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Extrapolation of a discrete collocation-type method of Hammerstein equations☆

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Abstract

In recent papers, Kumar and Sloan introduced a new collocation-type method for numerical solution of Hammerstein integral equations. Kumar studied a discretized version of this method and obtained superconvergence rate for the discrete approximation to the exact solution. In this paper, the asymptotic error expansion of a discrete collocation-type method for Hammerstein integral equations is obtained. We show that when piecewise polynomials of degree $p-1$ are used and numerical quadrature is used to approximate the definite integrals occurring in this method, the approximation solution admits an error expansion in powers of the step-size h. For a special choice of collocation points and numerical quadrature rule, the leading terms in the error expansion for the collocation solution contain only even powers of the step-size h, beginning with a term h^{2p}. Thus Richardson's extrapolation can be performed on the solution, and this will increase the accuracy of numerical solution greatly. Some numerical results are given to illustrate this theory.

Keywords: Nonlinear integral equations; Hammerstein equations; Discrete callocation-type method; Interpolatory quadrature rules; Superconvergence; Asymptotic error expansion; Richardson extrapolation

1. Introduction

Consider the Hammerstein integral equations of the second kind:

$$y(t) = f(t) + \int_a^b k(t,s)\, g(s, y(s))\, \mathrm{d}s, \quad t \in [a, b], \tag{1.1}$$

where $-\infty < a < b < \infty$, f, k, and g are known functions, with $g(s, y)$ nonlinear in y, and $y(t)$ is the solution to be determined.

Several numerical methods for approximating the solution of Hammerstein integral equations are known. The classical method of successive approximations was presented in the 1950s.

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A variation of the Nyström method was introduced in [10]. The classical method of the degenerate kernel was obtained in [6]. A new collocation-type method was developed in recent papers [7–9]. But the error expansions for numerical solutions of integral equations seem to have been discussed in only a few places. For linear Fredholm integral equations, Marchuk and Shaidurov [13, pp. 300–309], Baker [3, pp. 466–473], and Dobrovol'ski [5] obtained the asymptotic error expansion of the Nystrom method. Lin and Liu [11] analyzed the methods of extrapolation from the iterated collocation solutions of Fredholm integral equations whose kernels have lower degree smoothness. Ref. [2] dealt with error expansions for eigenvalues of integral equations. Under the assumption of a uniform partition, McLean [14] obtained asymptotic error expansion for numerical solutions of integral equations, including the Nyström method, iterated collocation method, the iterated Galerkin method. Lin et al. [12] gave a one-term asymptotic error expansion for the iterated collocation method on an arbitrary mesh. For nonlinear integral equations, the systematic derivation and analysis of error expansions for numerical methods have not received much attention.

The method of Kumar and Sloan [9] is a collocation method applied not to (1.1), but rather to an equivalent equation for the function z defined by

$$z(t) = g(t, y(t)), \quad t \in [a, b], \tag{1.2}$$

or

$$z(t) = g\left(t, f(t) + \int_a^b k(t, s) z(s) \, ds\right). \tag{1.3}$$

The desired approximation to the solution y of (1.1) is then obtained by using the equation

$$y(t) = f(t) + \int_a^b k(t, s) z(s) \, ds, \quad t \in [a, b]. \tag{1.4}$$

Kumar and Sloan [9] had shown that, under suitable conditions, the approximation to y converges to the exact solution. For a special choice of the collocation points, Kumar [7] showed that the approximation to y may exhibit (global) superconvergence. The discrete version of the Kumar and Sloan method was considered in [8]. The superconvergence results of Kumar [7] for the exact method were extended to the discrete case. The main aim of this paper is to give an asymptotic error expansion of a discrete collocation-type method for (1.1). Thus Richardson's extrapolation can be performed on the solution, and this will increase the accuracy of numerical solution greatly.

We assume throughout this paper that the following conditions are satisfied:

(i) $y^* \in \mathbb{C}$ is an exact solution of (1.1);
(ii) $f \in \mathbb{C}$;
(iii) the kernel $k(t, s)$ is continuous on $a \leqslant t, s \leqslant b$;
(iv) the function $g(t, y)$ is defined and continuous on $[a, b] \times \mathbb{R}$;
(v) the partial derivative $g_y(t, y) = (\partial/\partial y) g(t, y)$ exists and is continuous on $[a, b] \times \mathbb{R}$;
(vi) the function g_y satisfies the Lipschitz condition:

$$|g_y(t, y_1(t)) - g_y(t, y_2(t))| \leqslant \sigma | y_1(t) - y_2(t)|,$$

for some constant $\sigma > 0$, $t \in [a, b]$ and all $y_1, y_2 \in B(y^*, \delta)$, where

$$B(y^*, \delta) = \{y \in \mathbb{C}\colon \|y - y^*\|_\infty \leqslant \delta\}, \quad \delta > 0.$$

Under assumption (iii), the linear integral operator K, defined by

$$(Kw)(t) = \int_a^b k(t, s)w(s)\,\mathrm{d}s,$$

is a compact operator.

