion of a model solvmanifold has been introduced and studied (cf. Definition 2.19).

The paper is organized as follows. In Section 2 we present basic information about the solvmanifolds. By the Mostow theorem, each compact three-dimensional solvmanifold $M$, which is not a torus, either fibres over a 2-torus or over the circle. Our aim is to show a topological classification of three-dimensional solvmanifolds in terms of their fundamental groups (Theorem 2.4, Lemmas 2.6, 2.8, Corollary 2.10, Lemma 2.13). In Section 3 we consider the case $S^1 \hookrightarrow M \rightarrow \mathbb{T}^2$ giving a complete description of the possible $\text{HPer}(f)$ (Theorem 3.5). Here we will not need the assumption that the considered manifold is a non-$NR$-solvmanifold. Our argument gives a new direct proof of the main theorem of [21, Corollary 3.6]. In Section 4 we consider the case $\mathbb{T}^2 \hookrightarrow M \rightarrow S^1$. By discussing the non-$NR$-solvmanifolds case by case we are led to a full description of $\text{HPer}(f)$ (Theorem 4.2). In Section 5 we conclude with some applications and final remarks.

2. Solvmanifolds

2.1. General information

In this section we present some basic information about solvmanifolds with a special emphasis on what can occur in dimension three.

Definition 2.1. A solvmanifold is a homogenous space $M = G/H$, where $G$ is a connected solvable Lie group and $H$ is a co-compact subgroup.

If $G$ is nilpotent then $M$ is called a nilmanifold.

If $G$ is simply-connected and $H \subset G$, is a uniform lattice, i.e. a discrete co-compact subgroup, then $M = G/H$ is called a special solvmanifold.

We must emphasize that some authors use a definition in which $G$ is connected simply-connected solvable Lie group. The latter is equivalent to that of Definition 2.1. Indeed, if $\tilde{G}$ a connected simply-connected Lie group which is a universal cover of $G$, and $\tilde{G} \rightarrow G$ the covering projection, $\tilde{H} = p^{-1}(H)$, then $M = G/H = \tilde{G}/\tilde{H}$.

Due to the Malcev theorem (cf. [27]) every nilmanifold is special.

Note that the fundamental group $\Pi := \pi_1(M)$ is a solvable group (cf. [2,7]). This is obvious when $M$ is special, because $\pi_1(G/H) = H$ in this case. The group $\Pi := \pi_1(M)$ always determines a solvmanifold due to the following theorem of Auslander and Mostow (cf. [3], [7, II, Theorem 2.6, Corollary 2]).
Theorem 2.2. Let $M$ be a compact solvmanifold. Then the fundamental group $\Pi = \pi_1(M)$ classifies $M$ up to diffeomorphism. In particular if $\Pi$ is nilpotent then $M$ is a nilmanifold.

Recall that if $\dim M = 1$, then there is only one, up to diffeomorphism, compact connected smooth manifold $M = S^1 = \mathbb{R}/\mathbb{Z}$ which is obviously a solvmanifold.

If $\dim M = 2$ then there is a complete classification of all homogenous spaces by the Mostow theorem (cf. [7, II 6, Theorem 2.2]). Among these only the torus $T^2 = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$ (in fact the only 2-dimensional nilmanifold) and the Klein bottle $\mathbb{K}^2$, here denoted by $\mathbb{K}$, are solvmanifolds. Since a separate version of the Wecken theorem is needed for the Klein bottle in dimension 2, we will discuss this case in the forthcoming work [18]. The same description of the sets of possible homotopy minimal periods was given in [26] using [8] and geometric arguments.

Example 2.3. The Klein bottle was defined in [8] as follows. Let $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $A(x, y) = (x + 1, y)$, $B(x, y) = (-x, y + 1)$ and let $\mathbb{H}$ be the group of self-homeomorphisms of $\mathbb{R}^2$ generated by $A$ and $B$. We define the Klein bottle as the quotient space: $\mathbb{K} = \mathbb{H} \setminus \mathbb{R}^2$.

To show that the Klein bottle is a solvmanifold we can define it in another way.

We consider the twisted product $G = \mathbb{R} \ltimes_{\phi} \mathbb{C}$ where $\phi_t(\alpha) = e^{\pi i t} \alpha$, i.e.

$$(t, \alpha) \ast (s, \beta) = \left(t + s, e^{\pi i t} \alpha + \beta\right)$$

for $(t, \alpha), (s, \beta) \in G$. We consider the quotient group $G/H$ where $H = \{(m, n + ir) : m, n \in \mathbb{Z}, r \in \mathbb{R}\}$. Then the correspondence

$\mathbb{K} \ni [x, y] \to [y, x + 0i] \in G/H$

defines a continuous bijection between the compact spaces, hence it is a homeomorphism (for another description of the above construction see [7]).

Let us notice that the Klein bottle is nonorientable and hence it is not a special solvmanifold and it cannot be obtained as the homogenous space of solvable group of dimension less than three (as above).

It was noted by Mostow that every solvmanifold can be represented (i.e. is diffeomorphic to) the total space of a fibration

$$\mathbb{N} \hookrightarrow M \to \mathbb{T}^d$$

where $\mathbb{N}$ is a nilmanifold and $\mathbb{T}$ is a torus. This fibration (called a Mostow fibration of $M$) is important in studying self-maps of $M$ with respect to the following theorem (cf. [29,28]).

Theorem 2.4. To each solvmanifold $M$ there is a minimal Mostow fibration such that every self-map $f$ of $M$ is homotopic to a fibre map $f^\ast$ of this Mostow fibration.

Note that $\Gamma = \pi_1(\mathbb{N})$ is a nilpotent group without torsion elements, and thus $\Pi = \pi_1(M)$ is an extension of $\mathbb{Z}^d$ by $\Gamma$.

For a given group $H$ let $\text{Out}(H)$ denote the group $\text{Aut}(H)/\text{Inn}(H)$ of outer automorphisms of $H$. To each short exact sequence $1 \to \Gamma \to \pi_1(M) \to \mathbb{Z}^d \to 1$ there is a natural homomorphism (well-defined) $P : \mathbb{Z}^d \to \text{Out}(\Gamma)$ defined by conjugation of $\Gamma$, using any preimage of an element in $\mathbb{Z}^d$. Thus when $\Gamma$ is abelian, $P$ produces a well-defined action of $\mathbb{Z}^d$ on $\Gamma$ by conjugation.

The next lemma establishes that the geometrically defined action of the fundamental group of the base $\mathbb{T}^d$ on the fundamental group of the fibre $\mathbb{N}$, $\pi_1(\mathbb{N}) = \Gamma$, of a Mostow fibration is given by the homomorphism $\rho : \mathbb{Z}^d \to \text{Out}(\Gamma)$. The proof involves elementary techniques and is omitted.

Lemma 2.5. Let $\rho : E \to B$ be a Hurewicz fibration. For two fixed points $b_0 \in B$ and $x_0 \in \rho^{-1}(b_0)$ we denote $F = \rho^{-1}(b_0)$ and assume that the inclusion $i : F \to E$ induces the monomorphism $i_\# : \pi_1(F; x_0) \to \pi_1(E; x_0)$.

Then the natural, geometrically defined, action $\rho$ of $\pi_1(B; b_0)$ on $\pi_1(F; x_0)$ obtained by path lifting is equal to the algebraic action given by the exact sequence

$$1 \to \pi_1(F; x_0) \to \pi_1(E; x_0) \to \pi_1(B; b_0) \to 1.$$
2.2. 3-Solvmanifolds as fibre spaces

Let us observe, that in the case of a three-dimensional solvmanifold which is not a nilmanifold, the corresponding Mostow fibration must have one of the following two forms

\[ \mathbb{T}^2 \hookrightarrow M \to S^1 \quad \text{or} \quad S^1 \hookrightarrow M \to \mathbb{T}^2, \]

since the only two-dimensional nilmanifold is the torus \( \mathbb{T}^2 \). We shall call \( M \) of type \((2, 1)\), or \((1, 2)\), respectively.

This leads to the corresponding exact sequences of the fundamental groups

\[ 0 \to \mathbb{Z}^2 \to \Pi \to \mathbb{Z} \to 0 \quad \text{or} \quad 0 \to \mathbb{Z} \to \Pi \to \mathbb{Z}^2 \to 0. \]

Observe that every exact sequence \( 0 \to H \to \Pi \to K \to 0 \) with \( H \) abelian determines a unique \( K \)-module structure in \( H \). Indeed, we can put \( k \cdot h := \rho(k)(h) := ghg^{-1} \), where \( g \in \Pi \) is any element in the preimage of \( k \). Since \( H \) is abelian, there are no nontrivial inner automorphisms and \( \rho \) is well defined. The commutativity of \( H \) allows us to use group cohomology theory to study isomorphism classes of extensions of the form (3). More precisely, there is a correspondence between the set \( H^2(K; (H, \rho)) \) and isomorphism classes of extensions of \( K \) by \( H \) with the fixed module structure \( \rho \) (cf. [5]). Furthermore \( 0 \in H^2(K; (H, \rho)) \) corresponds to the twisted product \( H \rtimes_{\rho} K \) given by \( \rho \).

**Lemma 2.6.** In the case \((2, 1)\) of (3) the fundamental group \( \Pi \) is the twisted product \( \mathbb{Z}^2 \rtimes_{\rho} \mathbb{Z} \).

**Proof.** Indeed, let \( \rho: \mathbb{Z} \to \text{Aut}(\mathbb{Z}^2) \) be the module structure on \( H = \mathbb{Z}^2 \) given by the first exact sequence in (3). Note that such a structure corresponds to a choice of a \( 2 \times 2 \) matrix \( A = \rho(1) \in \text{Aut}(\mathbb{Z}^2) \). Since the classifying space of the group \( K = \mathbb{Z} \) is \( S^1 \), which is one-dimensional, we have that \( H^2(\mathbb{Z}; (\mathbb{Z}^2, \rho)) = 0 \) for every module structure \((\mathbb{Z}^2, \rho)\) of \( \mathbb{Z} \) in \( \mathbb{Z}^2 \). We thus get that there is only one class for our first extension in (3) corresponding to the twisted product \( \mathbb{Z}^2 \rtimes_{\rho} \mathbb{Z} \). \( \square \)

**Remark 2.7.** A countable family of pairwise non-isomorphic special solvmanifolds of type \((2, 1)\) are described in [7, II 6.2.3], and also in [19].

Before discussing the second case we recall a fact about nilmanifolds. For a nilpotent connected simply-connected Lie group \( G \) and its uniform lattice \( \Gamma \subset G \) we have finite towers

\[ G_{(n)} = e = [G_{(n-1)}, G] \cdots \lhd G_i = [G_{(i-1)}, G] \cdots \lhd G_1 = [G, G] \lhd G_0 = G, \]
\[ \Gamma_{(n)} = e \lhd \Gamma_{n-1} = G_{n-1} \cap \Gamma \cdots \lhd \Gamma_1 = G_1 \cap \Gamma \lhd \Gamma_0 = \Gamma \]

with compact nilmanifolds \( G_{(i)}/\Gamma_{(i)} \) and quotients \( G_{(i-1)}/G_{(i)} \cong \mathbb{R}^{d_i}, \Gamma_{i-1}/\Gamma_i \cong \mathbb{Z}^{d_i} \) and \( \sum d_i = \dim G = \dim G/\Gamma \).

The number \( n \) is called the nilpotency length of \( G \) (or of \( \Gamma, G/\Gamma \), respectively). Note that \( \Gamma \), and all \( \Gamma_i \), are nilpotent as subgroups of a nilpotent group.

Now we turn to the study of the case \((1, 2)\) of the second possible Mostow fibration in (2).

Observe that a module structure \( \rho \) of \( \mathbb{Z}^2 \) on \( H = \mathbb{Z} \) is a homomorphism \( \rho: \mathbb{Z}^2 \to \text{Aut}(\mathbb{Z}) = \{1, -1\} = \mathbb{Z}_2 \). We have two distinct possibilities: when \( \rho \) is trivial or when it is not.

First assume that the homomorphism \( \rho: \mathbb{Z}^2 \to \mathbb{Z} \) of the exact sequence (3) is trivial. Then we claim that the extension \( \Pi \) is nilpotent.

**Lemma 2.8.** Let

\[ 1 \to H \to \Pi \to K \to 1 \]

be an exact sequence of groups with \( H \) nilpotent. Let \( l(H) \) be the nilpotency length of \( H \) and suppose that \( K \) is abelian. If the \( K \)-module structure \( \rho: K \to \text{Out}(H) \) induced by this exact sequence is trivial, then \( \Pi \) is nilpotent with \( l(\Pi) \leq l(H) + 1 \).

In particular, any extension of \( \mathbb{Z}^p \) by \( \mathbb{Z}^n \) with the trivial action of \( \mathbb{Z}^p \) on \( \mathbb{Z}^n \) is a nilpotent group with nilpotency length \( \leq 2 \).
Proof. The trivial action means that for any \( s \in \Pi \) there is an \( \tilde{h} \in H \) such that \( s h s^{-1} = \tilde{h} h \tilde{h}^{-1} \) for all \( h \in H \).

Let us define inductively: \( \Pi_0 = \Pi, \Pi_{k+1} = [\Pi, \Pi_k] \). Analogously we define the sequence \( H_k \) (normal tower). We show inductively that \( \Pi_{k+1} \subset H_k \). Then \( \Pi_{k+1} \subset H_{\ell(H)} = e \) and the proof is complete.

For \( k = 0 \) we notice that \( \Pi_1 = [\Pi, \Pi] \subset H = H_0 \), since for \( s, s' \in \Pi \), \( K \) is abelian so \( p(s's^{-1} s'^{-1}) = 1 \).

