Some properties for a class of interchange graphs

Jingjing Jin

Basic Sector, Fujian Communications Technology College, Fuzhou, 350007, Fujian, PR China

A R T I C L E   I N F O

Article history:
Received 15 July 2010
Received in revised form 7 June 2011
Accepted 20 June 2011
Available online 22 July 2011

Keywords:
Interchange graph
Distance
(0, 1)-matrix
Vector
Clique

A B S T R A C T

The Wiener number is the sum of distances between all pairs of vertices of a connected graph. In this paper, we give an explicit algebraic formula for the Wiener number of a class of interchange graphs. Moreover, distance-related properties and cliques of this class of interchange graphs are investigated.

© 2011 Elsevier B.V. All rights reserved.

1. Introduction

In this paper, all graphs are finite, undirected, and without loops and multiple edges. The vertex set and edge set of a graph \( G \) are denoted by \( V(G) \) and \( E(G) \), respectively. The distance \( d_{C}(u, v) \) between vertices \( u, v \in V(G) \) is the number of edges on a shortest path connecting vertices \( u \) and \( v \) in \( G \). The distance of a vertex \( v \in V(G) \), \( d_{C}(v) \), is the sum of all distances between \( v \) and all other vertices of \( G \), i.e.,

\[
d_{C}(v) = \sum_{u \in V(G)} d_{C}(u, v).
\]

The Wiener number, which was introduced to define the boiling point of alkane by Harold Wiener [9], is one of the most important topological indices of chemical graphs. It is denoted by \( W(G) \) and defined as the sum of distances between all pairs of vertices in \( G \):

\[
W(G) = \sum_{u, v \in V(G)} d_{C}(u, v) = \frac{1}{2} \sum_{v \in V(G)} d_{C}(v).
\]

Wiener number has been widely applied in communications, equipment orientation and cryptography, and so on (cf. Refs. [9,6,5,8,4,7,1,2], and the references therein). Since it deals with distance properties of graphs, computing the Wiener number of a graph is itself an interesting mathematical problem. The research dealing with Wiener number has attracted both chemists and mathematicians, and is still active [6,5,8,4,7,1].

In this paper, we consider interchange graphs. Let \( m \geq 2 \) and \( n \geq 2 \) be two positive integers, and let \( R = (r_1, \ldots, r_m) \) and \( S = (s_1, \ldots, s_n) \) be two nonnegative integral vectors with \( \sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j \). Denote by \( U(R, S) \), the set of all \((0, 1)\)-matrices
A = (a_{ij})_{m \times n} with row sum vector \( R \) and column sum vector \( S \), i.e.,
\[
    a_{ij} = 0 \quad \text{or} \quad 1 (i = 1, \ldots, m; j = 1, \ldots, n)
\]
\[
    \sum_{j=1}^{n} a_{ij} = r_i \quad (i = 1, \ldots, m)
\]
\[
    \sum_{i=1}^{m} a_{ij} = s_j \quad (j = 1, \ldots, n).
\]

Let \( A \in U(R, S) \). An interchange of \( A \) is a transformation which replaces the \( 2 \times 2 \) submatrix \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) with another matrix in \( U(R, S) \) with another matrix in \( U(R, S) \).

2. The Wiener number of \( G(R^*, S^*) \)

Let \( A \in V(G(R^*, S^*)) \). Note that \( R^* = (r_1, r_2) \) and \( S^* = (1, \ldots, 1) \). Then \( a_{ij} = 1 \) (or \( a_{ij} = 0 \)) implies \( a_{2j} = 0 \) (or \( a_{2j} = 1 \)), \( j = 1, 2, \ldots, n \), where \( n = r_1 + r_2 \). This means that \( A \) is uniquely determined by its first row (or second row). Clearly, the first row of \( A \) is a vector of dimension \( n \) consisting of \( r_1 \) ones and \( n - r_1 \) zeros. In the following, let \( H(r, n) \) denote the set of all \( n \) dimension vectors of ones and zeros, where \( r(\geq 1) \) denotes the number of ones in each \( n \) dimension vector of \( H(r, n) \). Thus, the number of zeros in each \( n \) dimension vector of \( H(r, n) \) is \( n - r \). Suppose \( \tilde{A} \in H(r, n) \). An interchange of \( \tilde{A} \) is a transformation which replaces the subvector \( (1, 0) \) of \( \tilde{A} \) with \( (0, 1) \) or vice versa. Let \( G(r, n) \) denote the simple undirected graph whose vertex set \( V(G(r, n)) \) is just \( H(r, n) \), and where two \( n \) dimension vectors are adjacent if and only if one of them can be obtained from the other by a single interchange. Comparing the definitions of \( G(R^*, S^*) \) and \( G(r, n) \), we have the following lemmas.

