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On preserving full orientability of graphs

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ABSTRACT

Suppose that D is an acyclic orientation of the graph G. An arc of D is *dependent* if its reversal creates a directed cycle. Let $d_{\min}(G)$ ($d_{\max}(G)$) denote the minimum (maximum) of the number of dependent arcs over all acyclic orientations of G. We call G *fully orientable* if G has an acyclic orientation with exactly k dependent arcs for every k satisfying $d_{\min}(G) \leqslant k \leqslant d_{\max}(G)$. In this paper, we study conditions under which full orientability of a graph can be preserved when the graph is extended by attaching new paths or cycles. Preservation theorems are applied to prove full orientability of subdivisions of Halin graphs and graphs of maximum degree at most three. We also characterize their $d_{\min}(G)$.

1. Introduction

Graphs considered in this paper are finite, without loops or multiple edges unless otherwise stated. Let $G = (V, E)$ be a graph. We use $|G|$ and $\|G\|$ to denote the cardinalities of vertex set V and edge set E, respectively.

An orientation D of G assigns a direction to each edge of G. If there does not exist any directed cycle in D, then D is said to be *acyclic*. Suppose that D is an acyclic orientation of G. An arc of D is called *dependent* if its reversal creates a directed cycle, or equivalently a directed closed walk. This definition of a dependent arc is due to Edelman (as quoted in [11]). Note that $u \to v$ is a dependent arc if and only if there exists a directed walk of length at least 2 from u to v. Let $d(D)$ denote the number of dependent arcs in D. Define $d_{\min}(G)$ ($d_{\max}(G)$) to be the minimum (maximum) value of $d(D)$ over all acyclic orientations D of G. It is known that $d_{\max}(G) = \|G\| - |G| + k$ for a graph G having k components [4].

An interpolation question asks whether G has an acyclic orientation with exactly d dependent arcs for every d satisfying $d_{\min}(G) \leqslant d \leqslant d_{\max}(G)$. We call G *fully orientable* if its interpolation question has an affirmative answer. West [11] showed that complete bipartite graphs are fully orientable.

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Let $\chi(G)$ denote the *chromatic number* of G, i.e., the least number of colors to color the vertices of G so that adjacent vertices receive distinct colors. Let $g(G)$ denote the *girth* of G, i.e., the length of a shortest cycle of G if there is any, and ∞ if G possesses no cycles. Fisher, Fraughnaugh, Langley, and West [4] showed that G is fully orientable if $\chi(G) < g(G)$. They also proved that $d_{\min}(G) = 0$ when $\chi(G) < g(G)$. In fact, $d_{\min}(G) = 0$ if and only if G is a *cover graph*, i.e., the underlying graph of the Hasse diagram of a partially ordered set. ([9], Fact 1.1).

A graph G is called *k-degenerate* if each subgraph H of G contains a vertex of degree at most k in H. It is straightforward to prove the following characterization. A graph G is k-degenerate if and only if there exists an ordering $v_1, v_2, \ldots, v_n$ of all the vertices of G so that the degree of v_i in the subgraph induced by $v_i, v_{i+1}, \ldots, v_n$ is at most k for each i, $1 \leqslant i \leqslant n$. We call such an ordering a *characterizing ordering* of a k-degenerate graph. Recently, Lai, Chang, and Lih [7] have established the full orientability of 2-degenerate graphs that generalizes a previous result for outerplanar graphs (see Lih, Lin, and Tong [8]).

Let $K_{r(n)}$ denote the complete r-partite graph each of whose partite set has n vertices. This is an $n(r-1)$-degenerate graph. Chang, Lin, and Tong [2] proved that $K_{r(n)}$ is not fully orientable when $r \geqslant 3$ and $n \geqslant 2$. An immediate consequence is that, when k is a composite number, there exist k-degenerate graphs which are not fully orientable. The problem whether any p-degenerate (but not $(p-1)$-degenerate) graph, where p is an odd prime, is fully orientable remains unsettled. The graph $K_{3(2)}$ shows that a graph may not be fully orientable when chromatic number is equal to girth.

Fisher, Fraughnaugh, Langley, and West [4] also studied the parameter $r_{\chi,g}$ defined to be the supremum of $d_{\min}(G)/\|G\|$ over all graphs G with $\chi(G) = \chi$ and $g(G) = g$. They proved that $r_{\chi,g} \leqslant (\chi-2)/\chi$ and the equality holds when $g = 3$. Collins and Tysdal [3] gave further lower bounds for $r_{\chi,g}$ with specific χ and g. Recently, Rödl and Thoma [9] have established the equality $r_{\chi,g} = \binom{\chi-g+2}{2} / \binom{\chi}{2}$ for $\chi \geqslant 2$ and $g \geqslant 3$.

In this paper, we are mainly concerned with the preservation of full orientability after appropriately attaching a path or a cycle. The organization of this paper is as follows. In Section 2, we introduce and study several graph parameters that are lower bounds for $d_{\min}$. These parameters provide more interpretations for $d_{\min}$ when they coincide with it. In Section 3, we establish preservation results for attaching paths, especially for a particular type of edges. We prove our main preservation results in Section 4 with respect to attaching cycles of appropriate types. We apply preservation results to prove full orientability of subdivisions of Halin graphs and graphs with maximum degree at most three and give interpretations to their $d_{\min}(G)$ in the final Section.

2. Parameters lower bounding $d_{\min}$

In showing full orientability of graphs, we often get extra information about their $d_{\min}$. We are going to introduce several graph parameters which facilitate the interpretation of $d_{\min}$.

Define $\pi_E(G) = \min\{|E'| \mid E' \subseteq E(G) \text{ and } G - E' \text{ is a cover graph}\}$. This parameter is equal to the minimum number of new vertices whose addition to edges of G produces a cover graph. (See Holub [6].)

Each acyclic orientation of a triangle has a unique dependent arc. Hence, any cover graph must be triangle-free. We define $\pi_T(G) = \min\{|E'| \mid E' \subseteq E(G) \text{ and } G - E' \text{ is triangle-free}\}$. Note that $\pi_T(G)$ is also the minimum number of edges meeting all triangles in G.

For a graph G, let G^T be the graph whose vertex set is the set of triangles of G and two vertices of G^T are adjacent if and only if the corresponding triangles have a common edge. A clique is a set of mutually adjacent vertices. A set of cliques *covers* G if the union of its members is $V(G)$. The clique cover number $\theta(G)$ of G is the minimum number of cliques needed to cover G.

Since there is a one-to-one correspondence between edge-disjoint triangles in G and independent sets in G^T, the maximum number of edge-disjoint triangles in G is equal to $\alpha(G^T)$, where the parameter α denotes the maximum cardinality of an independent set. By convention, $\alpha(G^T) = \theta(G^T) = 0$ when G is triangle-free.