Now we assume that \( \Pi_k \subset H_{k-1} \). We consider a generator \([s, h] = s h s^{-1} h^{-1} \) of the group \( \Pi_{k+1} = [\Pi, \Pi_k], s \in \Pi \) and \( h \in \Pi_k \subset H_{k-1} \). Since the action of the group \( K \) on \( H \) is trivial, we have \([s, h] = (s h s^{-1}) \cdot h^{-1} = (\tilde{h} h \tilde{h}^{-1}) \cdot h^{-1} = [\tilde{h}, h] \). \( Q.E.D. \)

Remark 2.9. Note that for a trivial \( \rho: \mathbb{Z}^2 \to \mathbb{Z} \) we have that \( H^2(\mathbb{Z}^2; (\mathbb{Z}, \rho)) = H^2(\mathbb{Z}^2; \mathbb{Z}) = \mathbb{Z} \). One should remark that extensions corresponding to the elements \( r, s \in \mathbb{Z} \) are isomorphic if and only if \(|r| = |s| \) (i.e. the numbers are the same up to sign). In the case where \( r = 0 \), one gets the group \( \mathbb{Z}^3 \) (corresponding to \( \mathbb{T}^3 \)), while for \( r \neq 0 \) one gets a Heisenberg manifold as described below.

It is known, that if \( M \) is a three-dimensional nilmanifold then \( M \) is either diffeomorphic to the torus \( \mathbb{T}^3 \) or to a Heisenberg nilmanifold \( N_{3} = G/\Gamma_{1,1,r}, r \in \mathbb{N} \), where \( G \) is the group of unipotent \( 3 \times 3 \) real matrices and \( \Gamma_{1,1,r} \) is a certain discrete subgroup (cf. [7, II 4, Corollary 2]).

Now we turn to a discussion of the Mostow fibrations of type \((1, 2)\) with a nontrivial action \( \rho: \mathbb{Z}^2 \to \operatorname{Aut}(\mathbb{Z}) = \mathbb{Z}_2 \).

First we show that \( H^2(\mathbb{Z}^2; (\mathbb{Z}, \rho)) = \mathbb{Z}_2 \) for such a \( \rho \). Indeed, the classifying space for \( K = \mathbb{Z}^2 \) is \( \mathbb{T}^2 \). This implies that \( H^2(\mathbb{Z}^2; (\mathbb{Z}, \rho)) = H^2(\mathbb{T}^2; (\mathbb{Z}, \rho)) \) (with twisted coefficients). By the Poincaré duality for the cohomology with twisted coefficients and nontrivial action we have that

\[
H^2(K; (H, \rho)) = H^2(\mathbb{T}^2; (\mathbb{Z}, \rho)) = H_0(\mathbb{T}^2; (\mathbb{Z}, \rho)) = \mathbb{Z}_2,
\]

since the action is nontrivial. This implies

Corollary 2.10. For every nontrivial action \( \rho \) of \( \mathbb{Z}^2 \) on \( \mathbb{Z} \) we have two classes of isomorphisms of exact sequences

\[
0 \to \mathbb{Z} \to \Pi \to \mathbb{Z}^2 \to 0,
\]

one of these is a twisted product and the other is not.

Now we describe the groups \( \Pi \) which appear as the extension for the case \((1, 2)\) with nontrivial \( \rho \). The fibration involves the exact sequence of fundamental groups

\[
1 \to \mathbb{Z}c \to \Pi \xrightarrow{\rho} \mathbb{Z}a \times \mathbb{Z}b \to 1,
\]

where \( a, b, c \) are generators of cyclic groups. Let us fix \( a \in \rho^{-1}(\tilde{a}), b \in \rho^{-1}(\tilde{b}) \). Then it is easy to check that the elements \( a, b, c \) generate \( \Pi \). Since \( \Pi \) is not abelian we use the notation of multiplication for a product in \( \Pi \), but the notation of addition for a product in the quotient group \( \mathbb{Z}a \times \mathbb{Z}b \).

In fact for an \( x \in \Pi \) we denote \( p(x) = k\tilde{a} + l\tilde{b} \). Then \( p(x(a^k b^l)^{-1}) = 0 \) hence \( x(a^k b^l)^{-1} \in \ker p = \mathbb{Z}c \). Now \( x(a^k b^l)^{-1} = c^i \) which implies \( x = c^i \cdot (a^k b^l) \). Thus \( \Pi \) is a quotient of the free group \( \langle a, b, c \rangle \) of three generators by relations. Since \( \Pi/\mathbb{Z}c \) is abelian, \( [\Pi, \Pi] \subset \mathbb{Z}c \). Consequently a commutator of any two elements of \( \Pi \) belongs to \( \mathbb{Z}c \).

Theorem 2.11. Let the Mostow fibration of a 3-dimensional solvmanifold \( M \) be of the type \( 1 \to \mathbb{Z} \to \Pi \to \mathbb{Z} \times \mathbb{Z} \to 1 \). If the action \( \rho \) is trivial then \( M \) is a Heisenberg nilmanifold. If \( \rho \) is nontrivial then \( \pi_1(M) \) is one of the following groups:

\[
(-) \quad \Pi_0 = \langle a, b, c \rangle / \{ ac = ca; bc = c^{-1} b; [a, b] = 1 \} — \text{twisted product, or}
\]

or

\[
(1) \quad \Pi_1 = \langle a, b, c \rangle / \{ ac = ca; bc = c^{-1} b; [a, b] = c \} — \text{nontwisted product.}
\]

Since the fundamental group determines the solvmanifold, there are exactly two non-nilmanifold solvmanifolds of dimension 3, which fibre over \( \mathbb{T}^2 \).

Proof. The case of the trivial action was discussed in Remark 2.9. Now we assume that the action is not trivial. There are three actions \( \rho, \rho', \rho'' \):

\[
\rho(a) = 1, \quad \rho(b) = -1; \quad \rho'(a) = -1, \quad \rho'(b) = 1; \quad \rho''(a) = -1, \quad \rho''(b) = -1.
\]
We will show that in each case there are exactly two extensions with the given action and that in each case we obtain the same two groups.

Let us first concentrate on the case $\rho(a) = 1$, $\rho(b) = -1$. From what we said above $\Pi$ has the form

$$\Pi_k = \langle a, b, c | ac = ca; bc = c^{-1}b; [a, b] = c^k \rangle, \quad k \in \mathbb{Z}. $$

Indeed, note that $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z} \tilde{a} \times \mathbb{Z} \tilde{b}$ acts on $\mathbb{Z} c$ by inner automorphisms, i.e. $\rho(\tilde{h})(h) = kgk^{-1}$, where $h \in \mathbb{Z}$ and $k$ is any element of $\Pi$ which maps onto $\tilde{k}$ into the exact sequence. $\rho(a) = 1$ means that $aha^{-1} = h$ for any $h \in \mathbb{Z} c$, e.g. $aca^{-1} = c$. Analogously $\rho(b) = -1$ gives $bcb^{-1} = c^{-1}$. Finally, the inclusion $[\Pi, \Pi] \subset \mathbb{Z} c$ implies that $[a, b] = c^k$ for some $k \in \mathbb{Z}$, which is the last relation.

Proposition 2.14 shows that by the above construction only two groups corresponding to $k = 0, 1$ are non-isomorphic.

It is easy to see that the groups $\Pi_0, \Pi_1$ are not isomorphic. In fact we recall that these groups occur in the exact sequences

$$1 \to \mathbb{Z} c \xrightarrow{\iota_0} \Pi_0 \xrightarrow{\rho_0} \mathbb{Z} \tilde{a} \times \mathbb{Z} \tilde{b} \to 1$$

where $[a, b] = 1, a\gamma = \gamma a, \beta \gamma = \gamma^{-1} \beta$ and

$$1 \to \mathbb{Z} c \xrightarrow{\iota_1} \Pi_1 \xrightarrow{\rho_1} \mathbb{Z} \tilde{a} \times \mathbb{Z} \tilde{b} \to 1$$

where $[a, b] = c, ac = ca, bc = c^{-1}b$.

We notice that $[\Pi_0, \Pi_0] = (\gamma^2), [\Pi_1, \Pi_1] = (c)$ are cyclic groups. If $\Pi_0$ and $\Pi_1$ were isomorphic, $\gamma^2$ would correspond to $c^{-1}$. This gives an element $d \in \Pi_1$ satisfying $d^2 = c \in \Pi_1$. We may represent $d = a^pb^qc^r$ in $\Pi_1$ for some $p, q, r \in \mathbb{Z}$. Now $d^2 = a^{2p}b^{2q}c^{2r}$. On the other hand

$$1 = p_1(c) = p_1(d^2) = p_1(a^{2p}b^{2q}c^{2r}) = a^{2p}b^{2q}c^{2r} \in \mathbb{Z} \tilde{a} \times \mathbb{Z} \tilde{b}$$

which gives $2p = 2q = 0$, hence $p = q = 0$. Now $d = c^r$ hence $c = d^2$ implies $c = c^{2r}$ which is impossible, since $\mathbb{Z} c \subset \Pi_1$.

Finally, since $H^2(\mathbb{Z}^2; \mathbb{Z}) = \mathbb{Z}_2$, we have exactly two non-isomorphic extensions for a given $\rho$, namely $\Pi_0, \Pi_1$, in respect of above (i.e. $[a, b] = c^k$ gives $\Pi_0$ or $\Pi_1$ depending the parity of $k$; see Proposition 2.14). Lemma 2.12 shows that the two groups are not nilpotent. Thus it remains to notice that other two actions, produce the same groups. This will also follow from Lemma 2.13. □

**Lemma 2.12.** The groups $\Pi_0$ and $\Pi_1$ are solvable but not nilpotent.

**Proof.** Note that $\Pi_0^{(1)} = (\Pi_0)_{(1)} := \{[\Pi_0, \Pi_0] = (c^{-2}) \equiv 2\mathbb{Z} c \subset \mathbb{Z} c$, since the only nontrivial commutator is $bcb^{-1}c^{-1} = bcb^{-1}(bcb^{-1}) = c^{-1}c^{-1} = c^{-2}$. Thus $\Pi_0^{(2)} := \{[\Pi_0^{(1)}, \Pi_0^{(1)}] \subset \mathbb{Z} c, \mathbb{Z} c\} = \{0\}$ and $\Pi_0$ is solvable. Note that $(\Pi_0)_{(1)} = [\Pi_0, (\Pi_0)_{(1)}] = \{c^2\} \equiv 4\mathbb{Z} c \subset \mathbb{Z} c$, and consequently $(\Pi_0)_{(k)} = \{c^{2k}\} \equiv 2^k\mathbb{Z} c \subset \mathbb{Z} c \neq 0$. This shows that $\Pi_0$ is not nilpotent, because this is nonzero for every $k$.

Note that $\Pi_1^{(1)} = (\Pi_1)_{(1)} := \{[\Pi_1, \Pi_1] = \{c\} \equiv \mathbb{Z} c$, since the only nontrivial commutators are $bcb^{-1}c^{-1} = bcb^{-1}(bcb^{-1}) = c^{-2}$ as above and $[a, b] = c$. Thus $\Pi_1^{(2)} := \{[\Pi_1^{(1)}, \Pi_1^{(1)}] \subset \mathbb{Z} c, \mathbb{Z} c\} = \{0\}$ and $\Pi_1$ is solvable. Note that $(\Pi_1)_{(1)} = [\Pi_1, (\Pi_1)_{(1)}] = \{c^{-2}\} \equiv 2\mathbb{Z} c \subset \mathbb{Z} c$, because $bcb^{-1}c^{-1} = c^{-2}$. Consecutively $(\Pi_1)_{(k)} = \{c^{2k^{-1}}\} \equiv 2^{k^{-1}}\mathbb{Z} c \subset \mathbb{Z} c$ which, as above, shows that $\Pi_1$ is not nilpotent. □

**Lemma 2.13.** Denote by $\Pi_0', \Pi_1'$, and correspondingly $\Pi_0'', \Pi_1''$, groups obtained in an analogous construction as in (5) for the remaining two nontrivial homomorphisms $\rho'$ and $\rho''$, respectively. Then

$$\Pi_0' \equiv \Pi_0'' \equiv \Pi_0''' \quad \text{and} \quad \Pi_1' \equiv \Pi_1'' \equiv \Pi_1'''. $$

In particular we have two isomorphism classes of non-nilmanifold type $(1, 2)$. 
Proof. The actions $\rho$ and $\rho'$ are symmetric, hence the isomorphisms $\Pi_i \equiv \Pi'_i$ are evident.

Now we consider the actions $\rho$ and $\rho''$. We will show that the groups

$$
\Pi_1 = \langle a, b, c \rangle / \{ ac = ca; bc = c^{-1}b; [a, b] = c \}, \n$$

$$
\Pi''_1 = \langle a, b, c \rangle / \{ ac = c^{-1}a; bc = c^{-1}b; [a, b] = c \}
$$

are isomorphic. Henceforth let $a, b, c$ will denote the generators of $\Pi''_1$.

Now we consider new generators:

$$
\hat{a} := ab^{-1}, \quad \hat{b} := b, \quad \hat{c} := c
$$

of $\Pi''_1$. Note that then $\hat{a}$ acts trivially, and $\hat{b}$ nontrivially on $\hat{c}$, since we have

1. $\hat{a}\hat{c} = ab^{-1}c = ac^{-1}b^{-1} = cab^{-1} = \hat{c}\hat{a}$;
2. $\hat{b}\hat{c} = bc = c^{-1}b = \hat{c}^{-1}\hat{b}$;
3. $\hat{a}\hat{b}\hat{c} = \hat{b}^{-1}(ab^{-1}b^{-1} - 1) = \hat{a}(ba^{-1}b^{-1} - 1) = \begin{cases} 1 & \text{if } 1 = [a, b], \Pi''_0, \\ c & \text{if } c = [a, b], \Pi''_1. \end{cases}$

Consequently the assignment $a \mapsto \hat{a}, b \mapsto \hat{b}, c \mapsto \hat{c}$, defines an isomorphism of the group $\Pi''_1 = \langle a, b, c \rangle / \{ ac = c^{-1}a, bc = c^{-1}b, [a, b] = c \}$ with the group $\Pi_i$. \qed

**Proposition 2.14.** The groups $\Pi_k$ (respectively $\Pi'_k$, $\Pi''_k$) and $\Pi_i$ (respectively $\Pi'_i$, $\Pi''_i$) are isomorphic iff $k - l$ is even.

