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Existence and uniqueness of nontrivial collocation solutions of implicitly linear homogeneous Volterra integral equations

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ABSTRACT

We analyse collocation methods for nonlinear homogeneous Volterra—Hammerstein integral equations with non-Lipschitz nonlinearity. We present different kinds of existence and uniqueness of nontrivial collocation solutions and give conditions for such existence and uniqueness in some cases. Finally, we illustrate these methods with an example of a collocation problem and give some examples of collocation problems that do not fit in the cases studied previously.

1. Introduction

The aim of this paper is the numerical analysis of the nonlinear homogeneous Volterra–Hammerstein integral equation (HVHIE)

$$y(t) = (\mathcal{H}y)(t) := \int_0^t K(t,s)G(y(s))\,\mathrm{d}s, \quad t \in I := [0,T], \tag{1}$$

by means of collocation methods on spaces of local polynomials. This equation has multiple applications in physics and analysis, as for example, the study of viscoelastic materials, the renewal equation, seismic response, transverse oscillations or flows of heat (see [1,2]).

Functions K and G are called *kernel* and *nonlinearity*, respectively, and we will assume that the following *general conditions* are always held, even if they are not explicitly mentioned.

- *Over K*. The kernel $K : \mathbb{R}^2 \to [0, +\infty[$ is a locally bounded function and its support is in $\{(t,s) \in \mathbb{R}^2 : 0 \le s \le t\}$. For every $t > 0$, the map $s \mapsto K(t,s)$ is locally integrable, and $\int_0^t K(t,s)\mathrm{d}s$ is a strictly increasing function.
- *Over G*. The nonlinearity $G : [0, +\infty[\to [0, +\infty[$ is a continuous, strictly increasing function, and $G(0) = 0$.

Note that since $G(0) = 0$, the zero function is a solution of Eq. (1), known as *trivial solution*, and so, uniqueness of solutions is no longer a desired property for Eq. (1) because we are obviously interested in nontrivial solutions. Existence and uniqueness of nontrivial solutions of Eq. (1), as well as their properties, have been deeply studied in a wide range of cases for K and G [3–8], especially in the case of convolution equations, i.e. $K(t,s) = k(t-s)$. In general, a necessary and

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sufficient condition for the existence of a nontrivial solution is the existence of a nontrivial subsolution; that is, a positive function u such that $u(t) \leq (\mathcal{H}u)(t)$. So most of the results on existence of nontrivial solutions are, indeed, characterisations of the existence of subsolutions. For instance, in [5] the next result can be found: under the *general conditions*, Eq. (1) has a nontrivial solution if and only if there is a positive integrable function $f(x)$ such that $\int_0^x \mathcal{K}(\mathcal{F}(x)-\mathcal{F}(s))\,\mathrm{d}s \geq G^{-1}(x)$, $x \geq 0$, where $\mathcal{K}(x) := \int_0^x k(s)\,\mathrm{d}s$ and $\mathcal{F}(x) := \int_0^x f(s)\,\mathrm{d}s$.

It is important to note that usually, in the analysis of solutions for non-homogeneous Volterra integral equations (and their numerical approximations), most of the existence and uniqueness theorems require that a Lipschitz condition is held by the nonlinearity (with some exceptions, for instance [9]). This is not our case, since it is well known that if the nonlinearity is Lipschitz continuous, then the unique solution of (1) is the trivial one [10]. Thus, the case we are going to consider in this paper is beyond the scope of classical results of numerical analysis of non-homogeneous Volterra integral equations, in the sense that we need a non-Lipschitz nonlinearity.

Actually, there is a wide range of numerical methods available for solving integral equations (see [11] for a comprehensive survey on the subject): iterative methods, wavelet methods [12–15], generalised Runge–Kutta methods [16,17], or even Monte Carlo methods [18]. Collocation methods [10,19] have proved to be very suitable for a wide range of equations, because of their accuracy, stability and rapid convergence. In this work, we use collocation methods to solve the nonlinear HVHIE (1) written in its implicitly linear form (see below). We also give conditions for different kinds of existence and uniqueness of nontrivial collocation solutions for the corresponding collocation equations.

We organise this paper into four sections. In Section 2, we write Eq. (1) in its implicitly linear form and we describe the corresponding collocation equations; moreover, we define the concept of nontrivial collocation solution. In Section 3, we present different kinds of existence of nontrivial collocation solutions and we give conditions for their existence and uniqueness in some cases, considering convolution and nonconvolution kernels. In Section 4, we illustrate the collocation methods and their numerical convergence with an example, showing how the errors change as the collocation points vary. Moreover, we give some examples of collocation problems that do not fit in the cases studied in the paper. Finally, we present the proofs of the main results in an Appendix at the end of the paper, for the sake of readability.

2. Preliminary concepts

Let us consider the nonlinear homogeneous Volterra–Hammerstein integral equation (HVHIE) given by (1). Taking $z := G \circ y$, Eq. (1) can be written as an *implicitly linear* homogeneous Volterra integral equation (HVIE) for z:

$$z(t) = G((\mathcal{V}z)(t)) = G\left(\int_0^t K(t,s)z(s)\,\mathrm{d}s\right), \quad t \in I, \tag{2}$$

where $\mathcal{V}$ is the linear *Volterra operator*. So, if z is a solution of (2), then $y := \mathcal{V}z$ is a solution of (1). It is known (see [20, p. 143]) that, under suitable assumptions on the nonlinearity G, there is a one-to-one correspondence between solutions of (1) and (2). Particularly, if G is injective, then $y = G^{-1} \circ z$ and hence this correspondence is given, which is granted by the *general conditions* exposed above.

2.1. Collocation problems for implicitly linear HVIEs

First, we are going to introduce the collocation problem associated with Eq. (2) and give the equations for determining a collocation solution, that we will use for approximating a solution of (2) or (1) (see Remark 1).

Let $I_h := \{t_n : 0 = t_0 < t_1 < \cdots < t_N = T\}$ be a mesh (not necessarily uniform) on the interval $I = [0, T]$ and set $\sigma_n :=]t_n, t_{n+1}]$ with lengths $h_n := t_{n+1} - t_n (n = 0, \ldots, N-1)$. The quantity $h := \max\{h_n : 0 \leq n \leq N-1\}$ is called the *stepsize*.

Given a set of m *collocation parameters* $\{c_i : 0 \leq c_1 < \cdots < c_m \leq 1\}$, the *collocation points* are given by $t_{n,i} := t_n + c_i h_n (n = 0, \ldots, N-1)(i = 1, \ldots, m)$ and the set of collocation points is denoted by X_h.

All this defines a *collocation problem* for Eq. (2) (see [21], [10, p. 117]), and a *collocation solution* z_h is given by the *collocation equation*

$$z_h(t) = G\left(\int_0^t K(t,s)z_h(s)\,\mathrm{d}s\right), \quad t \in X_h, \tag{3}$$

where z_h is in the space of piecewise polynomials of degree less than m (see [10, p. 85]). Note that the identically zero function is always a collocation solution, since $G(0) = 0$.