The following theorem gives an ordering of parameters introduced above. We only give the proof of $\theta(G^T) \leqslant \pi_T(G)$ for the other inequalities are easy to prove.

Theorem 1. *For every graph G, we have* $\alpha(G^T) \leqslant \theta(G^T) \leqslant \pi_T(G) \leqslant \pi_E(G) \leqslant d_{\min}(G)$.

Proof. Choose $F \subseteq E(G)$ such that $G-F$ is triangle-free and $|F| = \pi_T(G)$. If $e \in F$, the set of vertices in G^T whose corresponding triangles containing e forms a clique, say K_e. Since $G - F$ is triangle-free, each triangle contains one edge in F and $\{K_e \mid e \in F\}$ is a set of cliques that covers G^T. Hence $\pi_T(G) = |F| \geqslant |\{K_e \mid e \in F\}| \geqslant \theta(G^T)$. □

Lai, Chang, and Lih [7] have shown that all inequalities in Theorem 1 hold with equality for 2-degenerate graphs. On the other hand, we can construct examples to demonstrate that these parameters are indeed non-identical parameters. Basic properties of these parameters, such as being monotone with respect to taking subgraphs, sub-additive over edge-disjoint subgraphs, and additive over components or 2-connected components, will be derived now.

Lemma 2. *If H is a subgraph of G, then* $\alpha(H^T) \leqslant \alpha(G^T)$, $\theta(H^T) \leqslant \theta(G^T)$, $\pi_T(H) \leqslant \pi_T(G)$, $\pi_E(H) \leqslant \pi_E(G)$, *and* $d_{\min}(H) \leqslant d_{\min}(G)$.

Proof. If H is a subgraph of G, then H^T is an induced subgraph of G^T. Hence $\alpha(H^T) \leqslant \alpha(G^T)$ and $\theta(H^T) \leqslant \theta(G^T)$. Choose $F \subseteq E(G)$ such that $G - F$ is triangle-free and $|F| = \pi_T(G)$. Let $F' = F \cap E(H)$. Then $H - F'$ is triangle-free and $\pi_T(H) \leqslant |F'| \leqslant |F|$. Similarly, $\pi_E(H) \leqslant \pi_E(G)$. If D is an acyclic orientation of G with $d(D) = d_{\min}(G)$, then $d_{\min}(H) \leqslant d(D') \leqslant d(D)$, where D' is the restriction of D to H. □

Lemma 3. *If* H_1 *and* H_2 *are two edge-disjoint subgraphs of G, then* $\alpha(H_1^T) + \alpha(H_2^T) \leqslant \alpha(G^T)$, $\theta(H_1^T) + \theta(H_2^T) \leqslant \theta(G^T)$, $\pi_T(H_1) + \pi_T(H_2) \leqslant \pi_T(G)$, $\pi_E(H_1) + \pi_E(H_2) \leqslant \pi_E(G)$, *and* $d_{\min}(H_1) + d_{\min}(H_2) \leqslant d_{\min}(G)$.

Proof. Since H_1 and H_2 are edge-disjoint, H_1^T and H_2^T are vertex-disjoint induced subgraphs of G^T, hence $\alpha(H_1^T) + \alpha(H_2^T) \leqslant \alpha(G^T)$ and $\theta(H_1^T) + \theta(H_2^T) \leqslant \theta(G^T)$. Choose $F \subseteq E(G)$ such that $G - F$ is triangle-free and $|F| = \pi_T(G)$. Let $F_1 = F \cap E(H_1)$ and $F_2 = F \cap E(H_2)$. Then $F_1 \cap F_2 = \emptyset$ and $H_i - F_i$ is a triangle-free subgraph of $G - F$ for $1 \leqslant i \leqslant 2$. Hence $\pi_T(H_1) + \pi_T(H_2) \leqslant |F_1| + |F_2| \leqslant |F|$. Similarly, $\pi_E(H_1) + \pi_E(H_2) \leqslant \pi_E(G)$. If D is an acyclic orientation of G with $d(D) = d_{\min}(G)$, then $d_{\min}(H_1) + d_{\min}(H_2) \leqslant d(D_1) + d(D_2) \leqslant d(D)$, where D_i is the restriction of D to H_i for $1 \leqslant i \leqslant 2$. □

Lemma 4. *Let* $G_1, G_2, \ldots, G_k$ *be all the components of G. Then the following properties hold.*

(1) *If* $G_1, G_2, \ldots, G_k$ *are fully orientable, so is G.*

(2) $\alpha(G^T) = \Sigma_{i=1}^k \alpha(G_i^T)$, $\theta(G^T) = \Sigma_{i=1}^k \theta(G_i^T)$, $\pi_T(G) = \Sigma_{i=1}^k \pi_T(G_i)$, $\pi_E(G) = \Sigma_{i=1}^k \pi_E(G_i)$, *and* $d_{\min}(G) = \Sigma_{i=1}^k d_{\min}(G_i)$.

Proof. The first four equalities of part (2) and $d_{\min}(G) \leqslant \Sigma_{i=1}^k d_{\min}(G_i)$ are easy to prove. If $d_{\min}(G) < \Sigma_{i=1}^k d_{\min}(G_i)$, then, for some j, there is an acyclic orientation D_j of G_j such that $d(D_j) < d_{\min}(G_j)$, which is impossible. Therefore, $d_{\min}(G) = \Sigma_{i=1}^k d_{\min}(G_i)$. For any d satisfying $d_{\min}(G) \leqslant d \leqslant d_{\max}(G) = \|G\| - |G| + k = \Sigma_{i=1}^k d_{\max}(G_i)$, we can rewrite d as $\Sigma_{i=1}^k d_i$, where $d_{\min}(G_i) \leqslant d_i \leqslant d_{\max}(G_i)$ for each i. Since every G_i is fully orientable, there is an acyclic orientation D_i of G_i with $d(D_i) = d_i$. The union of all these acyclic orientations is an acyclic orientation D of G with $d(D) = d$. □

A proof analogous to the above can show the following.

Lemma 5. *Assume that G has a cut vertex u and* $G_1, G_2, \ldots, G_k$ *are the components of* $G - u$. *For each i,* $1 \leqslant i \leqslant k$, *let* H_i *be the subgraph induced by u and* G_i. *Then the following properties hold.*

(1) *If* $H_1, H_2, \ldots, H_k$ *are fully orientable, so is G.*

(2) $\alpha(G^T) = \Sigma_{i=1}^k \alpha(H_i^T)$, $\theta(G^T) = \Sigma_{i=1}^k \theta(H_i^T)$, $\pi_T(G) = \Sigma_{i=1}^k \pi_T(H_i)$, $\pi_E(G) = \Sigma_{i=1}^k \pi_E(H_i)$, *and* $d_{\min}(G) = \Sigma_{i=1}^k d_{\min}(H_i)$.