**Proof.** We will show that the groups

$$
\Pi_0 = \langle a, b, c \rangle / \{ ac = ca, bc = c^{-1}b, [a, b] = 1 \}
$$

and

$$
\Pi_2 = \langle \hat{a}, \hat{b}, \hat{c} \rangle / \{ \hat{a}\hat{c} = \hat{c}\hat{a}, \hat{b}\hat{c} = \hat{c}^{-1}\hat{b}, [\hat{a}, \hat{b}] = \hat{c}^{2l} \}
$$

are isomorphic.

We define the homomorphisms $\phi : \Pi_0 \to \Pi_2$ and $\psi : \Pi_2 \to \Pi_0$ by

$$
\phi(a) = \hat{a}\hat{c}^{-l}, \quad \phi(b) = \hat{b}, \quad \phi(c) = \hat{c} \quad \text{and} \quad \psi(\hat{a}) = ac^l, \quad \psi(\hat{b}) = b, \quad \psi(\hat{c}) = c.
$$

To see that these homomorphisms are well defined, it remains to check that the relations are preserved. It is already evident that the homomorphisms are inverse to one another.

We will check only that $\phi(a)\phi(c) = \phi(c)^{-1}\phi(a)$ and $[\phi(a), \phi(b)] = 1$. The remaining cases are similar.

- $\phi(a)\phi(c) = \phi(c)\phi(a)$ means $\hat{a}\hat{c}^{-l}\hat{c} = \hat{c}\hat{a}\hat{c}^{-l}$ but this follows since $\hat{a}, \hat{c}$ commute.
- $[\phi(a), \phi(b)] = [\hat{a}\hat{c}^{-l}, \hat{b}] = \hat{a}\hat{c}^{-l}\hat{b}\cdot(\hat{a}\hat{c}^{-l})^{-1}, \hat{b}^{-1} = \hat{a}\hat{c}^{-2l}\hat{b}\hat{a}^{-1}\hat{b}^{-1} = \hat{c}^{-2l}[\hat{a}, \hat{b}] = \hat{c}^{-2l}\hat{c}^{2l} = 1$.

The isomorphisms between $\Pi_1$ and $\Pi_{2l+1}$ are given by the same formulae.

Finally, since $H^2(\mathbb{Z}^2; \rho) = \mathbb{Z}_2$, we have exactly two non-isomorphic extensions for a given $\rho$, since $\Pi_0, \Pi_1$ are not isomorphic as it is shown in the proof of Theorem 2.11 and the two isomorphism classes are represented by $\Pi_0 \not\cong \Pi_1$. As in the proof of Theorem 2.11 one can show that $\Pi_{2l} \not\cong \Pi_1$ and $\Pi_{2l+1} \not\cong \Pi_0$, which shows that $\Pi_{2l} \equiv \Pi_0$ and $\Pi_{2l+1} \equiv \Pi_1$ for every $l \geq 1$.

The arguments corresponding to the actions $\rho'$ and $\rho''$ are the same. \qed

Furthermore, note that $\Pi_i$ satisfy the so-called Wang’s condition, i.e. they are the mid-term of an exact sequence with the right-term $\mathbb{Z}^p$ and the left torsion free nilpotent term (cf. [7]). Consequently, they are fundamental group of a solvmanifold.

**Remark 2.15.** We can indicate these two solvmanifolds in terms of $\mathbb{R}^3$ and their deck transformations, i.e. we represent them as the orbit spaces of actions of the groups $\Pi_i$, $i = 0, 1$, on $\mathbb{R}^3$. 

Let \( A, B, C \) be affine transformations of \( \mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2 \) given by the formulas
\[
A(x, \bar{y}) = (x, \bar{y} + (1, 0)), \quad B(x, \bar{y}) = (-x, \bar{y} + (0, 1)), \quad C(x, \bar{y}) = (x + 1, \bar{y})
\]
and
\[
A(x, \bar{y}) = (x + 1, \bar{y} + (1, 0)), \quad B(x, \bar{y}) = (-x, \bar{y} + (0, 1)), \quad C(x, \bar{y}) = (x + 2, \bar{y}).
\]
It is easy to check that in the first case \( BA = AB, AC = CA, BC = C^{-1}B \), and correspondingly \( BA = AB, AC = CA, BC = C^{-1}B \) in the second case. This means that a discrete subgroup of the affine transformations of \( \mathbb{R}^3 \) generated by \( A, B, C \) is isomorphic to \( \Pi_0 \), or \( \Pi_1 \), respectively, so are the fundamental groups of the orbit space.

Thanks to a discussion with Karel Dekimpe, we can give another description of the group \( \Pi_1 \) showing that the corresponding solvmanifold is an infra-nilmanifold.

**Example 2.16.** Let us consider the subgroup \( \Pi \) of the group \( G = SO(3) \ltimes \mathbb{R}^3 \) of isometries of Euclidean 3-space generated by the standard lattice of translations:
\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
and the element
\[
\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.
\]
Note that each element consists of a rotational part and a translational part.

We note that since \( \beta^2 = -A \), we could have omit \( A \). The group \( \Pi \) is torsion free and is a Bieberbach group with translational part \( \mathbb{Z}^3 = (A, B, C) \) and holonomy group \( \Pi / \langle A, B, C \rangle \cong \mathbb{Z}_2 \). The compact flat manifold \( \mathbb{R}^3 / \Pi \) has \( \Pi \) as its fundamental group.

Now we claim that \( \Pi \) is isomorphic to the group \( \tilde{\Pi} = (a, b, c) / \{ ac = c^{-1}a; bc = cb; [a, b] = c \} \).

One can check that the group \( \tilde{\Pi} \) above is exactly the group determined by the presentation
\[
\Pi = \langle A, B, C, \beta \rangle / \{ [A, B] = [A, C] = [B, C] = 1, \beta^2 = A, \beta A = A \beta, \beta B = C \beta, \beta C = B \beta \}.
\]

Now, we define a homomorphism \( \varphi : \Pi \to \Pi_1 \), given by
\[
\varphi(a) = a, \quad \varphi(A) = a^2, \quad \varphi(B) = b \quad \text{and} \quad \varphi(C) = b c^{-1}.
\]
It is not difficult to check that the map determined above really is really an isomorphism (the inverse \( \psi \) is defined by \( \psi(a) = b, \psi(b) = B \) and \( \psi(c) = BC^{-1} \)).

Consider an arbitrary compact solvmanifold \( M \) with \( \Pi = \pi_1(M) \) and its Mostow fibration \( N \hookrightarrow M \to \mathbb{T} \). We will consider the action of \( \Lambda_0 = \pi_1(\mathbb{T}) \) on the fibre \( N_e \) over a chosen point in \( \mathbb{T} \). Each element \( \lambda \in \Lambda_0 \) determines, up to homotopy, an admissible map \( \tau_\lambda \) of the fibre \( N_e \). However the admissible maps do not preserve basic points, hence the induced homotopy group homomorphism is not well defined (unless \( \pi_1(N_e) = 0 \)). But \( \tau_\lambda \) does correctly define the isomorphism of the abelian quotient groups \( \Lambda_1 = \Gamma / \Gamma_{i+1} \) where \( \Gamma_i = \pi_1(\mathbb{N}_i) \) and \( \mathbb{N} = N_0 \to N_1 \to \cdots \to N_s = 1 \) is the Fadell–Husseini tower of the nilmanifold \( N = N_e \) (this consists of consecutive Fadell–Husseini fibrations (cf. [22])) with \( \Gamma_1 = \pi_1(\mathbb{N}_1) \). This action may be described as follows. Let \( \lambda \in \Lambda_0 = \pi_1(\mathbb{T}) = \pi_1(\mathbb{T}) \) and let \( g \in \Gamma_i \). Let \( \bar{\lambda} \in \Pi \) represent \( \lambda \in \Pi / \Gamma \). Since \( \bar{\lambda} \cdot \gamma \cdot \bar{\lambda}^{-1} \in \Gamma_i \) and the quotient \( \Gamma_i / \Gamma_{i+1} \) is abelian, the class \( [\bar{\lambda} \cdot \gamma \cdot \bar{\lambda}^{-1}] \in \Gamma_i / \Gamma_{i+1} = \Lambda_i \) does not depend on the choices of \( \gamma \), and \( \bar{\lambda} \). Thus each homotopy class \( \lambda \in \Lambda_0 = \pi_1(\mathbb{T}) \) determines an automorphism of \( \Lambda_i = \pi_1(\mathbb{N}_i) / \pi_1(\mathbb{N}_{i+1}) \). For each \( i = 1, \ldots, s \) this yields a homomorphism \( \Lambda_i : \Lambda_0 \to \text{Aut}(\Lambda_i) \), which is in essence a square matrix. We call this matrix \( A_{i}(\lambda) \), unique up to conjugation by a unimodular matrix, the linearization of the action.

**Definition 2.17.** (See cf. [25].) The solvmanifold \( X \), is called an \( NR \)-solvmanifold if no \( A_i(\lambda) \) as above has an eigenvalue which is a root of unity different than 1.
Every nilmanifold is an $NR$-solvmanifold. The special solvmanifolds mentioned in Remark 2.7 are $NR$-solvmanifolds (cf. [19]). Except for the trivial action, all the $(1, 2)$ possibilities discussed in the previous section will yield non-$NR$-solvmanifolds. As we shall soon see there are also solvmanifolds of type $(2, 1)$ which are not $NR$-solvmanifolds.

Directly from Definition 2.17 and Lemma 2.6 we get the following

**Corollary 2.18.** A three-dimensional solvmanifold of type $(2, 1)$ is an $NR$-solvmanifold if and only if the matrix $A \in M_{2 \times 2}(\mathbb{Z})$ defining the twisted product $\Pi = \pi_1(M) = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ does not have a root of unity other than $1$ in its spectrum.

We now define the notion of model solvmanifold introduced by P. Heath and the second author (cf. [10]).

**Definition 2.19.** A solvmanifold $M$ is called a model solvmanifold if

$$\pi_1(M) = \mathbb{Z}^u \rtimes_A \mathbb{Z}^p, \quad u, p > 0,$$

where $A : \mathbb{Z}^p \to \text{Aut}(\mathbb{Z}^u)$ is a homomorphism.

Obviously any group satisfying Definition 2.19 is solvable, satisfies the Wang condition, and thus is the fundamental group of a solvmanifold $M$. Moreover, by uniqueness (Theorem 2.2) the Mostow fibration of $M$ has the form $\mathbb{T}^u \hookrightarrow M \to \mathbb{T}^p$. The notion of model solvmanifold was introduced in [10], where a geometrical realization of any model solvmanifold $M$, as a quotient of $\mathbb{R}^u \times \mathbb{R}^p$, and many properties of the class of all solvmanifolds are discussed. We shall explain and use some of them in the next section.

**Remark 2.20.** Note that the three-dimensional solvmanifold of Example 2.16 is not a model solvmanifold. Indeed its fundamental group is not a twisted product $\mathbb{Z} \rtimes \mathbb{Z}^2$.

As a direct consequence of Lemma 2.6 we get the following which we shall use in Section 4.

**Corollary 2.21.** Every three-dimensional solvmanifold of type $(2, 1)$ is a model solvmanifold.

The following results will be essential for our analysis of $\text{HPer}(f)$ (cf. [9]):

**Property 2.22.** Let $f : M \to M$ be a fibre preserving map of a Mostow fibration $N \hookrightarrow M \to \mathbb{T}^2$. Let $\xi \subseteq \text{Fix}(f^n)$ consist of one point from each essential Nielsen class of $f^n$. Then

$$N P_n(f) = \sum_{b \in \xi} N P_{n} \circ \text{per}(b) \left( (f^{\text{per}(b)})_b \right).$$

The following is a consequence of the periodic Wecken theorem of the first author [14] and Mostow’s result that we always have a fibre preserving map in every homotopy class.

**Property 2.23.** Suppose that $f : M \to M$ is a self map of a three-dimensional solvmanifold. Then $n \in \text{HPer}(f)$ iff $N P_n(f) \neq 0$.

### 3. Fibrations over $T^2$ with fibre $S^1$

In this section we study the set of homotopy minimal periods of a self-map $f : M \to M$ of a three-dimensional solvmanifold $M$ whose Mostow fibration is over $T^2$. Without loss, the theory of Mostow fibrations tells us that we can assume that, up to homotopy, $f$ is fibre preserving.
In general, suppose that \( p : M \to B \) is a fibration over a compact connected polyhedron, with \( S^1 \) as fibre. The natural action of \( \pi_1(B) \) on the group \( \pi_1(S^1) \) (i.e. the homomorphism \( \epsilon : \pi_1(B) \to \text{Aut}(\pi_1(S^1)) = \mathbb{Z}_2 \)) is given by \( \epsilon(\alpha) = +1 \) iff \( \alpha \) preserves (reverses) the orientation of the fibre. This may be also described as follows. By Hurewicz theory we may subordinate to any path \( \omega : I \to B \) a map \( \tau_\omega : E_0(0) \to E_0(1) \), called an admissible map, so that for the constant path \( c \) at the point \( x_0 \), \( \tau_c = \text{id}_{E_0(0)} \) and the family of maps \( \tau_\omega \) depends continuously on the paths \( \omega \) [31]. For any loop \( \alpha \) based at the point \( b_0 \) we have that \( \epsilon(\alpha) = \deg \tau_\alpha \). We will also say that a loop \( \alpha \) has even (odd) parity iff \( \epsilon(\alpha) = +1 \) (\(-1\)).

On the other hand, we may define the parity of a fixed point \( b \in \text{Fix}(\tilde{f}) \) with \( \deg(f_b) \neq 0 \) as \( \epsilon(b) = \text{sign}(\deg(f_b)) \).