Remark 1. From now on, a "collocation problem" or a "collocation solution" will always be referred to the implicitly linear equation (2). So, if we want to obtain an estimation of a solution of the nonlinear HVHIE (1), then we have to consider $y_h := \mathcal{V}z_h$.

As is stated in [10], a collocation solution z_h is completely determined by the coefficients $Z_{n,i} := z_h(t_{n,i})(n = 0, \ldots, N-1)$ $(i = 1, \ldots, m)$, since $z_h(t_n + vh_n) = \sum_{j=1}^m L_j(v)Z_{n,j}$ for all $v \in]0, 1]$, where $L_j(v) := \prod_{k\neq j}^m \frac{v-c_k}{c_j-c_k}$ $(j = 1, \ldots, m)$

are the Lagrange fundamental polynomials with respect to the collocation parameters. The values of $Z_{n,i}$ are given by the systems

$$Z_{n,i} = G\left(F_n(t_{n,i}) + h_n \sum_{j=1}^{m} B_n(i,j) Z_{n,j}\right), \tag{4}$$

where

$$B_n(i,j) := \int_0^{c_i} K(t_{n,i}, t_n + sh_n) L_j(s)\, \mathrm{d}s \tag{5}$$

and

$$F_n(t) := \int_0^{t_n} K(t,s) z_h(s)\, \mathrm{d}s. \tag{6}$$

The term $F_n(t_{n,i})$ is the *lag term* and can be expressed in the form

$$F_n(t_{n,i}) = \sum_{l=0}^{n-1} h_l \sum_{j=1}^{m} B_n^l(i,j) Z_{l,j},$$

where

$$B_n^l(i,j) := \int_0^1 K(t_{n,i}, t_l + sh_l) L_j(s)\, \mathrm{d}s$$

with $n = 0, \ldots, N-1$, $l = 0, \ldots, n-1$, $i = 1, \ldots, m$, $j = 1, \ldots, m$.

Remark 2. For convolution kernels, $K(t,s) = k(t-s)$, expression (5) is given by

$$B_n(i,j) = \int_0^{c_i} k((c_i - s)h_n) L_j(s)\, \mathrm{d}s.$$

In this case, $B_n(i,j)$ is independent from t_n and given some collocation parameters, it only depends on h_n.

Remark 3. The coefficients $Z_{n,i} \geq 0$ given by (4) are positive, since G is a positive function. But it does not imply that z_h was positive.

The advantage of implicitly linear collocation methods (called *new collocation-type methods*, see [10, p. 118]) lies in the fact that, in contrast to direct collocation methods for (1), the integrals need not to be re-computed for every iteration step when solving the nonlinear algebraic system (4).

2.2. Nontrivial collocation solutions

In this section, we are going to recall the definition of *nontrivial solution* for the implicitly linear HVIE (2) with convolution kernel, and its corresponding collocation problem. Nevertheless, it can be easily extended for the original problem given by Eq. (1), and for nonconvolution kernels (see Remark 4 below), but first we need the following definition:

Definition 1. We say that a property $\mathcal{P}$ holds *near zero* if there exists $\epsilon > 0$ such that $\mathcal{P}$ holds on $]0, \delta[$ for all $0 < \delta < \epsilon$. On the other hand, we say that $\mathcal{P}$ holds *away from zero* if $\mathcal{P}$ holds on $]t, +\infty[$ for all $t > 0$.

Given an implicitly linear HVIE (2), the zero function is always a solution, as it happens with Eq. (1). Moreover, for convolution kernels, given a solution $z(t)$ of (2), and $0 < c < T$ the *c-translated function* of z given by

$$z_c(t) := \begin{cases} 0 & \text{if } 0 \leq t < c \\ z(t-c) & \text{if } c \leq t \leq T \end{cases}$$

is also a solution of (2). Thus, for convolution kernels, we say that a solution is *nontrivial* if it is neither identically zero nor a c-translated function of another solution.

In this case, z is nontrivial if and only if it is not identically zero near zero. This characterisation allows us to extend the concept of nontrivial solution to collocation problems with convolution kernels:

Definition 2. Given a collocation problem with convolution kernel, we say that a collocation solution is *nontrivial* if it is not identically zero in σ_0.

Remark 4. The concept of nontrivial collocation solution can be easily extended to nonconvolution kernels. Nevertheless, we have to take into account that the c-translation of a solution of an implicitly linear HVIE with nonconvolution kernel is not necessarily a solution, and there can exist solutions that are c-translations of functions, which are not solutions.

3. Existence and uniqueness of nontrivial collocation solutions

Given a kernel K, a nonlinearity G and some collocation parameters $\{c_1, \ldots, c_m\}$, our aim is to study the existence of nontrivial collocation solutions (of the corresponding collocation problem) in an interval $I = [0, T]$ using a mesh I_h. We are only interested in existence (and uniqueness) properties, where h can be arbitrarily small and N arbitrarily large, because collocation solutions should converge to solutions of (2) (if they exist) when $h \to 0^+$ (and $N \to +\infty$). Unfortunately, since uniqueness is not guaranteed, these convergence problems are, in general, very complex and they are not in the scope of this work. Taking this into account, we are going to define three different kinds of existence of nontrivial collocation solutions:

- We say that there is *existence near zero* if there exists $H_0 > 0$ such that if $0 < h_0 \leq H_0$, then there are nontrivial collocation solutions in $[0, t_1]$; moreover, there exists $H_n > 0$ such that if $0 < h_n \leq H_n$, then there are nontrivial collocation solutions in $[0, t_{n+1}]$ (for $n = 1, \ldots, N-1$ and given $h_0, \ldots, h_{n-1} > 0$ such that there are nontrivial collocation solutions in $[0, t_n]$). Note that, in general, H_n depends on $h_0, \ldots, h_{n-1}$.

  This is the most general case of existence that we are going to consider. Loosely speaking this definition means that *"collocation solutions can always be extended a bit more"*. Nevertheless, the existence of nontrivial collocation solutions for arbitrarily large T is not assured (for instance, if there is a *blow-up*). So, this is an existence *near zero*.
- We say that there is *existence for fine meshes* if there exists $H > 0$ such that if $0 < h \leq H$, then the corresponding collocation problem has nontrivial collocation solutions.

  This is a particular case of existence near zero, but in this case it is assured the existence of nontrivial collocation solutions in any interval I, using fine enough meshes I_h.
- We say that there is *unconditional existence* if there exist nontrivial collocation solutions in any interval I and for any mesh I_h.

We are going to study two cases of collocation problems:

- Case 1: $m = 1$ with $c_1 > 0$.
- Case 2: $m = 2$ with $c_1 = 0$.

In these cases, the system (4) is reduced to a single nonlinear equation, whose solution is given by the fixed points of $G(\alpha + \beta y)$ for some α, β.