Note that the graph $K_{3(2)}$ has acyclic orientations with 4, 6, or 7 dependent arcs. The disjoint union of the cycle C_4 of length four with $K_{3(2)}$ is fully orientable. The graph obtained by identifying a vertex of C_4 with a vertex of $K_{3(2)}$ is also fully orientable. These examples show that the converse of part (1) in either Lemma 4 or Lemma 5 is false.

Exact values of graph parameters studied above can be determined for cycles and complete graphs. When $n \geqslant 4$, the cycle C_n on n vertices is a cover graph since $\chi(C_n) \leqslant 3 < g(C_n)$. Then the following is easy to prove.

Lemma 6. *The cycle C_n is fully orientable. When $n = 3$, $\alpha(C_n^T) = \theta(C_n^T) = \pi_T(C_n) = \pi_E(C_n) = d_{\min}(C_n) = 1$. When $n \geqslant 4$, $\alpha(C_n^T) = \theta(C_n^T) = \pi_T(C_n) = \pi_E(C_n) = d_{\min}(C_n) = 0$.*

Lemma 7. *The complete graph K_n is fully orientable and satisfies $\pi_T(K_n) = \pi_E(K_n) = \lceil n(n-2)/4 \rceil$ and $d_{\min}(K_n) = (n-2)(n-1)/2$.*

Proof. West [11] proved $d_{\min}(K_n) = (n-2)(n-1)/2$ which is exactly $d_{\max}(K_n)$, hence K_n is fully orientable.

Choose $F \subseteq E(K_n)$ such that $K_n - F$ is triangle-free and $|F| = \pi_T(K_n)$, then $\|K_n\| - |F| = \|K_n - F\| \leqslant \lfloor |K_n - F|^2/4 \rfloor = \lfloor n^2/4 \rfloor$. The last inequality holds due to a well-known theorem of Turán [10]. The equality $\pi_E(K_n) = \lceil n(n-2)/4 \rceil$ was proved in Holub [6]. By Theorem 1, $\lceil n(n-2)/4 \rceil = \|K_n\| - \lfloor n^2/4 \rfloor \leqslant \pi_T(K_n) \leqslant \pi_E(K_n) = \lceil n(n-2)/4 \rceil$. □

3. Attaching a path

When we extend a fully orientable graph G to a graph H, we are interested in the preservation of full orientability from G to H. Lai, Chang, and Lih [7] proved the following two lemmas.

Lemma 8. *Assume that u and v are two adjacent vertices of the graph G. If H is obtained from G by adding a new vertex w adjacent to u and v, then the following properties hold.*

(1) *If G has an acyclic orientation D_1 with $d(D_1) = k$, then H has an acyclic orientation D_2 with $d(D_2) = k + 1$.*

(2) *If G is fully orientable, so is H.*

Lemma 9. *Assume that u and v are two non-adjacent vertices of the graph G. If H is obtained from G by adding a new vertex w adjacent to u and v, then the following properties hold.*

(1) *If G has an acyclic orientation D_1 with $d(D_1) = k$, then H has an acyclic orientation D_2 with $d(D_2) = k$.*

(2) *If G is fully orientable, so is H.*

Since attaching only one end of a path to a fully orientable graph preserves full orientability, a straightforward deduction from the above two lemmas gives the following.

Corollary 10. *Let H be obtained from the graph G by identifying the ends of a new path P of length at least two with two vertices u and v of G. If u and v are non-adjacent or the length of P is at least 3, then the following properties hold.*

(1) $d_{\min}(G) = d_{\min}(H)$.

(2) *G is a cover graph if and only if H is a cover graph.*

(3) *If G is fully orientable, so is H.*

Let $K_{3(2)} - e$ denote the graph obtained from $K_{3(2)}$ by deleting an edge. We can construct acyclic orientations of $K_{3(2)} - e$ with 3, 4, 5, or 6 dependent arcs. When we delete any triangle from $K_{3(2)}$, we get an outerplanar graph that possesses three edge-disjoint triangles, and hence its $d_{\min}$ is at least three. It follows from Lemma 2 that $K_{3(2)} - e$ is fully orientable. Consequently, restoring the edge e will not preserve full orientability. So far, it is not completely understood when full orientability is preserved with respect to adding a new edge. However, the next special case is sufficient for our later use.

Theorem 11. *Let a and b be two non-adjacent vertices of degree two of the graph G such that they have exactly one common neighbor. If G is fully orientable, so is $G + ab$.*

Proof. We know that $d_{\max}(G + ab) = d_{\max}(G) + 1$ and $d_{\min}(G + ab) \geqslant d_{\min}(G)$ by Lemma 2. Our theorem is established if the following claim is true.

Claim. If G has an acyclic orientation D with $d(D) = d$, then $G + ab$ has an acyclic orientation D' with $d(D') = d + 1$.

Assume that v is the unique common neighbor of a and b. Let a' and b' be the second neighbors of a and b, respectively. Without loss of generality, we may assume $a \to v$ in D.

Case 1. $v \to b$ in D.

We extend D to D' by defining $a \to b$. Obviously, D' is acyclic and ab is dependent in D'. Hence $d(D') \geqslant d(D) + 1$. If the reversal of an arc e in D produces a directed cycle passing through $a \to b$, then e must be distinct from $a \to v$ and $v \to b$, and hence we may replace $a \to b$ by $a \to v \to b$. Thus e is already a dependent arc in D. Hence $d(D') = d(D) + 1$.

Case 2. $b \to v$ in D.

Suppose that $a \to a'$ or $b \to b'$ in D. If $b \to b'$ in D, then b is a source vertex. By reversing the two arcs incident with b, we get the acyclic orientation $\overline{D}$. We have $d(\overline{D}) = d(D)$ and $v \to b$ in $\overline{D}$. By Case 1, we can find an acyclic orientation D' of $G + ab$ satisfying $d(D') = d(D) + 1$. The case for $a \to a'$ in D can be argued similarly.

Suppose that $a' \to a$ and $b' \to b$ in D. Observe that $a \to v$ and $b \to v$ are not dependent in D. Define the extension acyclic orientation D' as follows. If there exists a directed path from a' to b' in D, then define $a \to b$ in D'; otherwise define $b \to a$ in D'.