As the next shows, then this is consistent with the parity of the loop which defines the Nielsen class containing \( b \), if we take as the base point a fixed point \( b \). We will always choose the base point as above, provided that such point exists. Otherwise \( \deg(f_b) \) is negative, for each fixed point \( b \), and the notion of parity is unnecessary. Note that if \( \deg(f_b) = 0 \) then \( \forall b, \deg(f_b) = 0 \). Thus we may assume from the start of the discussion that \( \deg(f_b) \neq 0 \).

Lemma 3.1. Assume that for \( b_0 \in \text{Fix}(\tilde{f}) \), \( \deg(f_b) \neq 0 \). Then under the above notations we have:

1. For each fixed point \( b \in \text{Fix}(\tilde{f}) \), \( \epsilon(b) = \epsilon(\alpha \ast \tilde{f}(\omega^{-1})) \) where \( \omega \) is any path from the base point \( b_0 \) to \( b \), for each fixed point \( b \in \text{Fix}(\tilde{f}) \).
2. \( \epsilon(\tilde{f} \alpha) = \epsilon(\alpha) \) for any \( \alpha \in \pi_1(B, b_0) \), i.e. \( \tilde{f}_b \) preserves the parity.
3. The parity of two Nielsen related fixed points \( b, b' \in \text{Fix}(f) \) is the same.

Proof. Note that if \( \deg(f_{b_0}) = 0 \) then \( \forall b, \deg(f_b) = 0 \). Thus we may assume from the start of this discussion that \( \deg(f_b) \neq 0 \).

1. For a path \( \omega \), from \( b_0 \) to \( b \in \text{Fix}(\tilde{f}) \) and \( r, s \in [0, 1] \), we will denote by \( \omega^s \) the path given by \( \omega^s(t) = \omega(r(1-t) + ts) \). Consider the homotopy \( h_t : E_0(0) \to E_0(1) \) given by \( h_t = \tau_{\omega^t \omega} - \tau_0 \). Then \( h_0 = \tau_{\omega} \), \( h_1 = \tau_{\omega(1)} \).

This implies that
\[
\deg(h_t) = \deg(h_0) = \deg(h_1) = \deg(f_{\omega(1)}) \cdot \deg(\tau_\omega).
\]

Since \( \deg(f_{\omega(0)}) > 0 \),
\[
\epsilon(b) = \text{sign}(\deg(\tau_\omega)) = \text{sign}(\deg(\tau_\omega \ast \tau_{\tilde{f}(\omega^{-1})})) = \text{sign}(\deg(\tau_{\omega(\tilde{f}(\omega^{-1})}))) = \epsilon(\omega \ast (\tilde{f}(\omega^{-1})).
\]

2. In the above formula we replace the path \( \omega \) with the loop \( \alpha \), then
\[
+1 = \epsilon(b_0) = \epsilon(\alpha \ast \tilde{f}(\alpha^{-1})) = \epsilon(\alpha) \cdot \epsilon(\tilde{f}(\alpha^{-1})^{-1})
\]

which implies \( \epsilon(\tilde{f} \alpha) = \epsilon(\alpha) \).

3. Let \( b_1, b_2 \in \text{Fix}(\tilde{f}) \) be Nielsen related and let \( \delta \) be a path from \( b_1 \) to \( b_2 \) satisfying \( \tilde{f}(\delta) \sim \delta \). Let \( \omega_1, \omega_2 \) be paths from \( b_0 \) to \( b_1 \) and from \( b_0 \) to \( b_2 \), respectively. Now \( \alpha = \omega_1 \ast f \omega_1^{-1}, \beta = \omega_2 \ast f \omega_2^{-1} \) represent the Nielsen classes of the points \( b_1 \) and \( b_2 \), respectively. Let \( \gamma = \omega_2 \ast \delta^{-1} \ast \omega_1^{-1} \).

Then
\[
\gamma \ast \alpha \ast \tilde{f}(\gamma^{-1}) = (\omega_2 \ast \delta^{-1} \ast \omega_1^{-1}) \ast (\omega_1 \ast \tilde{f}(\omega_1^{-1}) \ast (\tilde{f} \omega_1 \ast \tilde{f}(\omega_2^{-1})) \sim \omega_2 \ast \delta^{-1} \ast \tilde{f}(\omega_2^{-1}) \ast \omega_2 \ast \tilde{f}(\omega_1^{-1}) = \beta.
\]

At last
\[
\epsilon(\alpha) = \epsilon(\gamma) \epsilon(\alpha) \epsilon(\tilde{f}(\gamma^{-1})) = \epsilon(\gamma \ast \alpha \ast \tilde{f}(\gamma^{-1})) = \epsilon(\beta)
\]

where the left most equality follows from 2. \( \square \)

Thus we may define the parity of a Nielsen class \( A \subseteq \text{Fix}(\tilde{f}) \) as the parity of any loop representing \( A \): \( \epsilon(A) = \epsilon(\omega \ast f(\omega^{-1})) \).

Now we consider a fibration over the 2-torus: \( p : M \to \mathbb{T}^2 \), with fibre \( S^1 \). We fix a base point \( b_0 \in \text{Fix}(\tilde{f}) \) as above (i.e. we assume that the degree of the fibre map over the base point \( d = \deg(f_{b_0}) > 0 \)). Let \( Y \in \mathcal{M}_{2 \times 2}(\mathbb{Z}) \) be the
Lemma 3.2. If the action of $\pi_1(\mathbb{T}^2)$ on $\pi_1(S^1)$ is nontrivial and the matrix $I - Y$ is non-singular then the number of Reidemeister classes in $\mathcal{R}(\bar{f})$ is even and the number of the classes of even parity is the same as the number of the classes of odd parity.

Proof. We may assume that $\bar{f} : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is induced by a linear map represented by the matrix $Y$. Let $v_1 \in \mathbb{Z}^2 = \pi_1(\mathbb{T}^2)$ be represented by a loop $\gamma$ of odd parity (the fact that the action is nontrivial forces $v_1$ to exist). Since $I - Y$ is non-singular, there is a point $y_1 \in \mathbb{R}^2$ such that $(I - Y)y_1 = -v_1$. Then $[y_1] \in \text{Fix}(f)$ and

$$
\epsilon([y_1]) = \epsilon(\gamma^{-1}) = -1.
$$

Thus the parity of $y_1$ is odd. Now we define inverse bijective self maps of $\text{Fix}(\bar{f}) = \{[y] \in \mathbb{R}^2/\mathbb{Z}^2; (I - Y)(y) \in \mathbb{Z}^2\}$:

$$
\phi[\gamma] = [y + y_1] \text{ and } \psi[\gamma] = [y - y_1].
$$

One may check that the maps are correctly defined, $\phi \psi = \psi \phi = \text{id}$ and $\epsilon(\phi[\gamma]) = -\epsilon(\psi[\gamma])$. If we split $\mathcal{R}(Y) = \mathcal{R}^+(Y) \cup \mathcal{R}^-(Y)$ into elements of even and odd parity, respectively, then $\phi(\mathcal{R}^+(Y)) = \mathcal{R}^-(Y)$ and hence the both parts have the same cardinality. \(\square\)

Lemma 3.3. If the action of $\pi_1(\mathbb{T}^2)$ on $\pi_1(S^1)$ is nontrivial and $n \in \text{HPer}(\bar{f})$, then there is an essential irreducible class of odd parity in $\mathcal{R}(\bar{f}^n)$.

Proof. Since $n \in \text{HPer}(\bar{f})$ and $\mathbb{T}^2$ is a Jiang space, $N(\bar{f}^n) > 0$ and it remains to show that the number of irreducible classes of odd parity in $\mathcal{R}(\bar{f}^n)$ is not less than the number of irreducible classes of even parity. By Lemma 3.2 there is a bijection between the subsets of elements of odd and even parity and hence it is enough to show that the number of reducible classes of even parity in $\mathcal{R}(\bar{f}^n)$ is not less than the number of reducible classes of odd parity.

We recall that the set of reducible classes is the union:

$$
\bigcup_p \text{Im}(\iota_{n/p, n} : \mathcal{R}(\bar{f}^{n/p}) \to \mathcal{R}(\bar{f}^n))
$$

where the union runs over the set of all prime divisors of $n$. Now we recall that the torus is essentially reducible (i.e. if an essential Reidemeister class $\mathcal{A}$ reduces to a class $\mathcal{B}$ then $\mathcal{B}$ is also essential). In particular all fixed point classes are essential.

We recall that the action of $\pi_1(\mathbb{T}^2)$ is nontrivial. If $b \in \text{Fix}(\bar{f}^k)$ then the parity of $b$ is the sign of $\deg((f^k)b)$, $b/k|n$ then $b$ can also be viewed as an element of $\text{Fix}(\bar{f}^n)$ where the parity will be the sign of $\deg(((f^k)b)^\frac{n}{k}) = \deg((f^k)b)^\frac{n}{k}$. Therefore, if $k|n$ then

1. If $b \in \text{Fix}(\bar{f}^k)$ has even parity, then the same $b$ viewed as an element of $\text{Fix}(\bar{f}^n)$ will have even parity.
2. If $b \in \text{Fix}(\bar{f}^k)$ has odd parity, then the same $b$ viewed as an element of $\text{Fix}(\bar{f}^n)$ will have even parity if $\frac{n}{k}$ is even and odd parity if $\frac{n}{k}$ is odd.

Let $k = n/(p_1 \cdots p_s)$ where $p_1, \ldots, p_s$ are all odd prime divisors of $n$. By Lemma 3.2 there is a class $\mathcal{A}^k \in \mathcal{R}(\bar{f}^k)$ of odd parity. Let $\mathcal{A}^{n/p_i} = \iota_{k,n/p_i}(\mathcal{A}^k) \in \mathcal{R}(\bar{f}^{n/p_i})$ and $\mathcal{A}^n = \iota_{k,n}(\mathcal{A}^k) \in \mathcal{R}(\bar{f}^n)$. Since $n/k = p_1 \cdots p_s$ and $(n/p_i)/k$ are odd, the above arguments imply that the parity of $\mathcal{A}^{n/p_i}$ and $\mathcal{A}^n$ is also odd.

Let $x \in \mathbb{R}^2$ represent the class $\mathcal{A}_k \in \mathcal{R}(\bar{f}^k) = \mathbb{R}^2/\text{Im}(\text{id} - \bar{f}^k)\mathbb{Z}^2$. Then for each multiple $l$ of $k$, $x \in \text{Fix}(\bar{f}^k) \subset \text{Fix}(\bar{f}^l)$ represents the Reidemeister class $\mathcal{A}_l = [x - l\bar{f}x]$ and we denote $y^l = x - l\bar{f}x$.

The $\iota_{m,n}$ treat this addition linearly so we notice that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{R}(f^{n/p_i}) & \xrightarrow{\iota_{n/p_i,n}} & \mathcal{R}(f^n) \\
\downarrow{+y^{n/p_i}} & & \downarrow{+y^n} \\
\mathcal{R}(f^{n/p_i}) & \xrightarrow{\iota_{n/p_i,n}} & \mathcal{R}(f^n)
\end{array}
\]
where the vertical arrows are given by $[x] \mapsto [x + y^n/p_i]$ and $[x] \mapsto [x + y^n]$ respectively. Let us notice that the horizontal arrows preserve parity while the vertical ones reverse it ($p_i$ is odd). Moreover, the vertical arrows are bijections. This implies that the number of classes of odd parity equals the number of classes of even parity in the sum $\bigcup_{p_i} \text{im}(i_{n/p_i,n})$, where the union runs over the set of odd prime divisors of $n$.

Since for $n$ odd the above sum coincides with the set of reducible classes $\mathcal{R}(f^n) \setminus T \mathcal{R}(f^n)$, the claim is proved for all odd $n$.

If $n$ is even then the reducible classes arise from $(\bigcup_{p_i} \text{im}(i_{n/p_i,n})) \cup \text{im}(i_{n/2,n})$, where the unions run over the set of odd prime divisors of $n$. By the above the first sum contains the same number of classes of both parities while the last summand contains only the elements of even parity. Thus the number of reducible elements of even parity in the sum is greater. □

In the case of a fibre map over the torus there is a simple relation between the Nielsen classes of all $f$, $\tilde{f}$ and $f_b$ [32,23].

**Lemma 3.4.** Let $p: E \to \mathbb{T}^n$ be the Hurewicz fibration over the torus where the fibres and total space are compact polyhedra. Let $f: E \to E$ be a fibre map such that $L(\tilde{f}) \neq 0$. Let $A \subset \text{Fix}(f)$ be a Nielsen class. Then $p(A) \subset \text{Fix}(\tilde{f})$ and $A \cap E_b \subset \text{Fix}(f_b)$ (for any $b \in \text{Fix}(\tilde{f})$) are also contained in Nielsen classes of $\tilde{f}$ and $f_b$, respectively. Moreover, the class $A \subset \text{Fix}(f)$ is essential $\iff$ all the classes containing $p(A) \subset \text{Fix}(\tilde{f})$, $A \cap E_b \subset \text{Fix}(f_b)$ are essential.

If the map $\tilde{f}: \mathbb{T}^n \to \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is induced by a linear map and $L(\tilde{f}) \neq 0$ then each point $b \in \text{Fix}(\tilde{f})$ forms an essential singleton Nielsen class.

Since each of two generators of $\mathbb{Z}^2$ can be sent to +1 or to $-1 \in \mathbb{Z}_2 = \text{Aut}(\mathbb{Z})$, there are four possible actions of $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$ on $\pi_1(S^1) = \mathbb{Z}$. Let the homotopy homomorphism $\tilde{f}: \pi_1(\mathbb{T}^2) \to \pi_1(\mathbb{T}^2) = \mathbb{Z}^2$ be given by the matrix

$$Y = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$ 

**Case** $(+1, +1)$. The action is trivial. The numbers $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ are arbitrary and all Nielsen classes have the same parity.