Let us state some lemmas, which will be needed. Taking into account the *general conditions* over G, these results can be easily proved (see Appendix).

Lemma 1. *The following statements are equivalent to the statement that* $\frac{G(y)}{y}$ *is unbounded (in* $]0, +\infty[$*):*

(i) *There exists* $\beta_0 > 0$ *such that* $G(\beta y)$ *has nonzero fixed points for all* $0 < \beta \leq \beta_0$.
(ii) *Given* $A \geq 0$, *there exists* $\beta_A > 0$ *such that* $G(\alpha + \beta y)$ *has nonzero fixed points for all* $0 \leq \alpha \leq A$ *and for all* $0 < \beta \leq \beta_A$.

Lemma 2. *If* $\frac{G(y)}{y}$ *is unbounded near zero but it is bounded away from zero, then there exists* $\beta_0 > 0$ *such that* $G(\alpha + \beta y)$ *has nonzero fixed points for all* $\alpha \geq 0$ *and for all* $0 < \beta \leq \beta_0$.

If, in addition, $\frac{G(y)}{y}$ *is a strictly decreasing function, then there exists a unique nonzero fixed point.*

Lemma 3. *If* $\frac{G(y)}{y}$ *is unbounded near zero and there exists a sequence* $\{y_n\}_{n=1}^{+\infty}$ *of positive real numbers and divergent to* $+\infty$ *such that* $\lim_{n\to+\infty} \frac{G(y_n)}{y_n} = 0$, *then* $G(\alpha + \beta y)$ *has nonzero fixed points for all* $\alpha \geq 0$ *and for all* $\beta > 0$.

If, in addition, $\frac{G(y)}{y}$ *is a strictly decreasing function, then there exists a unique nonzero fixed point.*

Remark 5. The nonzero fixed points are always strictly positive, since G is strictly positive in $]0, +\infty[$. If, in addition, $G(\alpha + \beta y)$ has fixed points for $\alpha > 0$, then these fixed points are necessarily nonzero and, hence, strictly positive.

So, we are going to study different kinds of existence and uniqueness of nontrivial collocation solutions by means of the Eq. (4) considering two special cases.

3.1. Case 1: $m = 1$ with $c_1 > 0$

First, we shall consider $m = 1$ with $c_1 > 0$. Note that if $m = 1$ with $c_1 = 0$, then the unique collocation solution is the trivial one.

We have the equations

$$Z_{n,1} = G(F_n(t_{n,1}) + h_n B_n Z_{n,1}) \quad (n = 0, \ldots, N-1), \tag{7}$$

where

$$B_n := B_n(1, 1) = \int_0^{c_1} K(t_{n,1}, t_n + s h_n)\, \mathrm{d}s, \tag{8}$$

and the lag terms $F_n(t_{n,1})$ are given by (6) with $i = 1$. From the *general conditions* imposed over K, it is assured that $B_n > 0$.

Remark 6. Since K is locally bounded, we have $h_n B_n \to 0$ when $h_n \to 0^+$.

Now, we are in position to give a characterisation of the existence near zero of nontrivial collocation solutions.

Proposition 4. *Let K be a kernel such that $K(t,s) \le K(t',s)$ for all $0 \le s \le t < t'$. Then, there is existence near zero if and only if $\frac{G(y)}{y}$ is unbounded.*

Next, we are going to give some sufficient conditions for the existence and uniqueness of nontrivial collocation solutions.

Proposition 5. *If $\frac{G(y)}{y}$ is unbounded near zero but it is bounded away from zero, then there is existence near zero.*

If, in addition, $\frac{G(y)}{y}$ is a strictly decreasing function, then there is at most one nontrivial collocation solution.

For convolution kernels $K(t,s) = k(t-s)$, we can assure existence for fine meshes.

Proposition 6. *If $\frac{G(y)}{y}$ is unbounded near zero and there exists a sequence $\{y_n\}_{n=1}^{+\infty}$ of positive real numbers and divergent to $+\infty$ such that $\lim_{n\to+\infty} \frac{G(y_n)}{y_n} = 0$, then there is unconditional existence.*

If, in addition, $\frac{G(y)}{y}$ is a strictly decreasing function, then there is at most one nontrivial collocation solution.

In the proofs of Propositions 4–6 (see Appendix), it is shown that the nontrivial collocation solutions z_h are strictly positive. Moreover, if $K(t,s) \le K(t',s)$ for all $0 \le s \le t < t'$ and taking into account (3), it can be easily proved that these collocation solutions are strictly increasing functions.

3.2. Case 2: $m = 2$ with $c_1 = 0$

Now, we are going to consider $m = 2$ with $c_1 = 0$. Hence, we have to solve the following equations:

$$Z_{n,1} = G(F_n(t_{n,1})) \tag{9}$$

$$Z_{n,2} = G(F_n(t_{n,2}) + h_n B_n(2,1) Z_{n,1} + h_n B_n(2,2) Z_{n,2}) \quad (n = 0, \dots, N-1), \tag{10}$$

where

$$B_n(2,j) = \int_0^{c_2} K(t_{n,2}, t_n + s h_n) L_j(s)\, \mathrm{d}s, \quad (j = 1,2) \tag{11}$$

and $F_n(t_{n,i})$ is given by (6) with $m = 2$. Note that for solving $Z_{n,2}$ this system of equations can be reduced to the single Eq. (10), since (9) gives us $Z_{n,1}$ directly.

From the *general conditions* imposed over K and taking into account that functions $L_1(s) = 1 - \frac{s}{c_2}$ and $L_2(s) = \frac{s}{c_2}$ are strictly positive in $]0, c_2[$, it is assured that $B_n(2,j) > 0$ for $j = 1, 2$.

Remark 7. As in Remark 6, $h_n B_n(2,j) \to 0$ when $h_n \to 0^+$. In particular, $h_n B_n(2,2) \to 0$.

Analogously to the previous case, we present similar results for existence and uniqueness of nontrivial collocation solutions.

Proposition 7. *Let K be a kernel such that $K(t,s) \le K(t',s)$ for all $0 \le s \le t < t'$. Then, there is existence near zero if and only if $\frac{G(y)}{y}$ is unbounded.*

Proposition 8. *Let K be a kernel such that $K(t,s) \le K(t',s)$ for all $0 \le s \le t < t'$. If $\frac{G(y)}{y}$ is unbounded near zero but it is bounded away from zero, then there is existence near zero.*

If, in addition, $\frac{G(y)}{y}$ is a strictly decreasing function, then there is at most one nontrivial collocation solution.

For convolution kernels $K(t,s) = k(t-s)$, the hypothesis on K means that k is increasing, and we can assure existence for fine meshes.

Proposition 9. *Let K be a kernel such that $K(t,s) \le K(t',s)$ for all $0 \le s \le t < t'$. If $\frac{G(y)}{y}$ is unbounded near zero and there exists a sequence $\{y_n\}_{n=1}^{+\infty}$ of positive real numbers and divergent to $+\infty$ such that $\lim_{n\to+\infty} \frac{G(y_n)}{y_n} = 0$, then there is unconditional existence.*

If, in addition, $\frac{G(y)}{y}$ is a strictly decreasing function, then there is at most one nontrivial collocation solution.