Suppose $a \to b$ in D'. Then $a \to v$ is dependent in D'. Hence, $d(D') \geqslant d(D) + 1$. If $b' \to b$ is dependent in D' but not in D, then there would be a directed path from b' to a', contradicting the acyclicity of D. It is easy to see that $a \to b$ and $b \to v$ are not dependent in D'. If the reversal of an arc e different from $a \to v$ and $b' \to b$ in D produces a directed cycle passing through $a \to b$, then it must pass through $b \to v$, and hence we may replace these two arcs by $a \to v$. Thus e is already a dependent arc in D. Hence $d(D') = d(D) + 1$.

The case for $b \to a$ in D' can be argued similarly. □

4. Attaching a cycle

We are going to establish our main preservation results in this section. Suppose that G is a graph and C is a cycle disjoint from G. Let $(l_1, l_2, \ldots, l_k)$ be a sequence of positive integers such that $\sum_{i=1}^{k} l_i$ is equal to the length of C. Then a family of graphs G_C can be constructed as follows. Let the vertices of C be denoted successively as $v_{1,1} \cdots v_{1,l_1} v_{2,1} \cdots v_{2,l_2} \cdots v_{k,1} \cdots v_{k,l_k} v_{1,1}$. For each vertex $v_{i,j}$ of C, we add at most one new edge $v_{i,j}u$, $u \in V(G)$, according to the following rules. If $1 \leqslant i \leqslant k$, then v_{i,l_i} and $v_{i+1,1}$ have no common neighbor in G. (Indices are modulo k.) If $1 \leqslant i \leqslant k$ and $1 \leqslant j < l_i$, then $v_{i,j}$ and $v_{i,j+1}$ have a common neighbor in G. Let u_i denote the common neighbor of $v_{i,1}, v_{i,2}, \ldots, v_{i,l_i}$ in G when $l_i \geqslant 2$. Note that $u_i \neq u_{i+1}$, yet $u_i = u_j$ may occur when $|i - j| > 1$. Any of these graphs G_C is said to be obtained from G by attaching the cycle C of type $(l_1, l_2, \ldots, l_k)$.

Let C be a cycle of a graph G. A *chord* of C is an edge of G joining non-consecutive vertices of C. A *chordless cycle* in G is a cycle that has no chord. If C is a chordless cycle and each vertex of C has degree at most three in G, then G can be obtained from $G - C$ by attaching the cycle C. Moreover, the type of C and the manner of attachment can be determined in a straightforward way.

If we attach a cycle C of type (l_1), $l_1 = n \geqslant 3$, to a single vertex graph G, then G_C, denoted W_n, is commonly known as the *wheel* with n spokes.

Theorem 12. *The wheel W_n is fully orientable. When $n = 3$, $\alpha(W_n^T) = \theta(W_n^T) = 1$, $\pi_T(W_n) = \pi_E(W_n) = 2$, and $d_{\min}(W_n) = 3$. When $n \geqslant 4$, $\alpha(W_n^T) = \lfloor n/2 \rfloor$ and $\theta(W_n^T) = \pi_T(W_n) = \pi_E(W_n) = d_{\min}(W_n) = \lceil n/2 \rceil$.*

Proof. When $n = 3$, $W_3 = W_3^T = K_4$. Hence $\alpha(W_3^T) = \theta(W_3^T) = \theta(K_4) = 1$. Other parameters were determined and full orientability was proved in Lemma 7.

For $n \geqslant 4$, assume that W_n is obtained by joining the vertex u to all vertices of the cycle $C_n = v_1 v_2 \cdots v_n v_1$. It is easy to see that $W_n - v_1 v_2$ is 2-degenerate, hence fully orientable. We observe that v_1 and v_2 have the unique neighbor u in $W_n - v_1 v_2$. By Theorem 11, W_n is fully orientable. We also observe that $W_n^T = C_n$. Thus $\alpha(W_n^T) = \lfloor n/2 \rfloor$ and $\theta(W_n^T) = \lceil n/2 \rceil$. It remains to construct an acyclic orientation D of W_n with $d(D) = \lceil n/2 \rceil$. The desired acyclic orientation D is defined as follows. For all j, let $v_j \to u$. For $1 \leqslant i \leqslant n$, let $v_i \to v_{i+1}$ for odd i and $v_i \leftarrow v_{i+1}$ for even i. The arcs $v_{2i-1} \to u$, $1 \leqslant i \leqslant \lceil n/2 \rceil$, are all the dependent arcs of D. □

Theorem 13. *Let G_C be a graph obtained from the graph G by attaching a cycle C of type $(l_1, l_2, \dots, l_k)$ such that $\sum_{i=1}^{k} l_i = 3$.*

(1) If G is fully orientable, so is G_C.

(2) When $k = 1$, $\alpha(G_C^T) - \alpha(G^T) = \theta(G_C^T) - \theta(G^T) = 1$, $\pi_T(G_C) - \pi_T(G) = \pi_E(G_C) - \pi_E(G) = 2$, and $d_{\min}(G_C) - d_{\min}(G) = 3$.

(3) When $k = 2$ or 3, $\alpha(G_C^T) - \alpha(G^T) = \theta(G_C^T) - \theta(G^T) = \pi_T(G_C) - \pi_T(G) = \pi_E(G_C) - \pi_E(G) = d_{\min}(G_C) - d_{\min}(G) = 1$.

Proof. Since $\sum_{i=1}^{k} l_i = 3$, C is a triangle. By Lemma 4, our theorem holds if C is a component of G_C. From now on, we assume that there is an edge between C and G.

(1) Using Lemma 5, Corollary 10, and Theorem 11, we can construct G_C from G, and hence full orientability is preserved.

(2) This follows from Lemma 5 and Theorem 12.

(3) We may assume that $v_{1,1}$ always has a neighbor in G and $l_1 = 2$ when $k = 2$.

When we extended G to G_C, we obtained one or two adjacent new triangles, all edge-disjoint from triangles in G. So G_C^T can be obtained from G^T by adding an isolated new vertex or edge. Hence $\alpha(G_C^T) - \alpha(G^T) = \theta(G_C^T) - \theta(G^T) = 1$, $\pi_T(G_C) - \pi_T(G) \geqslant 1$, and $\pi_E(G_C) - \pi_E(G) \geqslant 1$ by Lemma 3.

Let e denote the edge $v_{1,1}v_{1,2}$ when $k = 2$, or the edge $v_{1,1}v_{2,1}$ when $k = 3$.

Choose $F \subseteq E(G)$ such that $G - F$ is triangle-free and $|F| = \pi_T(G)$. Then $G_C - (F \cup \{e\})$ is a triangle-free graph. Hence $\pi_T(G_C) - \pi_T(G) \leqslant 1$.