Lemma 2.1. \( G(R^*, S^*) \) is isomorphic to \( G(r, n) : G(R^*, S^*) \cong G(r, n) \), where \( R^* = (r, n - r) \), \( S^* = (1, \ldots, 1) \).

Lemma 2.2. \( G(r, n) \cong G(n - r, n) \).

By the above two lemmas, we need only to investigate the property of \( G(r, n) \) with \( r \leq n/2 \). Recall that a graph \( G \) is said to be \( k \)-regular if the degree of each vertex of \( G \) is \( k \). The following lemma is straightforward.

Lemma 2.3. \( G(r, n) \) is \( r(n - r) \)-regular. Moreover,
\[
    |V(G(r, n))| = \binom{n}{r}, \quad |E(G(r, n))| = \frac{1}{2} r(n - r) \binom{n}{r}.
\]

Lemma 2.4. \( G(r, n) \) \( (r \leq \frac{n}{2}) \) is a complete graph if and only if \( r = 1 \).

Proof. If \( r = 1 \), it is easy to check that \( G(1, n) \) is a complete graph. If \( 2 \leq r \leq \frac{n}{2} \), there are two different vertices \( v_i \) and \( v_j \) in \( G(r, n) \):
\[
    v_i = (1 \ 1 \ 0 \ 0 \ \cdots \ \cdots)
\]
\[
    v_j = (0 \ 0 \ 1 \ 1 \ \cdots \ \cdots).
\]

Evidently, \( v_i \) cannot be obtained from \( v_j \) by exactly one interchange. This means that \( v_i \) and \( v_j \) are not adjacent in \( G(r, n) \). Hence, if \( 2 \leq r \leq \frac{n}{2} \), \( G(r, n) \) is not a complete graph. Therefore, \( G(r, n) \) \( (r \leq \frac{n}{2}) \) is a complete graph if and only if \( r = 1 \). \( \square \)

Lemma 2.5. \( G(r, n) \) \( (r \leq \frac{n}{2}) \) is bipartite if and only if \( n = 2 \).

Proof. When \( n = 2 \), it is easy to see that \( r = 1 \) and \( G(1, 2) \cong K_2 \) is bipartite. Now suppose \( n \geq 3 \). Then there are three vertices in \( G(r, n) \):
$$v_1 = (1 \ 0 \ 0 \ \cdots \ \cdots)$$
$$v_j = (0 \ 1 \ 0 \ \cdots \ \cdots)$$
$$v_k = (0 \ 0 \ 1 \ \cdots \ \cdots)$$

such that they have the same element in the $t$th ($3 < t \leq n$) position. Hence, a 3-cycle $C = v_iv_jv_k$ exists in $G(r, n)$. Therefore, $G(r, n)$ is not bipartite if $n \geq 3$. The proof is thus completed. \hfill \Box

**Lemma 2.6.** If $r_1 \leq r_2$ and $n_1 - r_1 \leq n_2 - r_2$, $1 \leq r_1 \leq \frac{n}{2}$ ($i = 1, 2$), then there is a subgraph of $G(r_2, n_2)$ which is isomorphic to $G(r_1, n_1)$.