**Case** $(-1, -1)$. The action of $(y_1, y_2) \in \mathbb{Z}^2 = \pi_1(\mathbb{T}^2)$ on $\pi_1(S^1) = \mathbb{Z}$ is multiplication by $(-1)^{y_1+y_2}$. In particular $e([0, 1]^2T) = -1$ and $e((Y[1, 0]^2) = e([\alpha, \gamma]) = (-1)^{\alpha+\gamma}$. Since the map $\tilde{f}_b = Y$ preserves the parity (Lemma 3.1), $-1 = (-1)^{\alpha+\gamma}$ which implies that $\alpha + \gamma$ is odd. Similarly for the second generator we conclude that $\beta + \delta$ is also odd.

**Case** $(+1, -1)$. Then the action of $(y_1, y_2)$ is the multiplication by $(-1)^{y_1}$. Following the above reasoning we conclude that $\alpha$ is odd and $\beta$ is even.

**Case** $(+1, -1)$ is similar to the previous case but here we conclude that $\gamma$ is even and $\delta$ is odd.

Now we are ready to prove the main theorem of this section.

**Theorem 3.5.** Let $S^1 \hookrightarrow M \to \mathbb{T}^n$ be a fibration and $f: M \to M$ a fibre map. Assume without loss that $\tilde{f}$ is induced from its $2 \times 2$ linearization matrix $Y$. Then, except for three exceptional cases below, the following formula holds,

$$\text{HPer}(f) = \begin{cases} \text{HPer}(\tilde{f}) & \text{for } |d| \leq 1, \\ T_Y & \text{for } |d| \geq 2, \end{cases}$$

where $T_Y = \{k \in \mathbb{N}; \det(I - Y^k) \neq 0\}$ and $d = \deg(f_0)$ for the identity point $0 \in \text{Fix}(\tilde{f})$.

The exceptional cases are when the action of $\pi_1(B)$ on the fibre is trivial and $d$ is one of the following small integers:

- If $\deg(f_0) = d = -2$ and $2 \not\in \text{Fix}(\tilde{f})$, then $\text{HPer}(f) = T_Y \setminus \{2\}$.
- If $\deg(f_0) = d = +1$, then $\text{HPer}(f) = \emptyset$.
- If $\deg(f_0) = d = -1$, then $\text{HPer}(f) = \text{HPer}(\tilde{f}) \setminus 2\mathbb{N}$. 

Proof. If \( \det(I - Y) = 0 \), then \( \tilde{f} \) is homotopic to a map with no periodic points. Hence the same is true for \( f \) and

\[ \text{HPer}(f) = \text{HPer}(\tilde{f}) = T_Y = \emptyset \]

regardless of the value of \( d \).

If \( d = 0 \), then \( f \) is homotopic to a map which collapses each fibre to a point and \( \text{HPer}(f) = \text{HPer}(\tilde{f}) \).

Now we may assume that \( \det(I - Y) \neq 0 \) and \( d \neq 0 \). We consider three cases: \( |d| \geq 3 \), \( |d| = 2 \), and \( |d| = 1 \).

- Suppose that \( |d| \geq 3 \). We will show that \( \text{HPer}(f) = T_Y \).

Let \( n \not\in T_Y \). Then there are no essential Reidemeister classes in \( \mathcal{R}(\tilde{f}^n) \). Now Lemma 3.4 implies that there are no essential Reidemeister classes in \( \mathcal{R}(f^n) \) and hence \( n \not\in \text{HPer}(f) \).

Now we assume that \( n \in T_Y \). Consider the identity point \( 0 \in \text{Fix}(\tilde{f}) \). Then \( 0 \in \text{Fix}(\tilde{f}^n) \) and the Nielsen class of \( \tilde{f}^n \) containing \( 0 \) is essential (since \( n \in T_Y \) and \( \mathbb{T}^2 \) is a Jiang space). On the other hand the fibre map \( f_n^0 \) has an essential irreducible class \( A^0_0 \) (since \( |d| \geq 3 \) implies \( \text{HPer}(f_0) = \mathbb{N} \)). The Nielsen class of \( f^n \) containing \( A^0_0 \) is essential and irreducible and hence \( n \in \text{HPer}(f) \).

- Let \( |d| = 2 \).

We will show that in this case

\[ \text{HPer}(f) = \begin{cases} T_Y \setminus \{2\} & \text{if } 2 \not\in \text{HPer}(\tilde{f}), \\ T_Y & \text{if } 2 \in \text{HPer}(\tilde{f}). \end{cases} \]

Then \( 2 \not\in \text{HPer}(\tilde{f}) \) gives the first exceptional case.

Now we prove the claim. Since \( \text{HPer}(f_b) = \mathbb{N} \) for \( d = 2 \), we see that the above reasoning can all be repeated, provided there is a point \( b \in \text{Fix}(\tilde{f}) \) such that \( \deg(f_b^0) = +2 \). Such points exist whenever the action of \( \pi_1(\mathbb{T}^2) \) is not trivial (Lemma 3.2). If the action is trivial then each fixed point \( b \in \text{Fix}(\tilde{f}) \) has this property provided \( d = +2 \). It remains to consider the remaining case where the action is trivial and \( d = -2 \).

First we notice that \( \text{HPer}(\tilde{f}) \setminus \{2\} = T_Y \setminus \{2\} \). In fact follows from the obvious inclusion \( \text{HPer}(\tilde{f}) \subset T_Y \). To prove \( \supset \) we consider a natural number \( 2 \neq k \in T_Y \). Now the assumptions \( (d = -2 \) and the action of \( \pi_1(\mathbb{T}^2) \) is trivial) imply an essential irreducible class \( A_0 \subset \mathcal{O}(f^k_b) \) (for any \( b \in \text{Fix}(\tilde{f}) \)). The class \( A \subset \text{Fix}(f^k) \) containing \( A_0 \) is irreducible and is also essential, since \( k \in T_Y \).

It remains to find out when \( 2 \in \text{HPer}(\tilde{f}) \).

Let \( 2 \in \text{HPer}(\tilde{f}) \). Then obviously \( 2 \in T_Y \) and it remains to show that \( 2 \in \text{HPer}(f) \). Now we have an essential irreducible class \( \bar{B} \subset \text{Fix}(\tilde{f}^2) \). Let \( b \in \bar{B} \). Then \( \deg((f^2)_{b_0}) = 4 \) implies \( N((f^2)_{b_0}) = 3 \) which gives an essential class \( B_0 \subset \text{Fix}(f^2_b) \). Now the Nielsen class \( B \subset \text{Fix}(f^2) \) containing \( B_0 \) is irreducible and essential which proves \( 2 \in \text{HPer}(f) \).

If \( 2 \not\in \text{HPer}(\tilde{f}) \) then each essential orbit \( \bar{B} \in \mathcal{O}(\tilde{f}^2) \) is either inessential or reducible. If \( \bar{B} \) is inessential then each orbit \( B \in \mathcal{O}(f^2) \) lying over \( B \) is also inessential. If \( \bar{B} \) is essential and reducible then it contains a fixed point (essential reducibility of torus) \( b \) and now each orbit in \( \mathcal{O}(f^2)_{b_0} \) is reducible. It remains to show that each orbit on \( \mathcal{O}(f^2) \) lying over \( B \) is reducible. Since \( b \in \bar{B} \) is the fixed point, we have the map \( \mathcal{R}(f_b) \to \mathcal{R}(f^2_b) \) and it remains to show that this map is onto. We recall that the fibre is homeomorphic to \( S^1 \) and \( \deg f_b = -2 \). Now

\[ \#\mathcal{R}(f_b) = |1 - (-2)| = 3 = |1 - (-2)^2| = \#\mathcal{R}(f^2_b) \]

hence the injective map \( \mathcal{R}(f_b) \to \mathcal{R}(f^2_b) \) is onto.

- \( d = +1 \) and the action is trivial.

In this case we have that for any \( n \) the degree in each fibre over elements of \( \text{Fix}(\tilde{f}^n) \) is \( d^n = 1^n = 1 \). By the fixed point index product formula all Nielsen classes in \( \text{Fix}(\tilde{f}^n) \) are inessential and hence \( n \not\in \text{HPer}(f) \). This implies that \( \text{HPer}(f) = \emptyset \). This gives the second exceptional case.

- \( d = -1 \) and the action is trivial.
Thus \( \deg(f_b) = -1 \) and \( N(f_b) > 0 \) for each \( b \in \fix(\tilde{f}) \). Consequently \( \deg(f^n_b) = (-1)^n \) for each fixed point and since the action is trivial, the same formula holds for each point \( b \in \fix(f^n) \). In particular, for \( n \)-even \( \deg(f^n_b) = +1 \) and hence all Nielsen classes of \( f^n \) are inessential and \( n \notin \nper(f) \). We will show that \( \nper(f) = \nper(\tilde{f}) \setminus 2\mathbb{N} \).

This gives the last exceptional case.

\[ \therefore \text{Let } n \in \nper(f) \text{ be odd. Then there is an essential irreducible class } \tilde{\mathbb{H}} \subset \fix(\tilde{f^n}). \text{ Let } b \text{ be a point in this class. Since } \deg(f^n_b) = -1 \text{, there is an essential class } \mathbb{H}_0 \subset \fix(f^n). \text{ Let } \mathbb{H} \subset \fix(f^n) \text{ be the Nielsen class containing } \mathbb{H}_0. \text{ By the index product formula, } \mathbb{H} \text{ is essential. Moreover since } \mathbb{H} \text{ is irreducible, so is } p(\mathbb{H}) = \tilde{\mathbb{H}}. \]

\[ \therefore \text{Now we assume that } n \in \nper(f). \text{ Thus there is an essential irreducible class } \mathbb{H}^n \text{ of } f^n. \text{ Since } T^2 \text{ is essentially reducible, } p(\mathbb{H}^n) \text{ will be reducible to an essential irreducible class at some level } k|n \text{ [12]. Let } b \in \fix(f^k) \text{ be an element of } \tilde{\mathbb{H}}^n. \text{ Then } \tilde{\mathbb{H}}^n \text{ is comprised of an essential irreducible class } (f^n)^b \text{ of degree } (-1)^n \text{ at level } \frac{n}{k}. \text{ Since the map } f \text{ of degree } -1 \text{ has } \nper(f) = \{1\} \text{ we see that } k = n \text{ and so } n \in \nper(\tilde{f}). \text{ Furthermore, since } (f^n)_b \text{ will be a degree of } (-1)^n \text{ it follows that } n \text{ must be odd.} \]

\[ \bullet \left| d \right| = 1 \text{ and the action is nontrivial.} \]

We will show that in this case \( \nper(f) = \nper(\tilde{f}) \).

\[ \therefore \text{Let } n \in \nper(\tilde{f}). \text{ By Lemma 3.3 there exists an irreducible essential class } \tilde{\mathbb{H}}^n \subset \fix(\tilde{f^n}) \text{ of odd parity. Then } \deg(f^n_b) = -1 \text{ for } b \in \tilde{\mathbb{H}}^n. \text{ Let } \mathbb{H}_0 \subset \fix((f^n)_b) \text{ be an essential Nielsen class and suppose that } \mathbb{H} \subset \fix(f^n) \text{ is the class containing } \mathbb{H}_0. \text{ By the index product formula [32, Theorem 4.1] } \mathbb{H} \text{ is essential. Since } \tilde{\mathbb{H}} \text{ is irreducible, so is } \mathbb{H}. \text{ Thus } n \in \nper(f). \]

\[ \therefore \text{Let } n \in \nper(f) \text{ and } \mathbb{H} \subset \fix(f^n) \text{ be an irreducible essential class. Then as above } p\mathbb{H} \text{ must be essential and irreducible at level } n. \]

**Remark 3.6.** Note that Theorem 3.5 holds for any fibre map of a fibration of the form \( S^1 \hookrightarrow M \to \mathbb{T}^2 \). In particular for a fibration where the action of \( \pi_1(\mathbb{T}^2) \) on \( \pi_1(S^1) \) is trivial, the corresponding manifold \( M \) is a nilmanifold (cf. Lemma 2.8). Then, in this situation, the fibration \( S^1 \hookrightarrow M \to \mathbb{T}^2 \) is the Fadell–Husseini fibration (cf. [6]) (the Mostow fibration of \( M \) is trivial since \( M \) is a nilmanifold). Consequently, every self-map of \( M \) is homotopic to a fibre map on this fibration (cf. [6]), and our consideration gives a new proof, and new formulation, of the main theorem of [21].

Observe next that the product \( S^1 \times \mathbb{K} \) is the model manifold, in the sense of [10] (cf. 2.19), associated to the flat manifold \( M \) of Example 2.16. Indeed, the model of a given solvmanifold is obtained by the linearization procedure in [10]. In this case we produce a solvmanifold which corresponds to the twisted product \( \mathbb{Z}^1 \times_\rho \mathbb{Z}^2 \), where \( \rho : \mathbb{Z}^2 \to \{-1, 1\} = \text{Aut}(\mathbb{Z}) \) is a nontrivial homomorphism. This is isomorphic to the group \( I' \) of (5) which by Theorem 2.2 means that it is diffeomorphic to \( S^1 \times \mathbb{K} \).

### 4. Fibration over the circle with 2-torus as the fibre

In this section we consider those three-dimensional solvmanifolds \( M \) which fibre as \( \mathbb{T}^2 \hookrightarrow M \to S^1 \). This case will be divided into three subcases according to the degree of the base map \( d \). Since many distinct situations appear, we confine our consideration to the non-\( NR \)-solvmanifolds. The case of \( NR \)-solvmanifolds was described in [25].

Here is the schedule for the description of \( \nper(f) \) of the 3-dimensional non-\( NR \)-solvmanifolds fibering over \( S^1 \).

1. \( d \) is regular \( (d \notin \{-2, -1, 0, 1\}) \) (Section 4.1, Theorem 4.2).
2. \( |d| \leq 1 \) (Section 4.2, Theorem 4.6).
3. \( d = -2 \) (Section 4.3, Theorem 4.7).