We define another completely continuous operator T:

$$(Tw)(t) = f(t) + (Kw)(t)$$

and a continuous, bounded operator G:

$$G(u)(t) = g(t, u(t)), \quad t \in [a, b], \quad u \in \mathbb{C}.$$

With the above notation, integral equation (1.1) may be written in operator form as

$$y = TG(y) \tag{1.5}$$

and $z(t)$ satisfies the following integral equation:

$$z = GT(z).$$

2. Collocation using piecewise polynomial functions

For any natural number N, let

$$\Delta_N\colon \quad a = t_0 < t_1 < \cdots < t_N = b$$

be an equidistant partition of $[a, b]$, and let $h = (b - a)/N$. For given integers p and d, with $p > d \geqslant 0$, $S_{p-1}^{(d-1)}(\Delta_N) \subset C^{d-1}[a, b]$ will denote the space of piecewise-polynomial functions of degree $p - 1$ whose knots are the mesh points $\{t_n\colon 1 \leqslant n \leqslant N - 1\}$. If $d = 0$, there is no continuity requirement at the knots. Note that the dimension of this space is given by $\dim S_{p-1}^{(-1)}(\Delta_N) = Np$. If $d = 1$, $S_{p-1}^{(0)}(\Delta_N)$ denotes the space of continuous piecewise polynomial functions of degree $p - 1$ whose dimension is equal to $N(p - 1) + 1$.

In this paper, we shall consider only the cases $d = 0$ and $d = 1$. Introduce the set

$$X(N) = \{t_{n,i}\colon t_{n,i} = t_n + c_i h,\ 0 \leqslant c_1 < c_2 < \cdots < c_p \leqslant 1,\ 0 \leqslant n \leqslant N - 1\}.$$

Clearly, $|X(N)| = \dim S_{p-1}^{(-1)}(\Delta_N)$, provided that the set of parameters $\{c_i\}$ does not contain both 0 and 1. When $d = 1$, we choose $c_1 = 0$ and $c_p = 1$. Note that the choice $c_1 = 0$ and $c_p = 1$ implies $|X(N)| = \dim S_{p-1}^{(0)}(\Delta_N)$.

The collocation approximation to z is $z_N \in S_{p-1}^{(d-1)}(\Delta_N)$ satisfying

$$z_N(t_{n,i}) = g\left(t_{n,i}, f(t_{n,i}) + \int_a^b k(t_{n,i}, s) z_N(s)\,ds\right), \quad n = 0, 1, \ldots, N-1, \; i = 1, 2, \ldots, p, \tag{2.1}$$

and this yields an approximation to y:

$$y_N(t) = (Tz_N)(t) = f(t) + \int_a^b k(t, s) z_N(s)\,ds. \tag{2.2}$$

We define an interpolatory projection operator P_N which satisfies:

(A) $P_N w \in S_{p-1}^{(d-1)}(\Delta_N)$;
(B) $(P_N w)(t) = w(t)$, $t \in X(N)$.

Using operator theoretic representations, (2.1) and (2.2) can be, respectively, written as

$$z_N = P_N GT(z_N), \quad z_N \in S_{p-1}^{(d-1)}(\Delta_N) \tag{2.3}$$

and

$$y_N = Tz_N. \tag{2.4}$$

Note that y_N is also a solution of the equation

$$y_N = TP_N G(y_N). \tag{2.5}$$

It is clear that the integrals occurring in (2.1) and (2.2) cannot in general be obtained in analytic form. Hence, a further discretization step is needed: the integrals have to be approximated by suitable quadrature formulas. When this is done, a discrete form of the above collocation-type method is obtained.

For a fixed positive integer q, let $\tau_1, \tau_2, \ldots, \tau_q \in [0, 1]$, and the weights $w_1, w_2, \ldots, w_q$ define the quadrature rule

$$Q(f) = \sum_{i=1}^{q} w_i f(\tau_i) \approx \int_0^1 f(t)\,dt, \tag{2.6}$$

which is exact for all polynomials of degree $\rho - 1$, but not exact for some polynomial of degree ρ, with $\rho \geqslant p$ (that is, the quadrature rule (2.6) has degree of precision $\rho - 1$).

Defining a discrete integral operator K_N by

$$(K_N \phi)(t) = \sum_{i=0}^{N-1} \sum_{j=1}^{q} h w_j k(t, s_{ij})\, \phi(s_{ij}), \quad t \in [a, b],$$

where $s_{ij} = t_i + \tau_j h$.