**Proof.** For each vertex $v$ (a vector of dimension $n_1$) of $G(r_1, n_1)$, there is a corresponding vertex $v'$ (a vector of dimension $n_2$) of $G(r_2, n_2)$ satisfying: $v$ and $v'$ have the same element for $i = 1, 2, \ldots, n_1$, the $j$th element of $v'$ is 1 for $j = n_1 + 1, n_1 + 2, \ldots, n_1 + r_2 - r_1$ and is 0 for $j = n_1 + r_2 - r_1 + 1, j = n_1 + r_2 - r_1 + 2, \ldots, n_2$. Denote by $\tilde{G}(r_2, n_2)$ the subgraph of $G(r_2, n_2)$ induced by those $v'$'s. One can check that $\tilde{G}(r_2, n_2) \cong G(r_1, n_1)$. \hfill \Box

**Remark.** The converse of Lemma 2.6 is not valid. For instance, $G(1, 6)$ a complete graph with 6 vertices, while $G(2, 4)$ is a non-complete graph with 6 vertices. So $G(2, 4)$ is isomorphic to a subgraph of $G(1, 6)$. But $2 \nmid 1$.

In the following, we make the convention that $\binom{n}{m} = 0$, when $m > n$.

**Lemma 2.7.** Let $r_1, r_2$ be two positive integers. Then

$$\sum_{i+j \leq \text{min}(r_1, r_2)} i \binom{r_1}{i} \binom{r_2}{j} = r_1 \binom{r_1 + r_2 - 1}{t - 1} - r_1 \binom{r_1 - 1}{t - 1}, \quad t = 2, \ldots, r_1 + r_2.$$

**Proof.** Let $y = f(x) = \{r_1x(1 + x)^{r_1 - 1}\}(1 + x)^{r_2 - 1} - 1$.

On the one hand,

$$y = \sum_{i=1}^{r_1} i \binom{r_1}{i} x^i \left\{ \sum_{j=1}^{r_2} \binom{r_2}{j} x^j \right\} = \sum_{i+j \leq \text{min}(r_1, r_2)} i \binom{r_1}{i} \binom{r_2}{j} x^i.$$

On the other hand,

$$y = r_1x(1 + x)^{1+r_2-1} - r_1x(1 + x)^{r_1-1} = \sum_{i=1}^{r_1+r_2} r_1 \binom{r_1 + r_2 - 1}{t - 1} x^i - \sum_{i=1}^{r_1} r_1 \binom{r_1 - 1}{t - 1} x^i.$$

Bear in mind our convention that $\binom{n}{m} = 0$ for $i > n$. The lemma follows by comparing the coefficients of $x^i$ in the above two expressions of $y$. \hfill \Box

**Lemma 2.8.** Let $r_1, r_2$ be two positive integers. Then

$$\sum_{i=1}^{\text{min}(r_1, r_2)} i \binom{r_1}{i} \binom{r_2}{i} = r_1 \binom{r_1 + r_2 - 1}{r_2 - 1}.$$

**Proof.** First we assume $r_1 < r_2$.

$$i \binom{r_1}{i} \binom{r_2}{i} = \sum_{i=1}^{r_1} i \binom{r_1}{i} \binom{r_2}{r_2 - i} = \sum_{i+j \leq \text{min}(r_1, r_2)} i \binom{r_1}{i} \binom{r_2}{j}.$$

Lemma 2.7

$$r_1 \binom{r_1 + r_2 - 1}{r_2 - 1} - r_1 \binom{r_1 - 1}{r_2 - 1} = r_1 \binom{r_1 + r_2 - 1}{r_2 - 1}.$$
Now we assume \( r_1 = r_2 \),
\[
\begin{align*}
\text{min}_{i=1}^{\min(|r_1|, |r_2|)} \binom{r_1}{i} \binom{r_2}{i} &= \sum_{1 \leq i \leq r_1} \binom{r_1}{i} \binom{r_1}{r_1 - i} \\
&= \sum_{1 \leq i \leq r_1} \binom{r_1}{i} \binom{r_1}{j} \\
&= \sum_{1 \leq i \leq r_1} \binom{r_1}{i} \binom{r_1}{j} + r_1 \binom{r_1}{r_1} \binom{r_1}{0} \\
\text{Lemma 2.8} & \quad r_1 \left( \frac{2r_1 - 1}{r_1 - 1} \right) = r_1 \left( \frac{r_1 + r_2 - 1}{r_2 - 1} \right).
\end{align*}
\]