Choose $F' \subseteq E(G)$ such that $G - F'$ is a cover graph and $|F'| = \pi_E(G)$. Since $G_C - (F' \cup \{e\})$ can be obtained from $G - F'$ by attaching paths between non-adjacent vertices, $G_C - (F' \cup \{e\})$ is a cover graph by Corollary 10. Hence $\pi_E(G_C) - \pi_E(G) \leqslant 1$.

It remains to prove that $d_{\min}(G_C) - d_{\min}(G) = 1$. Let D be an acyclic orientation of G with $d(D) = d_{\min}(G)$.

Case 1. $k = 2$.

We first extend D by defining $v_{1,1} \to u_1$, $u_1 \to v_{1,2}$, $v_{1,1} \to v_{1,2}$, $v_{1,1} \to v_{2,1}$, and $v_{2,1} \to v_{1,2}$. If $v_{2,1}$ has a neighbor u_2 in G and there is (or is not) a directed path from u_1 to u_2 in D, then we further extend D to D_C by defining $v_{2,1} \to u_2$ (or $v_{2,1} \leftarrow u_2$). We see that the unique new dependent arc is $v_{1,1} \to v_{1,2}$.

Case 2. $k = 3$.

The cycle C is just $v_{1,1}v_{2,1}v_{3,1}v_{1,1}$. If there is only one edge between G and C, then we extend D to D_C by defining $v_{1,1} \to u_1$ and making C an acyclic triangle. If there are only two edges, say $v_{1,1}u_1$ and $v_{2,1}u_2$, between G and C, then we may assume that there is no directed path from u_1 to u_2. We extend D to D_C by defining $v_{1,1} \to u_1$, $v_{2,1} \to u_2$, $v_{1,1} \to v_{3,1}$, $v_{2,1} \to v_{1,1}$, and $v_{2,1} \to v_{3,1}$. In both cases, we see that $d_{\min}(G_C) - d_{\min}(G) = 1$.

Now suppose that there are three edges between G and C. We first extend D by defining the following arcs: $v_{1,1} \to u_1$, $v_{2,1} \to u_2$, and $v_{3,1} \to u_3$, Consider the three ordered pairs (u_1, u_2), (u_2, u_3), and (u_3, u_1). There must be one ordered pair such that no directed path in D connects the first vertex to the second vertex. Assume that ordered pair is (u_1, u_2).

Suppose that there is a directed path from u_2 to u_3 and there is no directed path from u_1 to u_3 in D. Then we further extend D to D_C by defining $v_{2,1} \to v_{1,1}$, $v_{2,1} \to v_{3,1}$, and $v_{3,1} \to v_{1,1}$. The unique new dependent arc is $v_{2,1} \to v_{1,1}$.

Suppose that there is a directed path from u_2 to u_3 and there is a directed path from u_1 to u_3 in D. Then we further extend D to D_C by defining $v_{1,1} \to v_{3,1}$, $v_{2,1} \to v_{1,1}$, and $v_{2,1} \to v_{3,1}$. The unique new dependent arc is $v_{2,1} \to v_{3,1}$.

Suppose that there is no directed path from u_2 to u_3 and there is no directed path from u_1 to u_3 in D. Then we further extend D to D_C by defining $v_{2,1} \to v_{1,1}$, $v_{3,1} \to v_{1,1}$, and $v_{3,1} \to v_{2,1}$. The unique new dependent arc is $v_{3,1} \to v_{1,1}$.

Suppose that there is no directed path from u_2 to u_3 and there is a directed path from u_1 to u_3 in D. It follows that (a) there is no directed path from u_2 to u_1 in D, or (b) there is no directed path from u_3 to u_2 in D. For (a), we further extend D to D_C by defining $v_{1,1} \to v_{2,1}$, $v_{1,1} \to v_{3,1}$, and $v_{3,1} \to v_{2,1}$. The unique new dependent arc is $v_{1,1} \to v_{2,1}$. For (b), we further extend D to D_C by defining $v_{1,1} \to v_{3,1}$, $v_{2,1} \to v_{1,1}$, and $v_{2,1} \to v_{3,1}$. The unique new dependent arc is $v_{2,1} \to v_{3,1}$.

We conclude that $d_{\min}(G_C) - d_{\min}(G) = 1$. □

Theorem 14. *Let G_C be a graph obtained from the graph G by attaching a cycle C of type $(l_1, l_2, \ldots, l_k)$ such that $\sum_{i=1}^{k} l_i \geqslant 4$. If $l_i \geqslant 2$ for some i or $v_{j,1}$ has no neighbor in G for some j, then the following properties hold.*

(1) If G is fully orientable, so is G_C.

(2) When $k = 1, \alpha(G_C^T) - \alpha(G^T) = \lfloor l_1/2 \rfloor$ and $\theta(G_C^T) - \theta(G^T) = \pi_T(G_C) - \pi_T(G) = \pi_E(G_C) - \pi_E(G) = d_{\min}(G_C) - d_{\min}(G) = \lceil l_1/2 \rceil$.

(3) When $k \geqslant 2$, $\alpha(G_C^T) - \alpha(G^T) = \theta(G_C^T) - \theta(G^T) = \pi_T(G_C) - \pi_T(G) = \pi_E(G_C) - \pi_E(G) = d_{\min}(G_C) - d_{\min}(G) = \sum_{i=1}^{k} \lfloor l_i/2 \rfloor$.

Proof. Since $\sum_{i=1}^{k} l_i \geqslant 4$, the attached cycle C has length at least four. By Lemmas 4 and 6, our theorem holds if C is a component of G_C. From now on, we assume that there is an edge between C and G.

We first define a subgraph G_0 of G_C by deleting edges and vertices from G_C as follows. For $1 \leqslant i \leqslant k$, delete edge $v_{i,1}v_{i,2}$ if $l_i = 2$; delete vertices $v_{i,2}, \ldots, v_{i,l_i-1}$ if $l_i \geqslant 3$. Evidently, G_0 contains G as a subgraph.

(1) Starting with G, we will make a sequence of extensions to get G_0, and then G_C, while preserving full orientability along the way. So we can conclude in the end that G_C is fully orientable.

Case 1. If $l_i \geqslant 2$ for some i.

We can get G_0 from G by attaching "comb-like" pieces. A typical piece consists of a path $P = v_{i,l_i}v_{i+1,1}v_{i+2,1} \cdots v_{j-1,1}v_{j,1}$, where $i \neq j$, $l_i \geqslant 2$, $l_j \geqslant 2$, and $l_t = 1$ for $i < t < j$, together with those edges joining P to G. Each of these pieces can be obtained by successively attaching paths of length at least two. By Corollary 10, G_0 is fully orientable.