As in the previous case, the nontrivial collocation solutions z_h are strictly positive (see Appendix). Also, from the hypothesis on K and (3), it can be easily proved that the images under z_h of the collocation points form a strictly increasing sequence, i.e. $0 = Z_{0,1} < Z_{0,2} < Z_{1,1} < Z_{1,2} < \cdots < Z_{N-1,2}$. Particularly, if $c_2 = 1$, then we can assure that z_h is a strictly increasing function.

Remark 8. In the case $c_2 = 1$, we can remove the hypothesis "$K(t,s) \le K(t',s)$ for all $0 \le s \le t < t'$" from Propositions 8 and 9, because we do not need that $Z_{l,2} > Z_{l,1}$ with $l = 1, \dots, n-1$ for proving that z_h is strictly positive in $]0, t_n[$, and hence, we do not need to prove that $Z_{n,2} > Z_{n,1}$ (see Appendix). On the other hand, if we remove this hypothesis, we cannot assure that z_h is a strictly increasing function, as in the case $m = 1$ studied in Propositions 5 and 6.

3.3. Nondivergent existence and uniqueness

Given a kernel K, a nonlinearity G and some collocation parameters, we are interested in the study of existence of nontrivial collocation solutions using meshes I_h with arbitrarily small $h > 0$. Following this criterion, we are not interested in collocation problems whose collocation solutions "escape" to $+\infty$ in a certain σ_n when $h_n \to 0^+$, since this is a divergence symptom.

Let S be an index set of all the nontrivial collocation solutions of a collocation problem with mesh I_h. For any $s \in S$, we denote by $Z_{s;n,i}$ the coefficients verifying Eq. (4) (with $n = 0, \ldots, N-1$ and $i = 1, \ldots, m$) and such that, at least, one of the coefficients $Z_{s;0,i}$ is different from zero (for some $i \in \{1, \ldots, m\}$).

So, given K, G and some collocation parameters, we are going to define the concepts of *nondivergent existence* and *nondivergent uniqueness* of nontrivial collocation solutions.

Definition 3. Given $0 = t_0 < \cdots < t_n$ such that there exist nontrivial collocation solutions using the mesh $t_0 < \cdots < t_n$, we say that there is *nondivergent existence in* t_n^+ if

$$\mathcal{Z}_{h_n} := \inf_{s \in S_{h_n}} \{ \max_{i=1,\ldots,m} \{Z_{s;n,i}\} \}$$

exists for small enough $h_n > 0$ and it does not diverge to $+\infty$ when $h_n \to 0^+$.

Note that the index set of the nontrivial collocation solutions is denoted by S_{h_n} because if we change h_n then we change the collocation problem.

Definition 4. Given $I_h = \{0 = t_0 < \cdots < t_{N-1}\}$ such that there exist nontrivial collocation solutions using this mesh, we say that there is *nondivergent existence* if there is nondivergent existence in t_n^+ for $n = 0, \ldots, N-1$.

To study the concept of *nondivergent uniqueness*, we need to state the following definitions:

Definition 5. Given $0 = t_0 < \cdots < t_n$ such that there exist nontrivial collocation solutions using the mesh $t_0 < \cdots < t_n$, we say that there is *nondivergent uniqueness in* t_n^+ if

$$\min_{s \in S_{h_n}} \{ \max_{i=1,\ldots,m} \{Z_{s;n,i}\} \} = \mathcal{Z}_{h_n}$$

exists for small enough $h_n > 0$, and it does not diverge to $+\infty$ when $h_n \to 0^+$, but

$$\inf_{s \in S_{h_n}} (\{ \max_{i=1,\ldots,m} \{Z_{s;n,i}\} \} - \{\mathcal{Z}_{h_n}\})$$

diverges (note that $\inf \emptyset = +\infty$).

Definition 6. Given $I_h = \{0 = t_0 < \cdots < t_{N-1}\}$ such that there exist nontrivial collocation solutions using this mesh, we say that there is *nondivergent uniqueness* if there is nondivergent uniqueness in t_n^+ for $n = 0, \ldots, N-1$.

When "nondivergent uniqueness" is assured, but there is not "uniqueness" (of nontrivial collocation solutions), there is only one nontrivial collocation solution that makes sense, as is stated in the following definition.

Definition 7. In case of nondivergent uniqueness, the *nondivergent collocation solution* is the one whose coefficients $Z_{n,i}$ satisfy $\max_{i=1,\ldots,m}\{Z_{n,i}\} = \mathcal{Z}_{h_n}$ for $n = 0, \ldots, N-1$.

Next, we are going to study nondivergent existence and uniqueness for cases 1 ($m = 1$ with $c_1 > 0$) and 2 ($m = 2$ with $c_1 = 0$). Recall that the *general conditions* over K and G are always held, even if it is not explicitly mentioned.

But first, we are going to state a result that reduces the study of nondivergent existence (and uniqueness) for any mesh I_h to the study of nondivergent existence (and uniqueness) in t_0^+. Lemma 10 can be easily proved taking into account the *general conditions* over G.

Lemma 10. *Given $\alpha > 0$, the minimum of the nonzero fixed points of $G(\alpha + \beta y)$ exists for a small enough $\beta > 0$ and converges to $G(\alpha)$ when $\beta \to 0^+$.*

If, in addition, $\frac{G(\alpha+y)}{y}$ is strictly decreasing near zero, then the other nonzero fixed points (if they exist) diverge to $+\infty$.

As a consequence of this Lemma, we obtain the following Proposition:

Proposition 11. *In cases 1 and 2 with existence of nontrivial collocation solutions, "nondivergent existence" and "nondivergent existence in t_0^+" are equivalent.*

If, in addition, $\frac{G(\alpha+y)}{y}$ is strictly decreasing near zero for all $\alpha > 0$, then "nondivergent uniqueness" and "nondivergent uniqueness in t_0^+" are equivalent.

Remark 9. The condition "$\frac{G(\alpha+y)}{y}$ strictly decreasing near zero for all $\alpha > 0$" is very weak, since $\lim_{y\to 0^+} \frac{G(\alpha+y)}{y} = +\infty$. For example, if G is twice differentiable, i.e., without accumulation of non-differentiable points and without accumulation of sign changes of the second derivative, then this condition is held. So, if a collocation problem has nontrivial collocation solutions and, roughly speaking, G is "well-behaved", then nondivergent uniqueness in t_0^+ implies nondivergent uniqueness in any t_n^+.