We now define a discrete form of the operator T by

$$T_N(\phi)(t) = f(t) + (K_N\phi)(t).$$

Using the operators introduced so far, the discrete analogues of z_N and y_N may be written as

$$\tilde{z}_N = P_N G T_N(\tilde{z}_N), \quad \tilde{z}_N \in S_{p-1}^{(d-1)}(\Delta_N) \tag{2.7}$$

and

$$\tilde{y}_N = T_N(\tilde{z}_N), \tag{2.8}$$

respectively. Thus $\tilde{y}_N$ also satisfies the following equation:

$$\tilde{y}_N = T_N P_N G(\tilde{y}_N). \tag{2.9}$$

From [4], we obtain the following result.

Lemma 1. *Let $u \in C^{r+1}[a, b]$, $r \geqslant p$. Then, for any $t \in (t_n, t_{n+1})$, $n = 0, 1, \ldots, N-1$, we have*

$$u(t) - P_N u(t) = \sum_{l=p}^{r} h^l u^{(l)}(t)\, \varphi_l\left(\frac{t - t_n}{h}\right) + \mathrm{O}(h^{r+1}),$$

where $\varphi_l(t) = (t - c_1) \cdots (t - c_p)[c_1, \ldots, c_p, t]\,(\cdot - t)^l/l!$, and $[c_1, \ldots, c_p, t]f(\cdot)$ is a pth divided difference of $f(t)$.

Proof. For any $t \in (t_n, t_{n+1})$, $n = 0, 1, \ldots, N-1$, it is well known that $P_N u$ can be written as

$$P_N u(t) = \sum_{i=1}^{p} u(t_{n,i}) L_i\left(\frac{t - t_n}{h}\right), \tag{2.10}$$

where

$$L_i(t) = \prod_{j=1,\, j \neq i}^{p} (t - c_j)/(c_i - c_j).$$

Note that

$$g_1(y) = \sum_{i=1}^{p} g_1(c_i) L_i(y) + \prod_{i=1}^{p} (y - c_i)[c_1, \ldots, c_p, y]\, g_1(\cdot),$$

we obtain

$$\frac{(y-x)^l}{l!} - \sum_{i=1}^{p} \frac{(c_i - x)^l}{l!} L_i(y) = \prod_{i=1}^{p} (y - c_i)\,[c_1, \ldots, c_p, y]\,\frac{(\cdot - x)^l}{l!}. \tag{2.11}$$

Let $y = x$, from (2.11), we know that

$$\sum_{i=1}^{p} \frac{(c_i - x)^l}{l!} L_i(x) = \delta_{l,0}, \quad l < p, \tag{2.12}$$

and

$$-\sum_{i=1}^{p} \frac{(c_i - x)^l}{l!} L_i(x) = \prod_{i=1}^{p} (x - c_i)\,[c_1, \ldots, c_p, x]\frac{(\cdot - x)^l}{l!} = \varphi_l(x), \quad l \geqslant p. \tag{2.13}$$

If $u(t) \in C^{r+1}[a, b]$, then using Taylor's formula we can write

$$u(t_{n,i}) = \sum_{k=0}^{r} \frac{(t_{n,i} - t)^k}{k!} u^{(k)}(t) + \mathrm{O}(h^{r+1}). \tag{2.14}$$

Substituting (2.14) into (2.10) and using (2.12) and (2.13) we get

$$\begin{aligned} P_N u(t) &= \sum_{i=1}^{p} L_i\left(\frac{t - t_n}{h}\right) \sum_{k=0}^{r} \frac{(t_{n,i} - t)^k}{k!} u^{(k)}(t) + \mathrm{O}(h^{r+1}) \\ &= \sum_{k=0}^{r} h^k u^{(k)}(t) \sum_{i=1}^{p} \left(c_i - \frac{t - t_n}{h}\right)^k L_i\left(\frac{t - t_n}{h}\right)/k! + \mathrm{O}(h^{r+1}) \\ &= u(t) - \sum_{k=p}^{r} h^k u^{(k)}(t)\, \varphi_k\left(\frac{t - t_n}{h}\right) + \mathrm{O}(h^{r+1}). \end{aligned}$$

The lemma is proved. □

3. Asymptotic error expansion of the discrete collocation-type method

Throughout this paper, we assume that the sum $\sum_{n_1}^{n_2}$ equals to zero when $n_1 > n_2$. $[a]$ denotes the integer part of a.

Using Taylor's expansion, we can easily get the following lemma.