In this case, G_0 is different from G_C. Now we define an ascending sequence $G_1, G_2, \ldots, G_k$ of subgraphs of G_C as follows. Assume that G_{i-1}, $1 \leqslant i \leqslant k$, has been defined and is fully orientable. If $l_i = 1$, let $G_i = G_{i-1}$. If $l_i = 2$, let $G_i = G_{i-1} + v_{i,1}v_{i,2}$, and hence G_i is fully orientable by Theorem 11. If $l_i \geqslant 3$, let G_i be obtained from G_{i-1} by adding vertex $v_{i,j}$ and edges $v_{i,j-1}v_{i,j}$ and $v_{i,j}u_i$ for $2 \leqslant j < l_i$ together with the last edge $v_{i,l_i-1}v_{i,l_i}$. By Corollary 10 and Theorem 11, G_i is fully orientable. We see that $G_C = G_k$ is fully orientable.

Case 2. All $l_i = 1$.

By our assumption, there is some $v_{j,1}$ without a neighbor in G. We may assume $j = 1$. Let $a > 1$ (or $b \leqslant k$) be the smallest (or largest) index t such that $v_{t,1}$ has a neighbor in G. We can obtain the graph $G' = G_C - \{v_{b+1,1}, \ldots, v_{1,1}, \ldots, v_{a-1,1}\}$ by attaching the "comb-like" piece consisting of the path $P' = v_{a,1}v_{a+1,1} \cdots v_{b,1}$ together with those edges joining P' to G. Then we get G_C by attaching the path $v_{b+1,1} \cdots v_{1,1} \cdots v_{a-1,1}$ of length at least two to G'. Hence G_C is fully orientable.

(2) If $k = 1$, the vertices $v_{1,1}, \ldots, v_{1,l_1}$ and u_1 induce a wheel W_{l_1} in G_C and u_1 is a cut vertex. By Lemma 5 and Theorem 12, we are done.

(3) For each $1 \leqslant i \leqslant k$, if $l_i \geqslant 2$, let F_i be the subgraph of G_C induced by $u_i, v_{i,1}, \ldots, v_{i,l_i}$. Then F_i^T is a path on $l_i - 1$ vertices. Since F_i is a 2-degenerate graph, $\alpha(F_i^T) = \theta(F_i^T) = \pi_T(F_i) = \pi_E(F_i) = d_{\min}(F_i) = \lfloor l_i/2 \rfloor$.

Since F_i and G are edge-disjoint for each i with $l_i \geqslant 2$, $\sum_{i=1}^{k} \lfloor l_i/2 \rfloor$ is a lower bound of $\alpha(G_C^T) - \alpha(G^T)$, $\theta(G_C^T) - \theta(G^T)$, $\pi_T(G_C) - \pi_T(G)$, $\pi_E(G_C) - \pi_E(G)$, and $d_{\min}(G_C) - d_{\min}(G)$ by Lemma 3.

If a triangle is in G_C but not in G, then the triangle must belong to some F_i. The components of G_C^T are precisely all components of G^T together with all F_i^T with $l_i \geqslant 2$. Hence, $\sum_{i=1}^{k} \lfloor l_i/2 \rfloor$ is also an upper bound of $\alpha(G_C^T) - \alpha(G^T)$ and $\theta(G_C^T) - \theta(G^T)$.

Let $E' = \cup_{i=1}^{k} \{u_i v_{i,j} \mid j \text{ even and } 1 \leqslant j \leqslant l_i\}$. Clearly, $|E'| = \sum_{i=1}^{k} \lfloor l_i/2 \rfloor$. Choose $F \subseteq E(G)$ such that $G - F$ is triangle-free and $|F| = \pi_T(G)$. Then $G_C - (F \cup E')$ is a triangle-free graph. Hence $\pi_T(G_C) - \pi_T(G) \leqslant \sum_{i=1}^{k} \lfloor l_i/2 \rfloor$.

Choose $F' \subseteq E(G)$ such that $G - F'$ is a cover graph and $|F'| = \pi_E(G)$. Since $G_C - (F' \cup E')$ can be obtained from $G - F'$ by attaching paths between non-adjacent vertices, $G_C - (F' \cup E')$ is a cover graph by Corollary 10. Hence $\pi_E(G_C) - \pi_E(G) \leqslant \sum_{i=1}^{k} \lfloor l_i/2 \rfloor$.

It remains to prove that there is an acyclic orientation D_C of G_C with $d(D_C) = d_{\min}(G) + \sum_{i=1}^{k} \lfloor l_i/2 \rfloor$.

Case 1. For some s, say $s = 1$, $l_s \geqslant 2$.

Let D be an acyclic orientation of G with $d(D) = d_{\min}(G)$. In the first stage, we are going to extend D to an acyclic orientation D_0 of G_0. For each $v_{m,i}$ that has a neighbor u_m in G, define $v_{m,i} \to u_m$. It remains to orient the segments of the cycle C. A typical segment is a path like $v_{i,l_i}v_{i+1,1}v_{i+2,1}\cdots v_{j-1,1}v_{j,1}$ such that v_{i,l_i} and $v_{j,1}$ have neighbors u_i and u_j, respectively, in G and none of the internal vertices have neighbors in G. We orient each path as follows.

If there is a directed path from u_i to u_j, define $v_{i,l_i} \rightarrow v_{i+1,1} \leftarrow v_{i+2,1} \rightarrow \cdots$ alternately until $v_{j,1}$. Otherwise, define $v_{i,l_i} \leftarrow v_{i+1,1} \rightarrow v_{i+2,1} \leftarrow \cdots$ alternately until $v_{j,1}$.

In this way, we have extended D to an acyclic orientation D_0 of G_0 such that $d(D_0) = d(D) = d_{\min}(G)$. Moreover, each arc outside D is not dependent in D_0.

In the second stage, we are going to extend D_0 to an acyclic orientation D_C of G_C such that $d(D_C) = d_{\min}(G) + \sum_{i=1}^{k}\lfloor l_i/2\rfloor$.

(i) Suppose $l_i = 2$. Define $v_{i,1} \rightarrow v_{i,2}$ if $v_{i-1,l_{i-1}} \leftarrow v_{i,1}$ in D_0; define $v_{i,1} \leftarrow v_{i,2}$ if $v_{i-1,l_{i-1}} \rightarrow v_{i,1}$ in D_0.

(ii) Suppose $l_i \geqslant 3$. Define $v_{i,j} \rightarrow u_i$ for all j, $1 < j < l_i$. For $1 \leqslant j < l_i$, define $v_{i,j} \leftarrow v_{i,j+1}$ if j is odd and $v_{i,j} \rightarrow v_{i,j+1}$ if j is even.