However we start with some general remarks and some new notions. Let us remind the reader that for tori the family of all possible \( \nper(f) \) sets naturally splits up into three mutually disjoint subfamilies (E), (F), and (G), called empty, finite, and generic, respectively (cf. [24]). What is important is that the type of \( \nper(f) \) is determined by the spectrum of the matrix of the linearization \( A_f : \mathbb{Z}^n \to \mathbb{Z}^n \). The “generic” case occurs when the set \( \nper(f) \) is infinite. Such a characterization still holds for other types of solvmanifolds (nilmanifolds [20], special completely...
Theorem 4.2. Dimensional nilmanifold \([20]\), generic classes of matrices \(A_f\) for which HPer\((f)\) \(\neq \mathbb{N}\) are called special generic. We remind the reader that in the case of \(M = S^1\) the linearization is the degree \(\deg(f)\) of \(f\). Here we have that \((E) = \{1\}\), \((F) = \{-1, 0\}\), and the special generic situation corresponds to \(\{-2\}\) (cf. [4], and also [22]).

The above analysis justifies calling the linearization matrix \(A_f \in \mathcal{M}_{n \times n}(\mathbb{Z})\) regular generic, or just regular, when HPer\((f) = \mathbb{N}\) for a map \(f : M \to M\) of an \(NR\)-solvmannifold \(M\).

Since the homotopy periods of self-maps of \(NR\)-solvmannifolds are described in [22], we concentrate here on the non-\(NR\)-solvmannifolds.

Each fibration over \(S^1\), with fibre \(\mathbb{T}^2\), can be obtained from the trivial fibration \([0, 1] \times \mathbb{T}^2 \to [0, 1]\) by identifying the fibres over the points 0 and 1. This can be done with \((0, x) \sim (1, Ax)\) using an invertible matrix \(A \in \mathcal{M}_{2 \times 2}(\mathbb{Z})\).

The total space of the above fibration is a non-\(NR\)-solvmannifold iff the matrix \(A\) has an eigenvalue other than 1 which is root of unity (cf. Corollary 2.18). The next lemma says that each such matrix is similar, over \(\mathbb{C}\), to one of six possibilities. We will consider these six cases separately.

**Lemma 4.1.** If \(A \in \mathcal{M}_{2 \times 2}(\mathbb{Z})\) is a non-\(NR\)-matrix and \(\det A = \pm 1\), then \(A\) is similar, using the conjugation with a matrix with real entries, to one of the matrices given below:

\[
\begin{bmatrix}
-1 & \lambda \\
0 & -1
\end{bmatrix}, \begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}, \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}, \begin{bmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{bmatrix}
\]

for \(\alpha = \pm \pi/3, \pm \pi/2, \pm 2\pi/3\).

**Proof.** Note that since \(A \in \mathcal{M}_{2 \times 2}(\mathbb{Z})\), thus its characteristic polynomial \(\chi_A(\lambda) = \lambda^2 - \text{tr} A \lambda + \det A \in \mathbb{Z}[\lambda]\). Consequently a root of unity is an eigenvalue of \(A\) only if the corresponding cyclotomic polynomial divides \(\chi_A(\lambda)\). But there are only five cyclotomic polynomials of degree \(\leq 2\) corresponding to \(n\)th roots of unity with \(n\) equal to 1, 2, 3, 4 or 6. In each case there exists an invertible matrix \(A \in \mathcal{M}_{2 \times 2}(\mathbb{Z})\) giving such a cyclotomic polynomial. \(\square\)

Let \(M\) be a 3-dimensional non-\(NR\)-solvmannifold which fibres over \(S^1\). Recall from our considerations on the trivial group cohomology which is involved, that these are model solvmannifolds. Now from the theory of model solvmannifolds (cf. in [10]) it follows that each fibre map \(f : M \to M\), over the map \(\bar{f} : S^1 \to S^1\) of degree \(d\), is homotopic to a map whose restriction to the fibre is induced by a linear map \(X \in \mathcal{M}_{2 \times 2}(\mathbb{Z})\), where \(A^d X = X A\). The collection of \((X, d)\) which satisfy \(X A = A^d X\) correspond precisely to the periodic Nielsen theory of all possible self-maps of \(M\).

At first we will consider the case when the degree of the base map is regular, i.e. \(d \notin \{-2, -1, 0, +1\}\). We will then consider the remaining cases individually. With the exception of \(d = -2\) they will be easy. The case \(d = -2\) always involves unexpected irregularities. We start with some observations.

Let \(\tilde{f}_d : S^1 \to S^1\) be a self map of the circle \(S^1 = \mathbb{R}/\mathbb{Z}\) given by the formula \(\tilde{f}_d[t] = [dt]\). We assume that \(d\) is regular. Then

\[
\text{Fix}(\tilde{f}_d) = \{\epsilon_0^d, \epsilon_1^d, \ldots, \epsilon_{|d-1|-1}^d\}
\]

where \(\epsilon_s^d = \lfloor s/d \rfloor \lfloor d-1 \rfloor \rfloor\). Similarly Fix\((\tilde{f}_d^k) = \{\epsilon_0^d, \epsilon_1^d, \ldots, \epsilon_{|d-1|-1}^d\}\). If \(l|k\) then \(i_{k,l}\) sends Fix\((\tilde{f}_d^k) \subset \text{Fix}(\tilde{f}_d^l)\) via

\[
\epsilon_s^d \to \epsilon_{s+l}^d, \epsilon_{2s+1}^d, \ldots, \epsilon_{s+l-1}^d.
\]

**4.1.** \(d\) is regular

In this section we will show

**Theorem 4.2.** Under the above notations (\(d\) is regular):

- When \(A = -I\) and Spec\((X) = \{+1, -1\}\) then HPer\((f) = 2\mathbb{N}\).
• When \( A \) is similar to \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) then

\[
\text{HPer}(f) = \begin{cases} \\
\emptyset & \text{for } 1 \in \text{Spec}(X) \cap \text{Spec}(AX), \\
\mathbb{N} \setminus 2\mathbb{N} & \text{for } -1 \in \text{Spec}(X) \cap \text{Spec}(AX), \\
\mathbb{N} & \text{otherwise}.
\end{cases}
\]

• When \( \text{Spec}(A) \) is complex on the unit circle and \( \text{Spec}(X) = \{+1, -1\} \) then \( \text{HPer}(f) = 2\mathbb{N} \).

• Otherwise \( \text{HPer}(f) = \mathbb{N} \).

The proof will split into several parts, each for a type of the matrix \( A \). They will be presented in the next subsections.

The next lemma will allow us to pinpoint the fibres where all essential contributions to \( \text{HPer}(f) \) occur. It implies a sufficient condition for \( k \in \text{HPer}(f) \).

**Lemma 4.3.** If \( d \) is regular and \( k \geq 2 \) then the points \( \epsilon^{d^k}_1, \epsilon^{d^k}_2 \in S^1 \) represent irreducible essential classes in \( \text{HPer}(f^k) \).

**Proof.** Suppose \( l|k \). Since the inclusion \( \text{Fix}(f^k) \subset \text{Fix}(f^k) \) maps \( \epsilon^{d^k}_s \to \epsilon^{d^k}_{s+1} + d^k \), it is enough to notice that, for \( d \) regular all \( s = 1, \ldots, |d^k - 1| - 1 \),

\[
2s \cdot |1 + d^k + d^k + \cdots + d^k| < |d^k - 1|.
\]

This follows from

\[
2 < |d^k - 1| = s \cdot |1 + d^k + d^k + \cdots + d^k| \leq (|d^k - 1| - 1) \cdot \frac{|d^k - 1|}{|d^k - 1|} < |d^k - 1|.
\]

**Corollary 4.4.** If \( \mathbb{R}^2 \to M \to S^1 \) is the fibration given by the matrix \( A \) and \( f : M \to M \) a fibre map given by \( (X,d) \) with \( d \) regular then

\[
(\det(AX^k - 1) \neq 0 \text{ or } \det(A^2X^k - 1) \neq 0) \implies k \in \text{HPer}(f).
\]

**Proof.** By Lemma 4.3, \( \epsilon^{d^k}_1 \) and \( \epsilon^{d^k}_2 \) represent essential irreducible classes of \( f^k \). Therefore any essential classes in \((f^k)_b \) for \( b \in \epsilon^{d^k}_1 \) or \( \epsilon^{d^k}_2 \) will be essential and irreducible in \( f^k \). Since one of \( L((f^k)_b) \) must be nonzero, the result follows.

**Lemma 4.5.** Let \( A, B, X \in M_{2 \times 2}(\mathbb{R}) \) satisfy \( AX = XB \). If moreover \( A, B \) are represented by the matrices of the form \( \begin{pmatrix} a & -b \\ b & \alpha \end{pmatrix} \), with eigenvalues of modulus 1, then at least one of the following alternatives hold:

1. \( X = \begin{pmatrix} \alpha & -\beta \\ \beta & a \end{pmatrix} \) (for some \( \alpha, \beta \in \mathbb{R} \)) and \( A = B \),
2. \( X = \begin{pmatrix} a & \beta \\ \beta & -a \end{pmatrix} \), \( A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \), \( B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \),
3. \( A = B = I \),
4. \( A = B = -I \),
5. \( X = 0 \).

**Proof.** Note that the spectrum of \( \begin{pmatrix} a & -b \\ b & \alpha \end{pmatrix} \) is \( \{a \pm \beta i\} \). Let us suppose that \( X \) is neither of the form \( X = \begin{pmatrix} a & -b \\ \beta & a \end{pmatrix} \) nor \( X = \begin{pmatrix} a & \beta \\ \beta & -a \end{pmatrix} \). Then the image of the unit circle \( S^1 = \{v \in \mathbb{R}^2 : \|v\| = 1\} \) under \( X \) is an ellipse but not a circle. Hence \( \|Xv\| \) realizes its maximum on \( S^1 \) exactly in two antipodal points \( v \) and \(-v\). Now \( \|X(v)\| = \|AX(v)\| = \|XB(v)\| \)

implies that either \( B(v) = v \) or \( B(v) = -v \). Since the eigenvalues of \( B \) are \( a \pm \beta i \) we must therefore have either \( B = I \) or \( B = -I \), respectively. This gives \( AX = X \) or \( AX = -X \) respectively and hence we have the following alternatives: either \( A = B = I \) or \( A = B = -I \) or \( X = 0 \).

Now we assume that \( X = \begin{pmatrix} a & \beta \\ \beta & -a \end{pmatrix} \) and \( X \neq 0 \). This means that \( X \) is invertible. Now \( X = X' I' \) where \( X' = \begin{pmatrix} a & -\beta \\ \beta & a \end{pmatrix} \) and \( I' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Thus \( B = X^{-1}AX \), which by substitution gives \( B = I'^{-1}X'^{-1}AX'I' \). Since the twists \( A \) and \( X' \) commute, this simplifies to \( B = I'^{-1}AI' \). Now the spectra of the twists \( A \) and \( B \) coincide, thus giving us case 2.
Let $X = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$. If $X \neq 0$ then $X$ is an isomorphism and $B = X^{-1}AX$. This implies that the twists $A$ and $B$ have the same spectrum. One can then check that the relationship between $A$ and $B$ of case 2 will not work and thus $A = B$ and we have case 1. \qed

Now we are in a position to consider each of the cases for $A$ in Lemma 4.1. Lemma 4.5 tells us that only the conjugacy class of $A$ is of importance.

### 4.1.1. Spec($A$) = $\{-1, -1\}$, $A$ non-diagonalizable, $d$ regular

Let $A$ be non-diagonalizable, Spec($A$) = $\{-1, -1\}$ and suppose that $A^dX = XA$. We will show that $d$ even implies $X = 0$ so HPer($f$) = HPer($\tilde{f}$) = $\mathbb{N}$, since $d$ is regular. For $d$ odd we show that for each $k$ the assumptions of Corollary 4.4 hold. This gives us that HPer($f$) = $\mathbb{N}$ here as well.

Since $A$ is similar to $B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, there is a matrix $C \in \mathcal{M}_{2 \times 2}(\mathbb{R})$ such that $B = C^{-1}AC$. Let us denote $Y = C^{-1}X$. We notice that $B^d = (-1)^d \begin{pmatrix} 1 & -d \\ 0 & 1 \end{pmatrix}$. Then the condition $B^dY = YB$ implies that

$$(1) \begin{pmatrix} 1 & -d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix},$$

$$(2) \begin{pmatrix} \alpha - \gamma d & \beta - \delta d \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} -\alpha & \alpha - \beta \\ -\gamma & \gamma - \delta \end{pmatrix}.$$ 

If $d$ is even then we can conclude

$$\alpha - \gamma d = -\alpha, \quad \beta - \delta d = \alpha - \beta,$$

$$\gamma = -\gamma, \quad \delta = \gamma - \delta.$$ 

Thus $\alpha = \beta = \gamma = \delta = 0$ so $Y = 0$, hence $X = 0$.

If $d$ is odd then

$$\alpha - \gamma d = \alpha, \quad \beta - \delta d = -\alpha + \beta,$$

$$\gamma = \gamma, \quad \delta = \delta - \gamma.$$ 

This forces $\gamma = 0$ and $\alpha = \delta d$ with $\beta$ arbitrary and so $Y = \begin{pmatrix} \delta d & \beta \\ 0 & \delta \end{pmatrix}$. Thus $Y = \begin{pmatrix} (\delta d)k & * \\ 0 & \delta k \end{pmatrix}$, $BY = \begin{pmatrix} (\delta d)k & * \\ 0 & -\delta k \end{pmatrix}$ and $B^2Y = \begin{pmatrix} (\delta d)k & * \\ 0 & \delta k \end{pmatrix}$. Suppose that $1$ is the eigenvalue of both $BY$ and $B^2Y$. Then

$$0 = \det(BY - I) = (-(\delta d)k - 1)(-\delta k - 1) = ((\delta d)k + 1)(\delta k + 1),$$

$$0 = \det(B^2Y - I^2) = ((\delta d)k - 1)(\delta k - 1).$$

Since $|d| \geq 2$ the above two equalities have no common integer solutions. In other words, the assumptions of Corollary 4.4 are satisfied.