Lemma 2. *Let $r \geqslant p$ be a positive integer, $V(t) = V_0(t) + \sum_{l=p}^{r} h^l V_l(t)$, $V_0(t)$, $V_l(t) \in C[a, b]$, $l = p, \ldots, r$, and $g(t, y) \in C^{[r/p]+1}([a, b] \times \mathbb{R})$. Then for any $t \in [a, b]$, we have*

$$g(t, V(t)) = g(t, V_0(t)) + \sum_{l=p}^{r} h^l (g_y(t, V_0(t))\, V_l(t) + f_l(t)) + \mathrm{O}(h^{r+1}),$$

where

$$f_l(t) = \sum_{S=2}^{[l/p]} \frac{1}{S!}\left(\frac{\partial}{\partial y}\right)^S g(t, V_0(t)) \sum_{\substack{k_1 + \cdots + k_S = l \\ k_i \geqslant p}} \prod_{n=1}^{S} V_{k_n}(t).$$

From Lemmas 1 and 2 we obtain the following lemma.

Lemma 3. *Let* $V(t) = V_0(t) + \sum_{l=p}^{r} h^l V_l(t)$, $V_0(t) \in C^{r+1}[a,b]$, $V_l(t) \in C^{r-l+1}[a,b]$, $l = p, p+1, \ldots, r$, *and* $g(t,y) \in C^{r+1}([a,b] \times \mathbb{R})$. *Then for any* $t \in (t_n, t_{n+1})$, $n = 0, 1, \ldots, N-1$, *we have*

$$P_N g(t, V(t)) = g(t, V_0(t)) + \sum_{l=p}^{r} h^l \Bigg\{ g_y(t, V_0(t)) V_l(t) + f_l(t) - g^{(l)}(t, V_0(t)) \varphi_l\left(\frac{t-t_n}{h}\right) - \sum_{k=p}^{l-p} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k [g_y(t, V_0(t)) V_{l-k}(t) + f_{l-k}(t)] \times \varphi_k\left(\frac{t-t_n}{h}\right)\Bigg\} + \mathrm{O}(h^{r+1}),$$

where $\varphi_l(t)$ *and* $f_l(t)$ *are, respectively, defined in Lemma* 1 *and Lemma* 2.

Lemma 4. *Let* $f(t) \in C^{r+1}[a,b]$. *Then the expansion*

$$h \sum_{i=0}^{N-1} \sum_{j=1}^{q} w_j f(t_i + \tau_j h) \varphi(\tau_j) = Q(\varphi) \int_a^b f(t)\,\mathrm{d}t + \sum_{l=1}^{r} h^l \frac{Q(B_l \varphi)}{l!} [f^{(l-1)}(t)]_{t=a}^{b} + \mathrm{O}(h^{r+1}) \tag{3.1}$$

holds, where $B_l(t)$ *are Bernoulli polynomials, the operator* $Q(f)$ *is defined by* (2.6), $[f(t)]_{t=a}^{b} = f(b) - f(a)$.

Proof. To establish (3.1), we use the general Euler–MacLaurin summation formula:

$$h \sum_{n=0}^{N-1} f(t_n + \tau h) = \int_a^b f(t)\,\mathrm{d}t + \sum_{l=1}^{r} h^l \frac{B_l(\tau)}{l!} [f^{(l-1)}(t)]_{t=a}^{b} + \mathrm{O}(h^{r+1}) \tag{3.2}$$

valid for $0 \leqslant \tau \leqslant 1$ (see [14, p. 377]).

Setting $\tau = \tau_j$, multiplying (3.2) by $w_j \varphi(\tau_j)$, and then summing up from $j = 1$ to $j = q$ we can obtain Lemma 4. □

One of the principal results of this paper is the following theorem.

Theorem 5. *Suppose the hypotheses of Lemma* 3 *are satisfied, and* $k(t,s) \in C^{r+1}([a,b] \times [a,b])$. *Then, for any* $t \in [a,b]$, *we have*

$$T_N P_N g(t, V(t)) = f(t) + \int_a^b k(t,s) g(s, V_0(s))\,\mathrm{d}s$$

$$+\sum_{l=p}^{r} h^l \left\{ \frac{Q(B_l)}{l!} \left[\left(\frac{\partial}{\partial s}\right)^{l-1} (k(t,s)g(s,V_0(s))) \right]_{s=a}^{b} \right.$$

$$+\int_a^b k(t,s)[g_y(s,V_0(s))V_l(s)+f_l(s)]\,\mathrm{d}s$$

$$+\sum_{n=p}^{l-1} \frac{Q(B_{l-n})}{(l-n)!} \left[\left(\frac{\partial}{\partial s}\right)^{l-n-1} (k(t,s)(g_y(s,V_0(s))V_n(s)+f_n(s))) \right]_{s=a}^{b}$$

$$-\int_a^b k(t,s)\, g^{(l)}(s,V_0(s))\,\mathrm{d}t\, Q(\varphi_l)$$

$$-\sum_{n=p}^{l-1} \frac{Q(B_{l-n}\varphi_n)}{(l-n)!} \left[\left(\frac{\partial}{\partial s}\right)^{l-n-1} (k(t,s)g^{(n)}(s,V_0(s))) \right]_{s=a}^{b}$$