Next, we are going to give a characterisation of nondivergent existence, but first we need the following Lemma, that can be easily proved taking into account the *general conditions* over G:

Lemma 12. *If $\frac{G(y)}{y}$ is unbounded, then the minimum of the nonzero fixed points of $G(\beta y)$ exists for a small enough $\beta > 0$. In this case, this minimum does not diverge to $+\infty$ when $\beta \to 0^+$ if and only if $\frac{G(y)}{y}$ is unbounded near zero.*

If, in addition, $\frac{G(y)}{y}$ is strictly decreasing near zero, then the other nonzero fixed points (if they exist) diverge to $+\infty$.

Using Lemma 12 and Proposition 11, the next result can be proved:

Proposition 13. *In cases 1 and 2 with existence of nontrivial collocation solutions, there is nondivergent existence if and only if $\frac{G(y)}{y}$ is unbounded near zero.*

If, in addition, $\frac{G(\alpha+y)}{y}$ is strictly decreasing near zero for all $\alpha \geq 0$, then there is nondivergent uniqueness.

Remark 10. If $\frac{G(y)}{y}$ is unbounded near zero, then the condition "$\frac{G(\alpha+y)}{y}$ strictly decreasing near zero for all $\alpha \geq 0$" is very weak, using the same arguments as in Remark 9. So, if G is "well-behaved" (see Remark 9), nondivergent existence implies also nondivergent uniqueness.

To sum up, combining Proposition 13 with Propositions 4–6 (case 1) and 7–9 (case 2) and taking into account Remark 8:

- $K(t,s) \leq K(t',s)$ for all $0 \leq s \leq t < t'$;
  $\frac{G(y)}{y}$ is unbounded near zero $\Leftrightarrow$ Nondivergent existence near zero.
  Moreover, if G is "well-behaved" $\Rightarrow$ Nondivergent uniqueness near zero.
- (Hypothesis only for case 2: $c_2 = 1$, or $K(t,s) \leq K(t',s)$ for all $0 \leq s \leq t < t'$);
  $\frac{G(y)}{y}$ is unbounded near zero but bounded away from zero $\Rightarrow$ nondivergent existence near zero.
  Moreover, if G is "well-behaved" $\Rightarrow$ nondivergent uniqueness near zero.
  For convolution kernels $K(t,s) = k(t-s) \Rightarrow$ nondivergent existence or uniqueness (resp.) for fine meshes.
- (Hypothesis only for case 2: $c_2 = 1$, or $K(t,s) \leq K(t',s)$ for all $0 \leq s \leq t < t'$);
  $\frac{G(y)}{y}$ is unbounded near zero and there exists a sequence $\{y_n\}_{n=1}^{+\infty}$ of positive real numbers and divergent to $+\infty$ such that $\lim_{n\to+\infty} \frac{G(y_n)}{y_n} = 0 \Rightarrow$ unconditional nondivergent existence.
  Moreover, if G is "well-behaved" $\Rightarrow$ unconditional nondivergent uniqueness.

4. Examples

4.1. Numerical study of convergence

In this section, we are going to show an example of a collocation problem (with nondivergent uniqueness in cases 1 and 2) and study numerically how the nontrivial collocation solution z_h converges to a solution of the implicitly linear HVIE (2) when $h \to 0^+$. In fact, we are going to show how the function $y_h = \mathcal{V} z_h$ (see Remark 1) converges to a solution of the original nonlinear HVHIE (1).

We are going to consider the equation

$$y(t) = \int_0^t (t-s)(y(s))^{1/2}\,ds, \quad t \in [0,1]. \tag{12}$$

The kernel and the nonlinearity verify the *general conditions* and all the hypothesis of Propositions 6 (for case 1) and 9 (for case 2). Hence, there is unconditional existence and uniqueness; moreover, we can also apply Proposition 13 to assure that the unique nontrivial collocation solution is nondivergent.

Since the unique nontrivial solution of the nonlinear HVHIE is given by

$$y(t) = \frac{1}{144}t^4,$$

we can study the difference between this solution and y_h when $h \to 0^+$ (see Figs. 1 and 3).

In Figs. 2 and 4 it is represented the relative error $\frac{\int_I |y_h - y|}{\int_I y}$ varying c_1 (case 1) or c_2 (case 2), for different stepsizes h. It is shown that the rate of convergence is the same as h. Moreover, a convenient choice of the collocation parameter can reduce the relative error more than two orders, as we see in Tables 1 and 2. Nevertheless, how to find a good collocation parameter is an open problem.

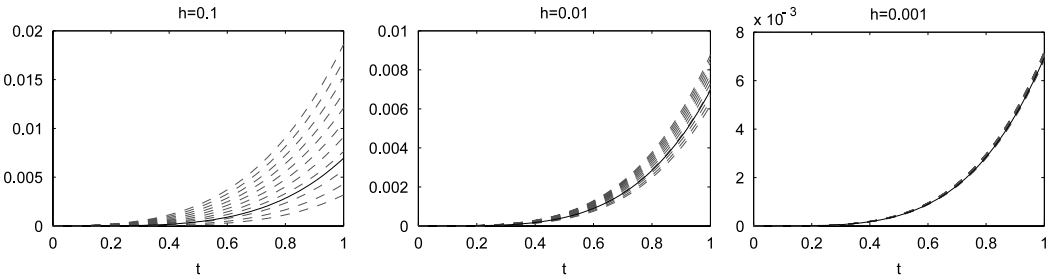

Fig. 1. Case 1 ($m = 1, c_1 > 0$). Approximation y_h (grey dots) for different stepsizes h, with c_1 from 0.01 (lower) to 1 (upper). The solution y is also represented (black solid line).

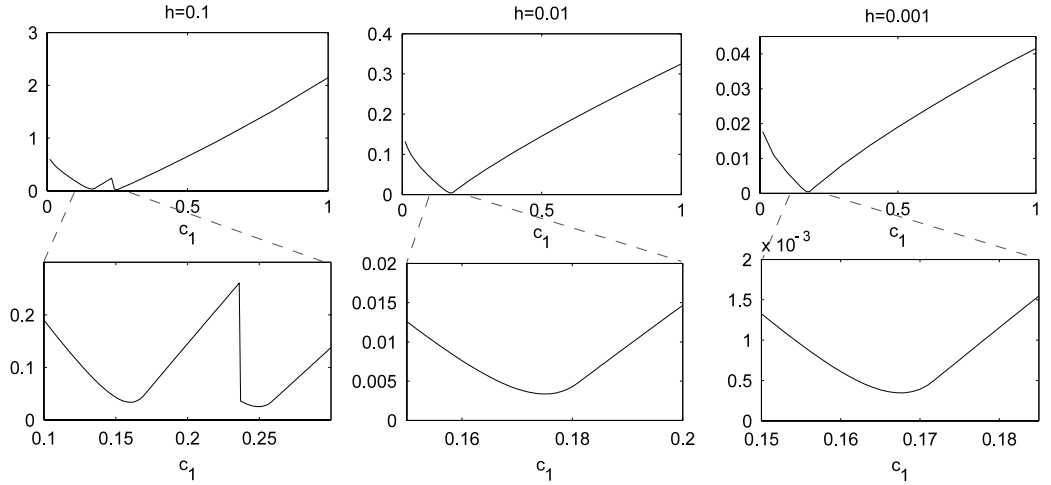

Fig. 2. Case 1. Relative error varying c_1 (from 0.01 to 1) for different stepsizes h.