In the orientation D_C, $v_{i-1,l_{i-1}} \leftarrow v_{i,1} \rightarrow v_{i,2}$ or $v_{i-1,l_{i-1}} \rightarrow v_{i,1} \leftarrow v_{i,2}$ when $l_i = 2$; $v_{i,1} \leftarrow v_{i,2} \rightarrow v_{i,3}$ when $l_i \geqslant 3$. Hence, C is not a directed cycle in D_C. If $C' \neq C$ is a cycle in G_C but not in G, then C' must use at least two edges between G and C. However, since these edges are all oriented toward G, C' cannot be a directed cycle in D_C. Therefore, D_C is an acyclic orientation.

The following set consists of dependent arcs of D_C: $E'' = \{$all dependent arcs in $D\} \cup \{v_{i,1} \rightarrow u_i \mid l_i = 2, v_{i-1,l_{i-1}} \leftarrow v_{i,1}\} \cup \{v_{i,2} \rightarrow u_i \mid l_i = 2, v_{i-1,l_{i-1}} \rightarrow v_{i,1}\} \cup \{v_{i,j} \rightarrow u_i \mid l_i \geqslant 3, j$ even, $1 \leqslant j \leqslant l_i\}$. Finally, we are going to show that the arcs in E'' exhaust all dependent arcs in D_C. Suppose that the reversal of an arc e in D_C produces a directed cycle C^*. If C^* lies entirely in G, then e is a dependent arc of D. If C^* lies entirely in some subgraph F_i defined at the beginning of (3), it is easy to see $e \in E''$. Since the length of C is at least four, from the way the orientation was defined, there exist at least two forward and two backward arcs when traversing the cycle C. Hence C cannot be C^*.

It remains to handle the case when C^* contains at least one edge in C and one edge in G. Since all arcs between C and G are oriented toward G, the arc e must be identical to some $v_{i,j} \rightarrow u_i$. Hence there exists a directed path P from $v_{i,1}$ or v_{i,l_i} to u_i in $D_C - \{v_{i,1} \rightarrow u_i, v_{i,l_i} \rightarrow u_i\}$. Let us assume that P begins with $v_{i,1}$ and P is of the shortest possible length.

Since $v_{m,1}$ is a source when $l_m = 2$ and $v_{m-1,l_{m-1}} \leftarrow v_{m,1}$, P cannot pass through the arc $v_{m,1} \rightarrow v_{m,2}$. If P uses an arc $v_{m,1} \leftarrow v_{m,2}$ when $l_m = 2$ and $v_{m-1,l_{m-1}} \rightarrow v_{m,1}$, then the length of P can be shortened by replacing its subpath $v_{m,2} \rightarrow v_{m,1} \rightarrow u_m$ by $v_{m,2} \rightarrow u_m$, contradicting the minimality of P.

Suppose that P passes through an arc in some subgraph F_m and the corresponding l_m is at least three. If l_m is odd, then that arc must be $v_{m,l_m} \rightarrow u_m$. If l_m is even, then either that arc is $v_{m,l_m} \rightarrow u_m$ or the length of P can be shortened by replacing its subpath $v_{m,l_m} \rightarrow v_{m,l_m-1} \rightarrow u_m$ by $v_{m,l_m} \rightarrow u_m$, contradicting the minimality of P. It follows that P lies entirely in $D_0 - \{v_{i,1} \rightarrow u_i, v_{i,l_i} \rightarrow u_i\}$. But this would imply that $v_{i,1} \rightarrow u_i$ is a dependent arc in D_0, which is impossible.

Therefore, the cycle C^* cannot contain one edge of C and one edge of G. Consequently, the arcs in E'' exhaust all dependent arcs in D_C.

Case 2. For all s, $l_s = 1$.

There exists j such that $v_{j,1}$ has no neighbor in G by our assumption. Without loss of generality, we may assume $j = k$. Orienting edges in $G_C - v_{k,1}$ according to the method used in Case 1, we obtain an acyclic orientation D_0 of $G_C - v_{k,1}$ such that $d(D_0) = d_{\min}(G)$. Since $v_{1,1}$ and $v_{k-1,1}$ are non-adjacent, we can find an acyclic orientation D_C of G_C such that $d(D_C) = d(D_0) = d_{\min}(G)$ by Lemma 9.

We conclude that $\sum_{i=1}^{k}\lfloor l_i/2\rfloor$ is an upper bound of $d_{\min}(G_C) - d_{\min}(G)$. □

Whether full orientability can be preserved when the type of the attached cycle is $(1, 1, \ldots, 1)$ and every vertex of the cycle is joined to a vertex outside the cycle remains undetermined.

5. Applications

A *Halin graph* H is a plane graph obtained by drawing a tree T in the plane, where T has no vertex of degree 2, and then drawing a cycle C through all leaves in the plane. We write $H = T \bigcup C$.

A *subdivision* of a given graph is obtained by replacing some edges by paths. Let G be a subdivision of a Halin graph $H = T \cup C$. Then we can rewrite G as $T' \cup C'$, where T' is a subdivision of T and C' is a subdivision of C. Let T_0 be the subtree of T' obtained by deleting all leaves of T'. We see that G can be obtained from T_0 by attaching the cycle C'.

Theorem 15. *Let $G = T' \cup C'$ be a subdivision of a Halin graph $H = T \cup C$. Then G is fully orientable. Moreover, if the type of C' is $(l_1, l_2, \ldots, l_k)$ when it is attached to T_0, then the following properties hold.*

(1) *If $G = K_4$, then $\alpha(G^T) = \theta(G^T) = 1$, $\pi_T(G) = \pi_E(G) = 2$, and $d_{\min}(G) = 3$.*

(2) *If $G = W_n \neq K_4$, then $\alpha(G^T) = \lfloor n/2 \rfloor$ and $\theta(G^T) = \pi_T(G) = \pi_E(G) = d_{\min}(G) = \lceil n/2 \rceil$.*

(3) *If G is not a wheel and $\sum_{i=1}^{k} l_i = 3$, then $\alpha(G^T) = \theta(G^T) = \pi_T(G) = \pi_E(G) = d_{\min}(G) = 1$.*

(4) *If G is not a wheel and $\sum_{i=1}^{k} l_i \geqslant 4$, then $\alpha(G^T) = \theta(G^T) = \pi_T(G) = \pi_E(G) = d_{\min}(G) = \sum_{i=1}^{k} \lfloor l_i/2 \rfloor$.*

Proof. (1) and (2) have been settled in Theorem 12.

(3) Since $\sum_{i=1}^{k} l_i = 3$, $C' = C$ is a triangle and G is a proper subdivision of K_4, hence 2-degenerate. It is also easy to check that G has at most two triangles and all triangles in G have a common edge. Hence, $\pi_T(G) = 1$. Since G is 2-degenerate, G is fully orientable and $\alpha(G^T) = \theta(G^T) = \pi_T(G) = \pi_E(G) = d_{\min}(G) = 1$.