$$-\sum_{k=p}^{l-p} \int_a^b k(t,s) \left[\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^{k} (g_y(s,V_0(s))V_{l-k}(s)+f_{l-k}(s)) \right] \mathrm{d}s\, Q(\varphi_k)$$

$$-\sum_{n=p}^{l-1}\sum_{k=p}^{n-p} \left[\left(\frac{\partial}{\partial s}\right)^{l-n-1} (k(t,s) \left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^{k} (g_y(s,V_0(s))V_{n-k}(s)+f_{n-k}(s))) \right]_{s=a}^{b}$$

$$\left. \cdot \frac{Q(B_{l-n}\varphi_k)}{(l-n)!} \right\} + \mathrm{O}(h^{r+1}). \tag{3.3}$$

Proof. From the definition of T_N and Lemma 3 we obtain

$$T_N P_N g(t,V(t)) = f(t) + \sum_{i=0}^{N-1}\sum_{j=1}^{q} h w_j k(t,s_{ij}) P_N g(s_{ij}, V(s_{ij}))$$

$$= f(t) + \sum_{i=0}^{N-1}\sum_{j=1}^{q} h w_j k(t,s) g(s,V_0(s))|_{s=s_{ij}}$$

$$+\sum_{l=p}^{r} h^l \left\{ \sum_{i=0}^{N-1}\sum_{j=1}^{q} h w_j k(t,s)(g_y(s,V_0(s))V_l(s)+f_l(s))|_{s=s_{ij}} \right.$$

$$-\sum_{i=0}^{N-1}\sum_{j=1}^{q} h w_j k(t,s) g^{(l)}(s,V_0(s))|_{s=s_{ij}}\, \varphi_l(\tau_j)$$

$$\left. -\sum_{i=0}^{N-1}\sum_{j=1}^{q} h w_j k(t,s) \sum_{k=p}^{l-p} \left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^{k} (g_y(s,V_0(s))V_{l-k}(s)+f_{l-k}(s))|_{s=s_{ij}}\, \varphi_k(\tau_j) \right\}$$

$$+ \mathrm{O}(h^{r+1}). \tag{3.4}$$

Using the summation formulation (3.1), we find

$$\sum_{i=0}^{N-1}\sum_{j=1}^{q} hw_j k(t,s)g(s,V_0(s))|_{s=s_{ij}}$$
$$= \int_a^b k(t,s)g(s,V_0(s))\,\mathrm{d}s + \sum_{l=1}^{r} h^l \frac{Q(B_l)}{l!}\left[\left(\frac{\partial}{\partial s}\right)^{l-1}(k(t,s)g(s,V_0(s)))\right]_{s=a}^{b} + \mathrm{O}(h^{r+1}), \tag{3.5}$$

$$\sum_{i=0}^{N-1}\sum_{j=1}^{q} hw_j k(t,s)(g_y(s,V_0(s))V_l(s)+f_l(s))|_{s=s_{ij}}$$
$$= \int_a^b k(t,s)(g_y(s,V_0(s))V_l(s)+f_l(s))\,\mathrm{d}s$$
$$+ \sum_{i=1}^{r-l} h^i \frac{Q(B_i)}{i!}\left[\left(\frac{\partial}{\partial s}\right)^{i-1}(k(t,s)(g_y(s,V_0(s))V_l(s)+f_l(s)))\right]_{s=a}^{b} + \mathrm{O}(h^{r+1-l}), \tag{3.6}$$

$$\sum_{i=0}^{N-1}\sum_{j=1}^{q} hw_j k(t,s)g^{(l)}(s,V_0(s))|_{s=s_{ij}}\,\varphi_l(\tau_j)$$
$$= Q(\varphi_l)\int_a^b k(t,s)\,g^{(l)}(s,V_0(s))\mathrm{d}s$$
$$+ \sum_{i=1}^{r-l} h^i \frac{Q(B_i\varphi_l)}{i!}\left[\left(\frac{\partial}{\partial s}\right)^{i-1}(k(t,s)g^{(l)}(s,V_0(s)))\right]_{s=a}^{b} + \mathrm{O}(h^{r+1-l}), \tag{3.7}$$

$$\sum_{i=0}^{N-1}\sum_{j=1}^{q} hw_j k(t,s)\sum_{k=p}^{l-p}\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^k (g_y(s,V_0(s))V_{l-k}(s)+f_{l-k}(s))|_{s=s_{ij}}\,\varphi_k(\tau_j)$$
$$= \int_a^b k(t,s)\sum_{k=p}^{l-p} Q(\varphi_k)\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^k (g_y(s,V_0(s))V_{l-k}(s)+f_{l-k}(s))\,\mathrm{d}s$$
$$+ \sum_{i=1}^{r-l}\frac{h^i}{i!}\left[\left(\frac{\partial}{\partial s}\right)^{i-1}(k(t,s)\sum_{k=p}^{l-p} Q(B_i\varphi_k)\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^k (g_y(s,V_0(s))V_{l-k}(s)\right.$$
$$\left. + f_{l-k}(s)))\right]_{s=a}^{b} + \mathrm{O}(h^{r+1-l}). \tag{3.8}$$