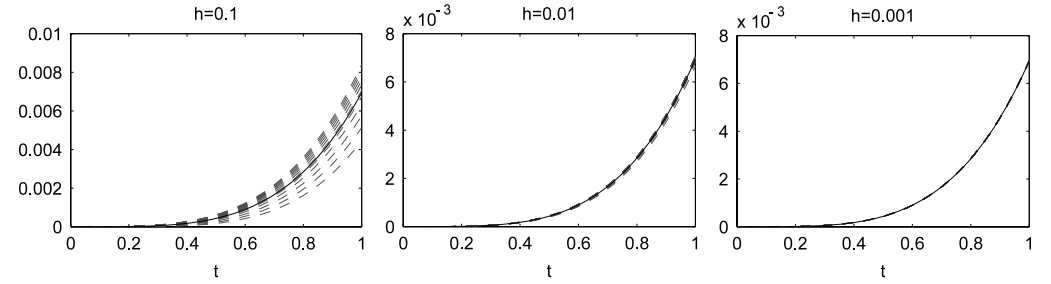

Fig. 3. Case 2 ($m = 2, c_1 = 0$). Approximation y_h (grey dots) for different stepsizes h, with c_2 from 0.01 (lower) to 1 (upper). The solution y is also represented (black solid line).

Table 1
Case 1. Maximum and minimum relative errors varying c_1, for different stepsizes h (see Fig. 2).

Case 1 ($m = 1, c_1 > 0$)	$h = 0.1$	$h = 0.01$	$h = 0.001$
Max. rel. error	$2.1\,(c_1 = 1)$	$3.2 \cdot 10^{-1}\,(c_1 = 1)$	$4.1 \cdot 10^{-2}\,(c_1 = 1)$
Min. rel. error	$2.5 \cdot 10^{-2}\,(c_1 = 0.25)$	$3.4 \cdot 10^{-3}\,(c_1 = 0.175)$	$3.4 \cdot 10^{-4}\,(c_1 = 0.168)$

Table 2
Case 2. Maximum and minimum relative errors varying c_2, for different stepsizes h (see Fig. 4).

Case 2 ($m = 2, c_1 = 0$)	$h = 0.1$	$h = 0.01$	$h = 0.001$
Max. rel. error	$4.2 \cdot 10^{-1}\,(c_2 = 0.01)$	$5.2 \cdot 10^{-2}\,(c_2 = 0.01)$	$5.4 \cdot 10^{-3}\,(c_2 = 0.01)$
Min. rel. error	$1.5 \cdot 10^{-3}\,(c_2 = 0.358)$	$1.4 \cdot 10^{-5}\,(c_2 = 0.369)$	$1.4 \cdot 10^{-7}\,(c_2 = 0.37)$

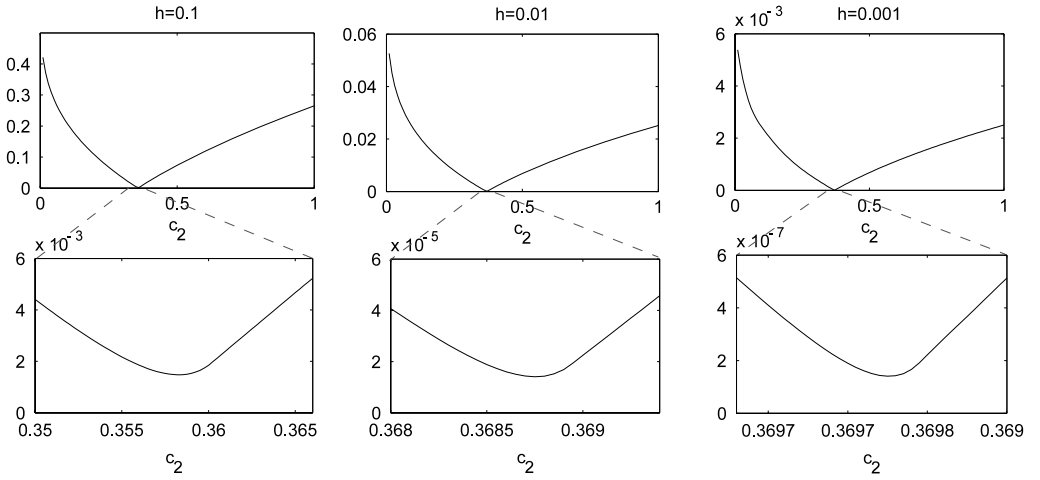

Fig. 4. Case 2. Relative error varying c_2 (from 0.01 to 1) for different stepsizes h.

4.2. Examples in other cases

If the collocation problem is not in the scope of cases 1 and 2, the existence of nontrivial collocation solutions is not assured, even if the kernel and the nonlinearity satisfy all the mentioned conditions.

For example, if we consider

- $K(t,s) = (t-s)^a,\ a > 0.$
- $G(y) = y^{1/b},\ b > 1,$

the *general conditions* over K and G are held, as well as all the conditions for assuring unconditional existence and uniqueness in cases 1 and 2. Moreover, by Proposition 13, the unique nontrivial collocation solution is nondivergent. Nevertheless, in other cases we cannot assure anything, as we will show in the next examples (where existence is considered at least in σ_0):

- Let $b = 2$. If $m = 2$ and $c_1 > 0$, then it can be proved that there is not any nontrivial collocation solution (for any $a > 0$).
  But, if $m = 3$, $a = 2$ and $c_1 > 0$, then there exists a unique nontrivial collocation solution at least in σ_0, independently of c_2 and c_3; on the other hand, if $c_1 = 0$, then there is not any nontrivial collocation solution (contrary to what happens in the case $m = 2$).
  Moreover, if $m = 3$ and $a = 1$, then there exists a unique nontrivial collocation solution at least in σ_0, independently of the collocation parameters.
- Let $b = 3$, $a = 1$ and $m \geq 2$. It can be proved that there exists a unique nontrivial collocation solution at least in σ_0, independently of the collocation parameters. This is a special case, since the collocation solution for $m \geq 2$ coincides with the solution of the corresponding implicitly linear HVIE, $z(t) = t/\sqrt{6}$.

5. Discussion and comments

A general theoretical analysis of numerical approximations of nontrivial collocation solutions for Eq. (1) is an outstanding problem which, to our knowledge, remains open. Among the reasons for such difficulty is the lack of uniqueness of solutions and the lack of Lipschitz-continuity of the nonlinearity, to name a few.

Nevertheless, this work is a first step towards such general analysis: we give conditions for different kinds of existence and uniqueness of nontrivial collocation solutions aiming also for the convergence analysis (e.g. nondivergent existence).