### 4.1.2. $A = -I$, $d$ regular

We show that then

$$\text{HPer}(f) = \begin{cases} 2\mathbb{N} & \text{for } \text{Spec}(X) = \{+1, -1\}, \\
\mathbb{N} & \text{otherwise.} \end{cases}$$

We notice that for any $2 \times 2$ matrix $X, 1 \in \text{Spec}(X) \cap \text{Spec}(-X)$ if and only if Spec($X$) = $\{+1, -1\}$.

Let $d$ be even. Then $A^dX = XA$ gives $+X = -X$ and hence $X = 0$. Then HPer($f$) = HPer($\tilde{f}$) = $\mathbb{N}$. This gives the claim, since then Spec($X$) = $\{0\} \neq \{+1, -1\}$.

Let $d$ be odd. Then $A^dX = XA$ gives $-X = -X$ and hence $X$ is arbitrary. Now $A^sX = -X$ for $s$ odd and $A^sX = X$ for $s$ even. Suppose that Spec($X$) = $\{+1, -1\}$ and $k$ is odd. Then we also have that, for any $s, \text{Spec}(A^sX) = \{+1, -1\}$. Thus there are no essential classes in the fibres, hence no essential irreducible classes cannot result from essential irreducible classes of $\tilde{f}$. Now suppose that $k$ is even. Then Spec($AX$) = Spec($A$) = $\{-1\}$ and Lemma 4.3 gives that $k \in \text{HPer}(f)$.

On the other hand, if Spec($X$) $\neq \{+1, -1\}$ then, for any $k, \text{Spec}(X) \neq \{+1, -1\}$ and as we have noticed, either $1 \notin \text{Spec}(X)$ or $1 \notin \text{Spec}(AX)$. Thus Lemma 4.3 tells us that $k \in \text{HPer}(f)$.
4.1.3. Spec(A) = \{+1, -1\} with d regular

Let A, X ∈ \mathcal{M}_{2 × 2}(\mathbb{Z}) satisfy A^dX = XA and suppose that Spec(A) = \{+1, -1\}.

Then we will show that then

\[
HPer(f) = \begin{cases}
\emptyset & \text{for } 1 \in \text{Spec}(X) \cap \text{Spec}(AX), \\
\mathbb{N} \setminus 2\mathbb{N} & \text{for } -1 \in \text{Spec}(X) \cap \text{Spec}(AX) \text{ but } 1 \notin \text{Spec}(X) \cap \text{Spec}(AX), \\
\mathbb{N} & \text{otherwise}.
\end{cases}
\]

The matrix A is similar to \(B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\) (i.e. \(B = C^{-1}AC\) for some \(C \in \mathcal{M}_{2 × 2}(\mathbb{R})\)). Let \(Y = C^{-1}XC\).

We notice that \(A^d = I\) for \(d\) even and \(A^d = A\) for \(d\) odd, and similarly for \(B\).

Suppose that \(d\) is odd. Then \(A^dX = XA\) gives \(AX = XA\). Similarly we get \(BY = YB\). We denote \(Y = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\). Now

\[
Y^k = \begin{pmatrix} \alpha^k & 0 \\ 0 & \delta^k \end{pmatrix}, \quad BY^k = \begin{pmatrix} \alpha^k & 0 \\ 0 & -\delta^k \end{pmatrix} \quad \text{and} \quad B^2Y^k = Y^k.
\]

This implies

\[
\text{Spec}(X^k) = \text{Spec}(Y^k) = \{\alpha^k, \delta^k\}, \quad \text{Spec}(AX^k) = \text{Spec}(BY^k) = \{\alpha^k, -\delta^k\} \quad \text{and} \quad A^2X^k = X^k.
\]

Suppose that \(1 \in \text{Spec}(X) \cap \text{Spec}(AX)\). Then \(\alpha = 1\) and \(1 \in \text{Spec}(A^kX^s)\) for all \(s, k \in \mathbb{N}\) hence \(HPer(f) = \emptyset\).

Suppose that \(-1 \in \text{Spec}(X) \cap \text{Spec}(AX)\). Then \(\alpha = -1\). For \(k\) even \(A^kX^k = \begin{pmatrix} 1 & 0 \\ 0 & (\gamma^2) \end{pmatrix}\) hence \(k \notin HPer(f)\). For \(k\) odd,

\[
\text{det}(AX^k - I) = \text{det}(BY^k - I) = \text{det} \begin{pmatrix} -1 & 0 \\ 0 & -\delta^k + 1 \end{pmatrix} = 2(\delta^k + 1),
\]

\[
\text{det}(A^2X^k - I) = \text{det}(B^2Y^k - I) = \text{det}(X^k - I) = -2(\delta^k - 1).
\]

\(k \notin HPer(f)\). Thus \(HPer(f) = \mathbb{N} \setminus 2\mathbb{N}\).

Now we suppose that \(\text{Spec}(X) \cap \text{Spec}(AX)\) contains neither \(+1\) nor \(-1\). Thus \(\alpha \neq \pm 1\) and

\[
\text{det}(AX^k - I) = \text{det}(BY^k - I) = \text{det} \begin{pmatrix} \alpha^k - 1 & 0 \\ 0 & -\delta^k + 1 \end{pmatrix} = -(\alpha^k - 1)(\delta^k + 1),
\]

\[
\text{det}(A^2X^k - I) = \text{det}(B^2Y^k - I) = \text{det} \begin{pmatrix} \alpha^k - 1 & 0 \\ 0 & \delta^k - 1 \end{pmatrix} = (\alpha^k - 1)(\delta^k - 1).
\]

Since these two equalities cannot hold simultaneously, we have that \(HPer(f) = \mathbb{N}\).

Now we assume that \(d\) is even. Thus \(A^dX = XA\) gives \(X = XA\) and \(X^T = A^T X^T\). Hence \(Y^T = B^T Y^T\), so the columns of \(Y^T\) are eigenvectors of \(B\) corresponding to eigenvalue 1. Now \(Y^T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\) which implies that \(Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\).

Moreover \(Y^k = \begin{pmatrix} \alpha^k & 0 \\ 0 & 0 \end{pmatrix}\). Now

\[
\text{det}(AX^k - I) = \text{det}(BY^k - I) = \text{det} \begin{pmatrix} \alpha^k - 1 & 0 \\ 0 & -1 \end{pmatrix} = 1 - \alpha^k,
\]

\[
\text{det}(A^2X^k - I) = \text{det}(B^2Y^k - I) = \text{det}(Y^k - I) = 1 - \alpha^k.
\]

We also notice that \(\text{Spec}(X) = \text{Spec}(AX) = \{\alpha, 0\}\).

If \(1 \in \text{Spec}(X) \cap \text{Spec}(AX)\) then \(\alpha = 1\) and \(1 \in \text{Spec}(A^kX^s)\) for all \(s, k \in \mathbb{N}\). Hence \(HPer(f) = \emptyset\).

If \(-1 \in \text{Spec}(X) \cap \text{Spec}(AX)\) but \(+1\) does not then \(\alpha = -1\). Then \(1 \in \text{Spec}(A^kX^k)\) exactly for \(k\) even and hence \(HPer(f) = \mathbb{N} \setminus 2\mathbb{N}\).

If \(\text{Spec}(X) \cap \text{Spec}(AX)\) does not contain any of \(\pm 1\) then \(\text{det}(AX^k - I) = (\alpha^k - 1)(-1) \neq 0\), for all \(k \in \mathbb{N}\) and \(HPer(f) = \mathbb{N}\).
4.1.4. Spec(A) is complex, d regular

Let $A \in M_{2 \times 2}(\mathbb{Z})$ have a complex eigenvalue. We will prove that, for each $k$, one of the determinants in Corollary 4.4 is not zero. This will give us $H(\bar{f}) = \mathbb{N}$. To count the determinants we will change coordinates. Since $A$ has a conjugate pair of complex eigenvalues, we may choose a basis and assume that $A = \begin{pmatrix} a & -\bar{b} \\ \bar{b} & a \end{pmatrix}$ where $a \pm bi$ are the eigenvalues of $A$. Since they are roots of unity, they lie on the unit circle.

If $X = 0$, then clearly $H(\bar{f}) = H(\bar{f}) = \mathbb{N}$. Assume that $X \neq 0$. Then, by the equality $A^d X = X A$, $X$ is either of the form $X = \begin{pmatrix} \alpha & -\bar{\beta} \\ \bar{\beta} & \alpha \end{pmatrix}$ or $X = \begin{pmatrix} \alpha & \bar{\beta} \\ -\bar{\beta} & \alpha \end{pmatrix}$ for some $\alpha, \beta \in \mathbb{R}$ (case 2 of Lemma 4.5). Let us notice that det $X > 0$ in the first case and det $X < 0$ in the second case. Let us fix $k \in \mathbb{N}$. We will show that at least one of the spectra $\text{Spec}(A X^k)$ or $\text{Spec}(A^2 X^k)$ does not contain 1.

Suppose otherwise and let us consider the case det $X > 0$. Since $A X^k$ and $A^2 X^k$ are also of the form $\begin{pmatrix} \alpha' & -\beta' \\ \beta' & \alpha' \end{pmatrix}$, $A X^k = I$ and $A^2 X^k = I$. This implies that $A = I$ contrary to our assumptions.

In the case where det $X < 0$, the form of $X^k$ can be of two possible types. When $k$ is odd this is the same form as $X$, and when $k$ is even $X^k$ is a scalar multiple of $I$. $A^2$ will have the same form as $A$ so when $k$ is odd, $A^2 X^k$ will be like $X$ and when $k$ is even $A^2 X^k$ will be like $A$ in form. For $k$ even if $1 \in \text{Spec}(A X^k) \cap \text{Spec}(A^2 X^k)$ then it means that $A X^k = A^2 X^k = I$ so $A = I$ which produces a contradiction as before. For $k$ odd, $A X^k = \begin{pmatrix} \delta & \gamma \\ -\gamma & \delta \end{pmatrix}$ whose eigenvalues are $\pm \sqrt{\delta^2 + \gamma^2}$. Thus $A^2 X^k$ will have eigenvalues $\pm \sqrt{(\delta^2 + \gamma^2)(\delta^2 + \gamma^2)} = \pm \sqrt{\delta^2 + \gamma^2}$ $(a \pm bi$ is a root of unity by the assumption). Thus $1$ can belong to both $\text{Spec}(A X^k)$ and $\text{Spec}(A^2 X^k)$ provided $X$ and has one (and hence both) eigenvalues on the unit circle. In this case, no odd number will belong to H(\bar{f}).

4.2. $|d| \leq 1$

In this short subsection we will prove:

**Theorem 4.6.** Let $f : M \to M$ be a map of a 3-solvmanifold which fibres over the circle $p : M \to S^1$ and let the action of the base on the fibre be given by the matrix $A$. Suppose that the linearization data for $f$ is given by $(X, d)$. Assume that $|d| \leq 1$. Then

$$H(\bar{f}) = \begin{cases} \emptyset & \text{for } d = 1, \\ H(X) & \text{for } d = 0, \\ (H(X) \cup H(AX)) \setminus 2\mathbb{N} & \text{for } d = -1. \end{cases}$$

**Proof.** The cases $d = 0, 1$ are evident. We assume that $d = -1$. Then $\bar{f}$ is the flip map. After a small deformation $\tilde{f}$ has two fixed points $b_0, b_1$ and no other periodic points. If $n$ is even then the unique Nielsen class of $\tilde{f}$ is inessential. Thus $f^n$ has no essential classes and we can conclude that $n \notin H(\bar{f})$. If $n$ is odd then the points $\{b_0\}$, $\{b_1\}$ are essential reducible classes. Since the fibre-maps $f_{b_0}$, $f_{b_1}$ have linearizations given by $X$ and $AX$, respectively, the formula follows. \hfill \Box

4.3. Case $d = -2$

Now we assume that $d = -2$. Then all the calculations from Section 4.1 are identical with the exception of $k = 2$. Thus $H(\bar{f}) \setminus \{2\} = H \setminus \{2\}$ where $H$ is given by Theorem 4.2 and the only question is to decide whether $2 \in H(\bar{f})$.

We will therefore focus our consideration on a fibration $\mathbb{T}^2 \to M \to S^1$ given by a non-$NR$-matrix $A \in M_{2 \times 2}(\mathbb{Z})$. Let $f : M \to M$ be a fibre map determined by $(X, -2)$ where $X \in M_{2 \times 2}(\mathbb{Z})$. This means $\text{deg}(f) = -2$ and $A^{-2} X = X A$. Under what condition is $2 \in H(\bar{f})$?

**Theorem 4.7.** Under the above assumptions, $2 \notin H(\bar{f})$ if and only if one of the following conditions holds:

1. $X = 0$,
2. $1$ is an eigenvalue of $A$ and $\text{Spec} X = \{0, a\}$ where $a = -2, -1, 0$, or $+1$,
3. $e^{2\pi i/3} \in \text{Spec} A$ and $X = I$ (identity matrix).
**Proof.** We may assume that \(\tilde{f}(z) = z^{-2}\). Then \(\text{Fix} (\tilde{f}) = \text{Fix}(\tilde{f}^2) = \{b_0, b_1, b_2\}\) consists of three points and each of them comprises a singleton essential Nielsen class. In particular \(\tilde{f}^2\) has no irreducible classes. The linearization of the restrictions of \(f\) over these points are given respectively by matrices \(X, A^{-1}X\) and \(A^{-2}X\). Then \(2 \notin \text{HPer}(f) \iff 2 \notin \text{HPer}(f_{b_i})\) for all \(i = 0, 1, 2\). This is equivalent to having

\[
N(\tilde{f}_0^2) \leq N(f_{b_i}) \quad \text{or equivalently} \quad |\det((A^{-1}X^2 - I)| \leq |\det(A^{-i}X - I)|
\]

for all \(i = 0, 1, 2\). Since \(\det((A^{-i}X)^2 - I) = \det(A^{-i}X - I) \cdot \det(A^{-i}X + I)\), the above holds \(\iff\) one of the following is true for each \(i = 0, 1, 2\):

\[
\det(A^{-i}X - I) = 0 \quad \text{or} \quad \det(A^{-i}X + I) = 0 \quad \text{or} \quad \det(A^{-i}X + I) = +1 \quad \text{or} \quad \det(A^{-i}X + I) = -1.
\]

Since each non-\(NR\)-matrix \(A \in M_{2 \times 2}(\mathbb{Z})\) is conjugate to one of the six possibilities listed (4.1) and each of the above equalities is independent of conjugation, it is enough to consider only these six possibilities.