Substituting (3.5)–(3.8) into (3.4), and writing them as polynomials in h we can obtain (3.3). □

Theorem 6. *Let $y^*(t) \in C^{r+1}[a,b]$ be the solution of* (1.1), *$z^*(t)$ be the corresponding solution of* (1.3), *$g(t,y) \in C^{r+1}([a,b]\times\mathbb{R})$, $k(t,s) \in C^{r+1}([a,b]\times[a,b])$, and assume that* 1 *is not an eigenvalue* *of the operator* $(GT)'(z^*)$. *Then, for sufficiently large* N, $\tilde{y}_N(t)$ *can be expanded as*

$$\tilde{y}_N(t) = y^*(t) + \sum_{l=p}^{r} h^l V_l(t) + O(h^{r+1}), \tag{3.9}$$

where $V_l(t)$ $(l = p, \dots, r)$ *are the solutions of the following linear Fredholm integral equations*:

$$\begin{aligned} V_l(t) &- \int_a^b k(t,s) g_y(s, y^*(s)) V_l(s)\,ds \\ &= \int_a^b k(t,s) f_l(s)\,ds + \frac{Q(B_l)}{l!}\left[\left(\frac{\partial}{\partial s}\right)^{l-1} (k(t,s) g(s, y^*(s)))\right]_{s=a}^{b} \\ &+ \sum_{n=p}^{l-1} \frac{Q(B_{l-n})}{(l-n)!}\left[\left(\frac{\partial}{\partial s}\right)^{l-n-1} (k(t,s)(g_y(s, y^*(s)) V_n(s) + f_n(s)))\right]_{s=a}^{b} \\ &- Q(\varphi_l) \int_a^b k(t,s)\, g^{(l)}(s, y^*(s))\,ds \\ &- \sum_{n=p}^{l-1} \frac{Q(B_{l-n}\varphi_n)}{(l-n)!}\left[\left(\frac{\partial}{\partial s}\right)^{l-n-1} (k(t,s) g^{(n)}(s, y^*(s)))\right]_{s=a}^{b} \\ &- \int_a^b k(t,s) \sum_{k=p}^{l-p} Q(\varphi_k)\left[\left(\frac{d}{ds}\right)^{k} (g_y(s, y^*(s)) V_{l-k}(s) + f_{l-k}(s))\right] ds \\ &- \sum_{n=p}^{l-1} \sum_{k=p}^{n-p} \frac{Q(B_{l-n}\varphi_k)}{(l-n)!}\left[\left(\frac{\partial}{\partial s}\right)^{l-n-1} (k(t,s) \left(\frac{d}{ds}\right)^{k} (g_y(s, y^*(s)) V_{n-k}(s) \right. \\ &\qquad \left. + f_{n-k}(s)))\right]_{s=a}^{b}. \end{aligned} \tag{3.10}$$

Proof. Defining an operator G_2:

$$G_2(w)(t) = g_y(t, y^*(t)) w(t),$$

from [7, p. 318], we know that $(I - KG_2)$ is invertible; also its norm and that of its inverse are uniformly bounded.

Let $V_0(t) = y^*(t)$. For any $p \leqslant l \leqslant r$, it is easily seen that $V_l(t)$ is defined by a linear Fredholm integral equation of the second kind (uniquely solvable since $(I - KG_2)$ is invertible) in which the inhomogeneous term is expressed in terms of $y^*(t)$, $V_p(t), \dots, V_{l-1}(t)$.

Clearly $V_l(t)$ is $r - l + 1$ times continuously differentiable if $(\partial/\partial t)^{r-l+1} k(t,s)$ is continuous for $a \leqslant t, s \leqslant b$, and the right-hand side of Eq. (3.10) is $r - l + 1$ times continuously differentiable. It is simple to prove by induction that the conditions of the theorem are sufficient to guarantee this.

Using Theorem 5 we get

$$V(t) - T_N P_N G(V)(t) = O(h^{r+1}). \tag{3.11}$$

Let $E(t) = V(t) - \tilde{y}_N(t)$. Subtracting (2.9) from (3.11) and then applying the mean value formula we obtain

$$(I - K_N P_N g_y(t, \eta_N))E = O(h^{r+1}), \tag{3.12}$$

where $\eta_N(t) = \theta \tilde{y}_N(t) + (1-\theta)V(t)$ and $\eta_N(t) - y^*(t) = O(h^p)$.