We have also studied numerically a concrete example showing the accuracy of the methods and how their errors depend on the collocation points (see Section 4.1). Therefore, in the case of nondivergent existence and uniqueness of collocation solutions, collocation methods have proved to be a valuable numerical tool for approximating solutions. Moreover, these methods are strengthened by the fact that, if $\frac{G(y)}{y}$ is a strictly decreasing function, the nonzero fixed points mentioned in Lemmas 2 and 3 are attractors, so they can be easily found numerically with iteration techniques. Furthermore, if there is nondivergent existence, the minimum of such nonzero fixed points is also an attractor.

It is worth pointing out that the conditions we have imposed over the nonlinearity in order to assure the nondivergent existence near zero of nontrivial collocation solutions, namely $\frac{G(y)}{y}$ must be unbounded near zero, agrees with the conditions for existence of nontrivial solutions of the original Eq. (1). To be more precise, if $\frac{G(y)}{y}$ is unbounded near zero, then it is not Lipschitz-continuous, which is a necessary condition for the existence of nontrivial solutions of Eq. (1).

Acknowledgements

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Appendix. Proofs

Proof of Lemma 1. ($\Rightarrow$ (i)) Let us prove that if $\frac{G(y)}{y}$ is unbounded (in $]0,+\infty[$), then there exists $\beta_0 > 0$ such that $G(\beta y)$ has nonzero fixed points for all $0 < \beta \leq \beta_0$. Let us take $y_0 > 0$, $y_1 := G(y_0) > 0$ and $\beta_0 := \frac{y_0}{y_1} > 0$. So, given $0 < \beta < \beta_0$, we have

$$G(\beta y_1) < G(\beta_0 y_1) = G(y_0) = y_1, \tag{13}$$

since G is strictly increasing. Moreover, by hypothesis, there exists $y_2 > 0$ such that $\frac{G(y_2)}{y_2} > \frac{1}{\beta}$. Let us define $y_3 := \frac{y_2}{\beta}$, then

$$G(\beta y_3) = G(y_2) > \frac{y_2}{\beta} = y_3. \tag{14}$$

So, taking into account (13) and (14), $G(\beta y)$ has fixed points between y_1 and y_3, since it is continuous.

($\Leftarrow$ (i)) Let us prove the other implication. Given $M > \frac{1}{\beta_0} > 0$, we take $0 < \beta < \frac{1}{M} < \beta_0$. Then, by hypothesis, there exists $y_0 > 0$ such that $G(\beta y_0) = y_0$, and so, taking $y_1 := \beta y_0$ we have $\frac{G(y_1)}{y_1} = \frac{1}{\beta} > M$.

((i) $\Rightarrow$ (ii)) Let us prove that (i) implies (ii), for $A > 0$. Let us take $y_0 > 0$, $y_1 := G(A + y_0) > 0$ and $\beta_A := \frac{y_0}{y_1} > 0$. So, given $0 < \beta < \beta_A$ and $0 < \alpha \leq A$ (we can suppose $\alpha > 0$ taking $\beta_A \leq \beta_0$), we have

$$G(\alpha + \beta y_1) < G(A + \beta_A y_1) = G(A + y_0) = y_1, \tag{15}$$

since G is strictly increasing. Moreover

$$G(\alpha + \beta 0) = G(\alpha) > 0. \tag{16}$$

So, taking into account (15) and (16), $G(\alpha + \beta y)$ has nonzero fixed points between 0 and y_1, since it is continuous.

((ii) $\Rightarrow$ (i)) Trivial. □

Proof of Lemma 2. Given $\alpha \geq 0$ and a small enough $\beta > 0$, it is easy to prove (taking analogous arguments as in the proof of Lemma 1) that there exists $y_1 > 0$ such that $G(\alpha + \beta y_1) < y_1$, since $\frac{G(y)}{y}$ is bounded away from zero. On the other hand, it is easy to prove that there exists $y_3 \geq 0$ such that $G(\alpha + \beta y_3) > y_3$, since $\frac{G(y)}{y}$ is unbounded near zero (note that the case $\alpha > 0$ is trivial taking $y_3 = 0$). So, $G(\alpha + \beta y)$ has nonzero fixed points between y_3 and y_1, since it is continuous. Moreover, it is clear that if $\frac{G(y)}{y}$ is a strictly decreasing function, this fixed point is unique. □

Proof of Lemma 3. Analogous to the proof of Lemma 2, but taking any $\beta > 0$. □

Proof of Proposition 4. ($\Leftarrow$) Let us prove that if $\frac{G(y)}{y}$ is unbounded, then there is existence near zero. So, we are going to prove by induction over n that there exist $H_n > 0$ ($n = 0, \ldots, N-1$) such that if $0 < h_n \leq H_n$, then there exist solutions of the system (7) with $Z_{0,1} > 0$:

- For $n = 0$, taking into account Remark 6 and Lemma 1-(i), we choose a small enough $H_0 > 0$ such that $0 < h_0 B_0 \leq \beta_0$ for all $0 < h_0 \leq H_0$. So, since the lag term is 0, we can apply Lemma 1-(i) to the Eq. (7), concluding that there exist strictly positive solutions for $Z_{0,1}$.
- Let us suppose that, choosing one of those $Z_{0,1}$, there exist $H_1, \ldots, H_{n-1} > 0$ such that if $0 < h_i \leq H_i$ ($i = 1, \ldots, n-1$), then there exist coefficients $Z_{1,1}, \ldots, Z_{n-1,1}$ fulfilling the Eq. (7). Note that these coefficients are strictly positive by Remark 5, and hence, it is guaranteed that the corresponding collocation solution z_h is strictly positive in $]0, t_n[$.
- Finally, we are going to prove that there exists $H_n > 0$ such that if $0 < h_n \leq H_n$, then there exists $Z_{n,1} > 0$ fulfilling the Eq. (7) with the previous coefficients $Z_{0,1}, \ldots, Z_{n-1,1}$:

  Let us define $A := F_n(t_n + c_1)$. So, taking into account Remark 6 and Lemma 1-(ii), we choose a small enough $0 < H_n \leq 1$ such that $0 < h_n B_n \leq \beta_A$ for all $0 < h_n \leq H_n$. Then, we have $0 < F_n(t_{n,1}) \leq F_n(t_n + c_1) = A$ because the hypothesis over K (and the *general conditions*), z_h is strictly positive in $]0, t_n[$, and $t_{n,1} \leq t_n + c_1$. Hence, we can apply Lemma 1-(ii), obtaining the existence of $Z_{n,1}$ (that is strictly positive by Remark 5).

($\Rightarrow$) For proving the other condition, we use Lemma 1-(i), taking into account Remark 6. □

Proof of Proposition 5. First, we are going to consider a convolution kernel $K(t,s) = k(t-s)$. Taking into account Remarks 2, 6, and Lemma 2, we choose a small enough $H > 0$ such that

$$h \int_0^{c_1} k((c_1 - s)h)\, \mathrm{d}s \leq \beta_0 \tag{17}$$

for all $0 < h \leq H$.