- Let \(A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\).
  Then \(A^{-2} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\) hence \(A^{-2}X = AX\) gives
  \[
  \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}
  \]
  hence
  \[
  \begin{bmatrix} a + 2c & b + 2d \\ c & d \end{bmatrix} = \begin{bmatrix} -a & a-b \\ -c & c-a \end{bmatrix}
  \]
  which in turn implies that \(a = b = c = d = 0\) and hence that \(X = 0\).

- Let \(A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\).
  Then \(A^{-2} = I\) hence \(A^{-2}X = AX\) gives \(X = -X\) which implies \(X = 0\).

- Let \(A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\). We denote \(X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\). Now \(A^{-2} = I\) so \(A^{-2}X = AX\) implies \(X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & -b \\ c & -d \end{bmatrix}\) hence \(X = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}\).
  We show that the four requirements hold exactly for \(a \in \{-2, -1, 0, +1\}\) and arbitrary \(c\).
  For \(i = 0\) we have:
  \[
  \begin{align*}
  \det(X-I) &= 0 \quad \Rightarrow \quad 1-a = 0 \quad \Rightarrow \quad a = 1, \\
  \det(X+I) &= 0 \quad \Rightarrow \quad a+1 = 0 \quad \Rightarrow \quad a = -1, \\
  \det(X+I) &= +1 \quad \Rightarrow \quad a+1 = +1 \quad \Rightarrow \quad a = 0, \\
  \det(X+I) &= -1 \quad \Rightarrow \quad a+1 = -1 \quad \Rightarrow \quad a = -2.
  \end{align*}
  \]
  Now we consider \(A^{-1}X = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}\) and \(A^{-2}X = X\). Now the prescribed conditions are exactly the same as in the previous case and hence the same numbers satisfy the equations. This gives the second case of the theorem.

- Let \(A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\).
  This matrix represents the twist with the angle \(\pi/2\). Then \(A^{-2} = -I\) and hence \(A^{-2}X = AX\) which gives that \(X = XA\). Therefore \(A^T X^T = -X^T\) hence the columns of \(X^T\) are eigenvectors of \(A^T\) corresponding to \(\lambda = -1\).
  However, since \(-1 \notin \text{Spec} A = \text{Spec} A^T\) we must again conclude that \(X = 0\).

- \(A\) represents the twist with the angle \(\pi/3\). Now \(A^{-2}\) is the twist with angle \(-2\pi/3\) and hence \(A^{-2} = -A\). Now \(-AX = XA\) which implies that \(X = 0\) (Lemma 4.5). A similar reasoning applies to the twist with the angle \(-\pi/3\) because it is also conjugate to \(A\).

- Twist with angle \(2\pi/3\).
  Then \(A^{-2}\) is the twist with angle \(-4\pi/3\) and hence \(A^{-2} = A\). This yields \(AX = XA\). Moreover \(A = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}\) and examining the entries of \(AX = XA\) we see that \(X\) has the form \(X = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\) for some \(a, b \in \mathbb{R}\) (Lemma 4.5).
  We will show that either \((a, b) = (0, 0)\) or \((a, b) = (1, 0)\).

We note that since \(X\) is conjugate to a matrix over \(\mathbb{Z}\), \(\text{tr}(X) = 2a\) and \(\det(X) = a^2 + b^2\) must be integers. We observe that \(\text{Spec}(X) = \{a + bi, a - bi\}\). Since \(A^{-i}X\) will have the same form as \(X\) it follows that if \(\det(A^{-i}X \pm I) = 0\) then \(A^{-i}X = \pm I\) and therefore that \(X = \pm A^{-i}\). This yields possible \((a, b)\) pairs:
First we consider the case when

\[
(1,0), (-1,0) \quad \text{for } i = 0,
\]

\[
(-1/2, \sqrt{3}/2), (1/2, -\sqrt{3}/2) \quad \text{for } i = 1,
\]

\[
(-1/2, -\sqrt{3}/2), (1/2, \sqrt{3}/2) \quad \text{for } i = 2.
\]

For each of the other possibilities where \(\det(A^{-i}X + I) = \pm 1\) we must do the following:

1. Solve \(\det(A^{-i}X + I) = \pm 1\) for \(a\).
2. Use this expression to compute \(\det(X) = a^2 + b^2\) and \(\text{tr}(X) = 2a\) in terms of \(b\).
3. Look at the possibilities forced on \(b\) (and hence \(a\)) for both these quantities to be integers. (In some instances this cannot happen.)

The resulting possibilities are

\[
(-1,-1), (-1,1), (\pm 1/\sqrt{2}, \pm \sqrt{3}/2), (0,0), (-2,0) \quad \text{for } i = 0,
\]

\[
(1,0), (3/2, -\sqrt{3}/2), (-1/2, -\sqrt{3}/2) \quad \text{for } i = 1,
\]

\[
(1,0), (0,0), (3/2, \sqrt{3}/2), (-1/2, \sqrt{3}/2), (1, \sqrt{3}), (0,\sqrt{3}) \quad \text{for } i = 2.
\]

Therefore, the only way for at least one of the required equalities to hold for each of \(i = 0,1,2\) is when \((a,b) = (1,0)\) which clearly implies that \(X = I\). \(\square\)

5. Applications and final discussion

As in the previous papers (see [24,21,19,22] for the 3-torus, 3-dimensional nilmanifolds, and 3-dimensional non-\(NR\)-solvmanifolds, respectively) we would like to formulate a Šarkovskii type theorem for self-maps of 3-dimensional non-\(NR\)-solvmanifolds. Moreover we would like to describe the collection of possible homotopy minimal periods for homeomorphisms of 3-dimensional non-\(NR\)-solvmanifold as in [13,21,19]. The next result follows from our previous analysis.

**Theorem 5.1.** Let \(f : M \to M\) be a homeomorphism of a 3-dimensional solvmanifold which is not diffeomorphic to the torus \(T^3\), i.e. it fibres over \(S^1\) or \(T^2\).

1. If \(M\) is of type \((2,1)\) then depending of the action of the base on the fibre \(A : \mathbb{Z} \to \text{Aut}(\mathbb{Z}^2)\) we have the following possibilities for \(\text{HPer}(f)\):

   \[
   \begin{array}{c|c|c|c}
   A(1) & \text{Empty} & \text{Finite} & \text{Generic} \\
   \hline
   -I & \emptyset & \{1\}, \{1,3\} & \mathbb{N} \setminus 2\mathbb{N} \\
   \begin{bmatrix} -1 & \lambda \\ 0 & -1 \end{bmatrix}_{\lambda \in \mathbb{R}, \lambda \neq 0} & \emptyset & \text{Impossible} & \text{Impossible} \\
   \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \emptyset & \{1\} & \mathbb{N} \setminus 2\mathbb{N} \\
   \begin{bmatrix} \pm \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2} \\ \mp \frac{1}{\sqrt{2}} & \pm \frac{\sqrt{3}}{2} \end{bmatrix} & \emptyset & \text{Impossible} & \text{Impossible} \\
   \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \emptyset & \text{Impossible} & \text{Impossible} \\
   \end{array}
   \]

2. If \(M\) is of type \((1,2)\) then we have the same possibilities for \(\text{HPer}(f)\) as for homeomorphisms of \(T^2\).

**Proof.** First we consider the case when \(M\) is of type \((2,1)\). In this situation the homeomorphism requirement assures us that for \(A = I\) we are forced to have \(\deg(f) = \pm 1\). Of course in the case of degree 1 we always have an empty \(\text{HPer}(f)\).
So henceforth we shall assume that deg(\(\tilde{f}\)) = -1. Thus if the map on the principal fibre has linearization \(X\) then \(\text{HPer}(f) = \text{HPer}(X) \cup \text{HPer}(AX) \setminus 2\mathbb{N}\) by Theorem 4.6 which says that if \(|d| \leq 1\) then

\[
\text{HPer}(f) = \begin{cases} 
\emptyset & \text{for } d = 1, \\
\text{HPer}(X) & \text{for } d = 0, \\
(\text{HPer}(X) \cup \text{HPer}(AX)) \setminus 2\mathbb{N} & \text{for } d = -1.
\end{cases}
\]

Also we use the condition

\[
A^{-1}X = XA
\]

which is the specification, for \(d = -1\), of the condition \(A^dX = XA\) for the linearization \(X\) of a self-map of model given by the pair \((A, d)\).

- When \(A(1) = -I\) we have that \(X\) can be an arbitrary automorphism with determinant \(\Delta\) and trace \(t\), since condition (7) is always satisfied. Then \(-X\) has the same determinant \(\Delta\) and trace \(-t\). Checking the possibilities in [1] (which are also listed as ordered pairs in Theorem 6.5.1 of [22] with a notation \((a, b) = (\text{trace}, \text{determinant})\) there) for maps on \(\mathbb{T}^2\), and subtracting \(2\mathbb{N}\) in respect of (7), we confirm the above list. The case \(A = -I\) is proved.

- In the second case when \(\text{Spec}(A) = \{-1, -1\}\) but \(A\) is not diagonalizable we show that \(\text{HPer}(f) = \emptyset\).

Indeed, in a basis \(A = \begin{bmatrix} -1 & \lambda \\ 0 & -1 \end{bmatrix}\) and \(X = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\) for \(\lambda \neq 0\). The condition (7) gives \(\gamma = 0\) and \(\delta = -\alpha\), hence \(X = \begin{bmatrix} \alpha & \beta \\ 0 & -1 \end{bmatrix}\). On the other hand det = ±1 implies \(\alpha = \pm 1\). Now

\[
\begin{align*}
\alpha = +1 & \text{ implies } X = \begin{bmatrix} 1 & \beta \\ 0 & -1 \end{bmatrix} \text{ and } AX = \begin{bmatrix} -1 & * \\ 0 & 1 \end{bmatrix}, \\
\alpha = -1 & \text{ implies } X = \begin{bmatrix} -1 & \beta \\ 0 & -1 \end{bmatrix} \text{ and } AX = \begin{bmatrix} 1 & * \\ 0 & -1 \end{bmatrix}.
\end{align*}
\]

It remains to notice that spectrum of each of the above four matrices contains +1 which implies that \(\text{HPer}(X) = \text{HPer}(AX) = \emptyset\).

- In the third case \(A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\).

Let \(X = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\) (\(\alpha, \beta, \gamma, \delta \in \mathbb{R}\)). The condition (7) implies \(\beta = \gamma = 0\). Since \(X\) is a linearization of a homeomorphism of the fibre \(\mathbb{T}^2\) we have det \(X = \alpha\delta = \pm 1\), which gives \(X = \begin{bmatrix} \alpha & 0 \\ 0 & \pm \alpha^{-1} \end{bmatrix}\). Next note that \(\text{HPer}(X)\) is finite but nonempty if \(X\) corresponds to one of the six cases on page 268 in [22, Theorem 6.5.1]. Since det(\(X\)) = ±1, only the cases \((t, \Delta) = (-2, 1), (-1, 1), (0, 1), (1, 1)\) (i.e. det(\(X\)) = (k, 1) for \(k = -2, -1, 0, 1\)) should be considered. Notice that now det(\(X\)) = +1, hence \(X = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}\).

Now \(\text{tr}(X) = \alpha + 1/\alpha = k\) for \(k = -2, -1, 0, +1\). But only \(k = -2\) admits a solution and then \(\alpha = -1\) which implies \(X = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\) and \(AX = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\). Now \(\text{HPer}(X) \cup \text{HPer}(AX) = \{1\} \cup \emptyset = \{1\}\) which shows that in this case we have only \(\text{HPer}(X) = \{1\}\).

- Finally, in the last two cases the form of \(X\) is forced to be \(\begin{bmatrix} \alpha & b \\ b & -\alpha \end{bmatrix}\) (Lemma 4.5) and so \(\text{Spec}(X) = \text{Spec}(AX) = \{\sqrt{\alpha^2 + b^2}, -\sqrt{\alpha^2 + b^2}\}\). Since det(\(X\)) = \(-\alpha^2 - b^2\) must be -1 for a homeomorphism, the spectrum contains \(+1\) and \(\text{HPer}(f) = \emptyset\).

When \(M\) is of the type \((1, 2)\) situation we use Theorem 3.5 with \(d = -1\) to conclude that \(\text{HPer}(f) = \text{HPer}(\tilde{f})\).

We notice that any \(\text{HPer}\) which can be realized by a homeomorphism \(\tilde{f}: \mathbb{T}^2 \to \mathbb{T}^2\) can be also realized by a homeomorphism of a solvmanifold with \(S^1\) fibre over \(\mathbb{T}^2\). Take the trivial bundle \(\mathbb{T}^2 \times S^1 \to \mathbb{T}^2\) and the map \(f(u, z) = (\tilde{f}(u), -z)\). Then \(\text{HPer}(f) = \text{HPer}(\tilde{f})\). \(\Box\)

A direct analysis for any self-map of a solvmanifold yields the following Šarkovskii type theorem, for all self-maps of any solvmanifold of dimension 3.

**Theorem 5.2.** Each self-map \(f\) of a 3-dimensional solvmanifold \(X\) has the property: if for some \(k \geq 4\) we have that \(k \in \text{HPer}(f)\) then \(\text{HPer}(f)\) is infinite.

**Proof.** The case of non-NR-solvmanifolds was already discussed in [19,22]. We will also use the fact that: for any self-map \(\tilde{f}: \mathbb{T}^2 \to \mathbb{T}^2\), if
It remains to notice that also in this case all enlisted sets have the property in the theorem.

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References