For any fixed $w \in L_\infty$, let $H(w)$ be the multiplicative linear operator defined by

$$(H(w)v)(t) = g_y\left(t, f(t) + \int_a^b k(t,s)w(s)\,ds\right)v(t), \quad t \in [a, b],\ v \in C,$$

then (3.12) becomes

$$(I - K_N P_N H(\xi_N))E = O(h^{r+1}), \tag{3.13}$$

where $\xi_N(t) - z^*(t) = O(h^p)$.

We shall now show that $(I - K_N P_N H(\xi_N))^{-1}$ exists for N sufficiently large and satisfies

$$\|(I - K_N P_N H(\xi_N))^{-1}\| \leqslant C_1 < \infty.$$

We recall from [15, Lemma 1] that, for sufficiently large N, $(I - K_N P_N H(z^*))^{-1}$ exists and is uniformly bounded. Note that

$$\|(I - K_N P_N H(\xi_N)) - (I - K_N P_N H(z^*))\| \leqslant C_2 \|H(\xi_N) - H(z^*)\| \leqslant C_3 \|\xi_N - z^*\|,$$

which approaches zero as $N \to \infty$. Using [1, Proposition 1.3], we conclude that $(I - K_N P_N H(\xi_N))^{-1}$ exists and is uniformly bounded for all sufficiently large N. So we have

$$\|E\| = O(h^{r+1}).$$

The theorem is thus proved. □

Remark. When $p = q$, and $c_i = \tau_i$, $i = 1, 2, \ldots, p$. Then $Q(\varphi_l) = 0$, $Q(B_i \varphi_l) = 0$. In this case, Eqs. (3.10) become

$$\begin{aligned} V_l(t) &- \int_a^b k(t,s) g_y(s, y^*(s)) V_l(s)\,ds \\ &= \int_a^b k(t,s) f_l(s)\,ds + \frac{Q(B_l)}{l!}\left[\left(\frac{\partial}{\partial s}\right)^{l-1} (k(t,s) g(s, y^*(s)))\right]_{s=a}^b \\ &\quad + \sum_{n=p}^{l-1} \frac{Q(B_{l-n})}{(l-n)!}\left[\left(\frac{\partial}{\partial s}\right)^{l-n-1} (k(t,s)(g_y(s, y^*(s)) V_n(s) + f_n(s)))\right]_{s=a}^b . \end{aligned} \tag{3.10'}$$

Corollary 7. *Suppose that the hypotheses of Theorem 6 are satisfied. We have the following results:*

(i) *If $\rho \geqslant p$, the collocation parameters $\{c_i\}$ have been chosen so that $c_{p-i+1} = 1 - c_i$, $i = 1, 2, \ldots, p$, the basic quadrature rule (2.6) is symmetric, i.e., suppose $\tau_{q-i+1} = 1 - \tau_i$ and $w_{q-i+1} = w_i$ for $1 \leqslant i \leqslant q$. Then $V_l(t) = 0$ whenever l is odd. In this case, expression (3.11) becomes*

$$\tilde{y}_N(t) = y^*(t) + \sum_{l=[(p+1)/2]}^{[r/2]} h^{2l} V_{2l}(t) + \mathrm{O}(h^{r+1}).$$

(ii) *If $p = q$, $c_i = \tau_i$, $i = 1, 2, \ldots, p$, are the Gauss points in the interval (0, 1), the basic quadrature rule (2.6) is an interpolatory quadrature rule. Then, for sufficiently large N, $\tilde{y}_N(t)$ can be expanded as*

$$\tilde{y}_N(t) = y^*(t) + \sum_{l=p}^{[r/2]} h^{2l} V_{2l}(t) + \mathrm{O}(h^{r+1}).$$

(iii) *If $p = q$, $c_1 = \tau_1 = 0$, $c_p = \tau_p = 1$, $c_i = \tau_i$, $i = 2, 3, \ldots, p-1$, are the Lobatto points in the interval (0, 1), the basic quadrature rule (2.6) is an interpolatory quadrature rule. Then, for sufficiently large N, $\tilde{y}_N(t)$ can be expanded as*

$$\tilde{y}_N(t) = y^*(t) + \sum_{l=p-1}^{[r/2]} h^{2l} V_{2l}(t) + \mathrm{O}(h^{r+1}).$$

Proof. (i) Suppose that $c_{p-i+1} = 1 - c_i$, $i = 1, 2, \ldots, p$, $\tau_{q-j+1} = 1 - \tau_j$ and $w_{q-j+1} = w_j$ for all $1 \leqslant j \leqslant q$, then using the facts

$$B_i(t) = (-1)^i B_i(1-t), \qquad \varphi_l(t) = (-1)^l \varphi_l(1-t),$$

it follows that

$$Q(B_i) = 0 \quad \text{when } i \text{ is odd}, \tag{3.14}$$

$$Q(B_i \varphi_l) = 0 \quad \text{when } i + l \text{ is odd}. \tag{3.15}$$

From (3.14) and (3.15), it is easily seen that the right-hand side term of Eq. (3.9) is equal to zero whenever l is odd. So, $V_l(t) = 0$ when l is odd.