So, if $0 < h \leq H$, we are going to prove by induction over n that there exist solutions of the system (7) with $Z_{0,1} > 0$:

- For $n = 0$, by (17), we have $0 < h_0 B_0 \leq \beta_0$, because $0 < h_0 \leq h \leq H$. Since the lag term is 0, we can apply Lemma 2 to the Eq. (7), concluding that there exist strictly positive solutions for $Z_{0,1}$.
- Let us suppose that, choosing one of those $Z_{0,1}$, there exist coefficients $Z_{1,1}, \ldots, Z_{n-1,1}$ fulfilling the Eq. (7). Note that these coefficients are strictly positive by Remark 5, and hence, it is guaranteed that the corresponding collocation solution z_h is strictly positive in $]0, t_n[$.
- Finally, we are going to prove that there exists $Z_{n,1} > 0$ fulfilling the Eq. (7) with the previous coefficients $Z_{0,1}, \ldots, Z_{n-1,1}$:
  On one hand, taking into account (6), the lag term $F_n(t_{n,1})$ is strictly positive because z_h is strictly positive in $]0, t_n[$ and k satisfies the *general conditions*. On the other hand, $0 < h_n B_n \leq \beta_0$ because $0 < h_n \leq h \leq H$. Hence, we can apply Lemma 2 to the Eq. (7), obtaining the existence of $Z_{n,1}$ (that is strictly positive by Remark 5).

If, in addition, $\frac{G(y)}{y}$ is a strictly decreasing function, the uniqueness is an immediate consequence of Lemma 2.

For general (nonconvolution) kernels, the proof is analogous but B_n is given by (8) and it does not only depend on h_n. The existence of h_n lies in choosing a small enough H_n such that $h_n B_n \leq \beta_0$ for all $0 < h_n \leq H_n$. □

Proof of Proposition 6. The proof is analogous to the proof of Proposition 5, but in this case $H > 0$ is any positive real number, and using Lemma 3 instead of Lemma 2. □

Proof of Proposition 7. ($\Leftarrow$) Let us prove that if $\frac{G(y)}{y}$ is unbounded, then there is existence near zero. So, we are going to prove by induction over n that there exist $H_n > 0$ ($n = 0, \ldots, N-1$) such that if $0 < h_n \leq H_n$, then there exist solutions of the system (10) with $Z_{0,2} > 0$:

- For $n = 0$, taking into account Remark 7 and Lemma 1-(i), we choose a small enough $H_0 > 0$ such that $0 < h_0 B_0(2,2) \leq \beta_0$ for all $0 < h_0 \leq H_0$. So, since the lag terms are 0 and $Z_{0,1} = G(0) = 0$, we can apply Lemma 1-(i) to the Eq. (10), concluding that there exist strictly positive solutions for $Z_{0,2}$.
- Let us suppose that, choosing one of those $Z_{0,2}$, there exist $H_1, \ldots, H_{n-1} > 0$ such that if $0 < h_i \leq H_i$ ($i = 1, \ldots, n-1$), then there exist coefficients $Z_{1,2}, \ldots, Z_{n-1,2}$ fulfilling the Eq. (10). Moreover, let us suppose that these coefficients satisfy $Z_{l,2} > Z_{l,1} > 0$ for $l = 1, \ldots, n-1$, and hence, it is guaranteed that the corresponding collocation solution z_h is strictly positive in $]0, t_n[$.
- Finally, we are going to prove that there exists $H_n > 0$ such that if $0 < h_n \leq H_n$, then there exists $Z_{n,2} > 0$ fulfilling the Eq. (10) with the previous coefficients, and $Z_{n,2} > Z_{n,1} > 0$:
  Let us define

$$A := F_n(t_n + c_2) + \int_0^{c_2} k((c_2 - s))L_1(s)\,\mathrm{d}sG(F_n(t_n + c_1)).$$

So, taking into account Remark 7 and Lemma 1-(ii), we choose a small enough $0 < H_n \leq 1$ such that $0 < h_n B_n(2,2) \leq \beta_A$ for all $0 < h_n \leq H_n$. Taking into account (6), the lag terms $F_n(t_{n,i})$ are strictly positive for $i = 1, 2$, because z_h is strictly positive in $]0, t_n[$ and K satisfies the *general conditions*. Therefore, by (9), $Z_{n,1} = G(F_n(t_{n,1}))$ is strictly positive, because G is strictly positive in $]0, +\infty[$. Moreover, $h_n B_n(2,1) > 0$, and hence, $h_n B_n(2,1) Z_{n,1} > 0$. So,

$$0 < F_n(t_{n,2}) + h_n B_n(2,1) Z_{n,1} \leq A,$$

because K satisfies the hypothesis, z_h is strictly positive in $]0, t_n[$, the polynomial $L_1(s)$ is strictly positive in $]0, c_2[$, the nonlinearity G is a strictly increasing function, and $t_{n,j} \leq t_n + c_j$ for $j = 1, 2$. Hence, we can apply Lemma 1-(ii) to the Eq. (10), obtaining the existence of $Z_{n,2}$.

Concluding, we have to check that $Z_{n,2} > Z_{n,1}$. Since K satisfies the hypothesis in (6), $F_n(t_{n,2}) \geq F_n(t_{n,1})$, and hence, by the properties of G, we have

$$\begin{aligned} Z_{n,2} &= G\left(F_n(t_{n,2}) + h_n \sum_{j=1}^{2} B_n(2,j) Z_{n,j}\right) \\ &> G(F_n(t_{n,2})) \geq G(F_n(t_{n,1})) = Z_{n,1}. \end{aligned}$$

($\Rightarrow$) For proving the other condition, we use Lemma 1-(i), taking into account Remark 7. □

Proof of Proposition 8. First, we are going to consider a convolution kernel $K(t,s) = k(t-s)$. Taking into account Remarks 2, 7, and Lemma 2, we choose a small enough $H > 0$ such that

$$h \int_0^{c_2} k((c_2 - s)h) \frac{s}{c_2}\,\mathrm{d}s \leq \beta_0 \tag{18}$$

for all $0 < h \leq H$. From here, the proof is analogous to the proof of Proposition 7, where $H_n = H$ and using Lemma 2 instead of Lemma 1. So, we do not need any A, and we do not need to check that $F_n(t_{n,2}) + h_n B_n(2,1) Z_{n,1} \leq A$.

If, in addition, $\frac{G(y)}{y}$ is a strictly decreasing function, the uniqueness is an immediate consequence of Lemma 2.

For general (nonconvolution) kernels, the proof is analogous but $B_n(2,j)$ is given by (11) and it does not only depend on h_n. The existence of h_n lies in choosing a small enough H_n such that $h_n B_n(2,2) \leq \beta_0$ for all $0 < h_n \leq H_n$. □

Proof of Proposition 9. The proof is analogous to the proof of Proposition 8, but in this case $H > 0$ is any positive real number, and using Lemma 3 instead of Lemma 2. □

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