(4) Every triangle in H contains exactly one edge of C. If there is one triangle in H any of whose edges has not been subdivided, then there exists some $l_i \geqslant 2$ in the type of C'. By Theorem 14, G is fully orientable and $\alpha(G^T) = \theta(G^T) = \pi_T(G) = \pi_E(G) = d_{\min}(G) = \sum_{i=1}^{k} \lfloor l_i/2 \rfloor$. Suppose that all triangles in H have at least one subdivided edge. Then G is a triangle-free planar graph. By a well-known theorem of Grötzsch [5], $\chi(G) \leqslant 3 < g(G)$. Hence G is fully orientable and $\alpha(G^T) = \theta(G^T) = \pi_T(G) = \pi_E(G) = d_{\min}(G) = 0 = \sum_{i=1}^{k} \lfloor l_i/2 \rfloor$. The last equality holds since now the type of C' is $(1, 1, \ldots, 1)$. □

Our last application deals with graphs whose maximum degree is at most three.

Lemma 16. *If G is a connected graph such that its minimum degree $\delta(G) \leqslant 2$ and its maximum degree $\Delta(G) \leqslant 3$, then G is 2-degenerate.*

Proof. Let u be a vertex of degree at most two. Let $S_i = \{v_1^i, v_2^i, \ldots, v_{k_i}^i\}$ denote the set of vertices that are at distance i to u. We claim that $u, v_1^1, \ldots, v_{k_1}^1, v_1^2, \ldots, v_{k_2}^2, \ldots$ is a characterizing ordering for G. Assuming that $i \geqslant 1$ and $1 \leqslant j \leqslant k_i$, the vertex v_j^i must have a neighbor in S_{i-1}. Since $\Delta(G) \leqslant 3$, the degree of v_j^i is at most two in the subgraph induced by $v_j^i, \ldots, v_{k_i}^i, v_1^{i+1}, \ldots$. Hence G is 2-degenerate. □

Theorem 17. *If $G \neq K_4$ is a connected graph with $\Delta(G) \leqslant 3$, then G is fully orientable and $\alpha(G^T) = \theta(G^T) = \pi_T(G) = \pi_E(G) = d_{\min}(G)$.*

Proof. If $\delta(G) \leqslant 2$, then G is 2-degenerate by Lemma 16, and hence we are done. Now assume that $\delta(G) = \Delta(G) = 3$, i.e., G is a cubic graph. If G is triangle-free, then G is 3-colorable by the well-known Brooks' Theorem [1]. Hence $\chi(G) < g(G)$. It follows that G is fully orientable and $\alpha(G^T) = \theta(G^T) = \pi_T(G) = \pi_E(G) = d_{\min}(G) = 0$. Suppose that G contains a triangle uvw.

Case 1. One of u, v or w, say u, is a cut vertex.

Let G_1 and G_2 be the two components of $G - u$. Let H_i be the subgraph induced by G_i and u. Since u has degree at most two in each H_i, each H_i is 2-degenerate by Lemma 16. Hence, each H_i is fully orientable and $\alpha(H_i^T) = \theta(H_i^T) = \pi_T(H_i) = \pi_E(H_i) = d_{\min}(H_i)$. By Lemma 5, G is fully orientable and $\alpha(G^T) = \theta(G^T) = \pi_T(G) = \pi_E(G) = d_{\min}(G)$.

Case 2. None of u, v and w is a cut vertex.

We claim that there is a cycle C of length at least four such that C contains an edge of uvw, but misses the remaining vertex of uvw.

Let u', v', and w' be the neighbors of u, v, and w outside the triangle uvw, respectively. Since $G \neq K_4$, the vertices u', v', and w' cannot all be identical. Suppose $u' = w' \neq v'$. If there exists a path P in

$G - \{u, w\}$ from u' to v', then $uPvu$ is the desired cycle. Otherwise, u' can only be connected to v' through u or w, i.e., v is a cut vertex, contradicting the case assumption. Suppose that u', v', and w' are all distinct. Since $G - u$ is connected, there exists a path from u' to v' in $G - u$. If all paths from u' to v' in $G - u$ pass through the edge ww', then v is again a cut vertex, contradicting the case assumption. So there must be a path Q from u' to v' in $G - u$ avoiding the edge ww'. Then $uQvu$ is the desired cycle.

Without loss of generality, we may assume that C has the shortest length among cycles containing the edge uv, yet missing the vertex w. It follows that no chord of C is incident with u or v. Actually, C is chordless since the existence of any chord of C would contradict the minimality of C. Since G is cubic, it can be obtained from $G - C$ by attaching the cycle C. Let the type of C be $(l_1, \ldots, l_k)$. Since C contains uv and misses w, we know $l_i \geqslant 2$ for some i. If $k = 1$, then G contains a wheel. Since C is of length at least four, the wheel has a vertex of degree at least four, which is impossible. Hence $k \geqslant 2$. Since G is cubic and connected, each component of $G - C$ has a vertex of degree at most two. By Lemma 16, $G - C$ is 2-degenerate. Therefore, $G - C$ is fully orientable and $\alpha((G - C)^T) = \theta((G - C)^T) = \pi_T(G - C) = \pi_E(G - C) = d_{\min}(G - C)$.

By Theorem 14, G is fully orientable and $\alpha(G^T) - \alpha((G - C)^T) = \theta(G^T) - \theta((G - C)^T) = \pi_T(G) - \pi_T(G - C) = \pi_E(G) - \pi_E(G - C) = d_{\min}(G) - d_{\min}(G - C)$. Hence $\alpha(G^T) = \theta(G^T) = \pi_T(G) = \pi_E(G) = d_{\min}(G)$. □

Corollary 18. *If G is a graph with $\Delta(G) \leqslant 3$ and t is the number of K_4 components of G, then G is fully orientable and $\alpha(G^T) + 2t = \theta(G^T) + 2t = \pi_T(G) + t = \pi_E(G) + t = d_{\min}(G)$.*

Proof. If $G_i = K_4$ is a component of G, $\alpha(G_i^T) = \theta(G_i^T) = 1$, $\pi_T(G_i) = \pi_E(G_i) = 2$ and $d_{\min}(G_i) = 3$ by Theorem 12. If $G_j \neq K_4$ is a component of G, $\alpha(G_j^T) = \theta(G_j^T) = \pi_T(G_j) = \pi_E(G_j) = d_{\min}(G_j)$ by Theorem 17. Our corollary follows from Lemma 4. □

We conclude this paper by making the following remark. The degree bound in Theorem 17 cannot be loosened since the graph $K_{3(2)}$ is a graph of maximum degree four and is not fully orientable.

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