(ii) Now consider the special case when $c_i = \tau_i$, $i = 1, 2, \ldots, p$, are the Gauss points in the interval (0, 1). For $p \leqslant l \leqslant 2p - 1$, note that $f_l(t) = 0$, from (3.10′) we have

$$V_l(t) - \int_a^b k(t,s) \frac{\partial}{\partial u} g(s, y^*(s)) V_l(s) \, \mathrm{d}s = \frac{Q(B_l)}{l!} \left[\left(\frac{\partial}{\partial S} \right)^{l-1} (k(t,s)\, g(s, y^*(s))) \right]_{s=a}^{b}.$$

Using the fact of $Q(B_l) = 0$, we get $V_l(t)$ whenever $l \leqslant 2p - 1$. From this and (i) we obtain (ii).

Similarly, we can prove (iii). The corollary is proved. □

4. Richardson extrapolation and numerical illustration

We now present a method, the Richardson extrapolation method, for increasing accuracy based on the asymptotic error expansion. We assume that, $p = q$, $c_i = \tau_i$, $i = 1, 2, \ldots, p$, are the Gauss points in the interval (0, 1), the basic quadrature rule (2.6) is an interpolation quadrature rule. Other cases are similar. We construct two grids Δ_N and Δ_{2N} with mesh-sizes h and $\frac{1}{2}h$ and solve approximations (2.7) and (2.9). Let $\tilde{y}_N^{(0)}$ and $\tilde{y}_{2N}^{(0)}$ be the solutions of these problems (the accuracy of each solution being of order $O(h^{2p})$).

The first step Richardson extrapolation gives

$$\tilde{y}_N^{(1)} = \frac{4^p \tilde{y}_{2N}^{(0)} - \tilde{y}_N^{(0)}}{4^p - 1}.$$

From the asymptotic error expansion, it is easy to see that the function $\tilde{y}_N^{(1)}$ approximates y^* with accuracy of order $O(h^{2p+2})$.

For general positive integer m, we have the mth step Richardson extrapolation:

$$\tilde{y}_N^{(m)} = \frac{4^{p+m-1}\tilde{y}_{2N}^{(m-1)} - \tilde{y}_N^{(m-1)}}{4^{p+m-1} - 1}.$$

The function $\tilde{y}_N^{(m)}$ approximates y^* with accuracy of order $O(h^{2p+2m})$.

We now give an example which illustrates the results of the previous section.

Consider the example

$$y(t) = t^2 + \sin(t) \int_{-1}^{1} \exp(-2s)(y(s))^2 \, ds, \quad t \in [-1, 1].$$

The equation has two solutions, one of which is

$$y^*(t) = t^2 + C \sin(t),$$

where $C = 1.95778398647\ldots$.

We choose uniform partitions with mesh length $h = 2/N$, $N = 2, 4, 8, 16, 32, 64$. The above equation was solved using the discrete collocation-type method, with a collocation method for z^* based on discontinuous piecewise linear functions ($d = 0$, $p = 2$), and a set of collocation points consisting of the two Gauss points in the interval (0, 1), the basic quadrature rule (2.6) selecting the Gauss 2-point rule. The maximum absolute errors of the example are given in Table 1.

Table 1

N	$E_N^{(0)}$	$\alpha^{(0)}$	$E_N^{(1)}$	$\alpha^{(1)}$	$E_N^{(2)}$	$\alpha^{(2)}$
2	$2.18\cdot10^{-2}$	4.71	$5.67\cdot10^{-4}$	4.85	$1.10\cdot10^{-5}$	7.58
4	$8.31\cdot10^{-4}$	4.63	$1.97\cdot10^{-5}$	5.75	$5.84\cdot10^{-8}$	7.90
8	$3.35\cdot10^{-5}$	4.28	$3.65\cdot10^{-7}$	5.94	$2.44\cdot10^{-10}$	7.98
16	$1.75\cdot10^{-6}$	4.08	$5.95\cdot10^{-9}$	5.99	$9.65\cdot10^{-13}$	
32	$1.04\cdot10^{-7}$	4.02	$9.38\cdot10^{-11}$			
64	$6.40\cdot10^{-9}$					

Notation: $E_N^{(m)} = \max\{|y^*(t) - \tilde{y}_N^{(m)}(t)| : t \in [a, b]\}$; $\alpha^{(i)} = \log_2(E_N^{(i)}/E_{2N}^{(i)})$.

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