By the symmetry of \( r_1 \) and \( r_2 \), we obtain immediately: when \( r_2 < r_1 \)
\[
\begin{align*}
\text{min}_{i=1}^{\min(|r_1|, |r_2|)} \binom{r_1}{i} \binom{r_2}{i} &= r_2 \left( \frac{r_1 + r_2 - 1}{r_1 - 1} \right).
\end{align*}
\]

One can check that \( r_1 \left( \frac{r_1 + r_2 - 1}{r_1 - 1} \right) = r_2 \left( \frac{r_1 + r_2 - 1}{r_1 - 1} \right) \). The lemma is thus proved. \( \square \)

If \( r_1 \) is replaced by \( r \), and \( r_1 + r_2 \) is replaced by \( n \), then \( r_2 = n - r \). So we can restate Lemma 2.8 as follows.

**Lemma 2.8**. Let \( r, n \) be positive integers with \( r < n \). Then
\[
\begin{align*}
\text{min}_{i=1}^{\min(r, n-r)} \binom{r}{i} \binom{n-r}{i} &= r \left( \frac{n-1}{n-r-1} \right).
\end{align*}
\]

**Theorem 2.9.** The Wiener number of \( G(r, n) \) \((r \leq \frac{n}{2})\) is given by
\[
W(G(r, n)) = \frac{1}{2} r \left( \frac{n-1}{n-r-1} \right) \binom{n}{r}.
\]

**Proof.** It is not difficult to see that for each vertex \( v \) in \( G(r, n) \), there are \( \binom{r}{i} \binom{n-r}{i} \) vertices \( v' \) in \( G(r, n) \) satisfying \( d(v, v') = i \), \( i = 1, \ldots, \min(r, n-r) \).

Therefore,
\[
W(G(r, n)) = \frac{1}{2} \sum_{v \in V(G(r, n))} d(v) = \frac{1}{2} \binom{n}{r} d(v)
\]
\[
= \frac{1}{2} \binom{n}{r} \sum_{i=1}^{\min(r, n-r)} i \binom{r}{i} \binom{n-r}{i} \text{ Lemma 2.8} \quad \frac{1}{2} r \left( \frac{n-1}{n-r-1} \right) \binom{n}{r}. \quad \square
\]

3. Some distance-related properties of \( G(R^*, S^*) \)

Since \( G(R^*, S^*) \cong G(r, n) \), we need only to investigate the distance-related properties of \( G(r, n) \). Let \( v_1, v_2 \) be two vertices in \( G(r, n) \). Denote by \( \text{dn}(v_1, v_2) \) the number of positions where the vectors \( v_1 \) and \( v_2 \) possess different elements.

The following lemma is straightforward.

**Lemma 3.1.** Let \( v_1, v_2 \in V(G(r, n)) \), then the number \( \text{dn}(v_1, v_2) \) is even.

The relation between the distance \( d(v_1, v_2) \) and the number \( \text{dn}(v_1, v_2) \) is given by the following lemma.

**Lemma 3.2.** Let \( v_1, v_2 \in G(r, n) \). Then \( d(v_1, v_2) = \frac{1}{2} \text{dn}(v_1, v_2) \).

**Proof.** By the above lemma, we may assume \( \text{dn}(v_1, v_2) = 2k \). We prove by induction on the number \( k \). \( \text{dn}(v_1, v_2) = 2k \).

If \( k = 1 \), then it is evident that \( d(v_1, v_2) = \frac{1}{2} \text{dn}(v_1, v_2) = 1 \), which implies that the lemma holds.
Now assume that \(dn(v_1, v_2) = 2(k + 1)\). Denote by \(v'_1\) the vector which is obtained by a single interchange from \(v_1\) such that \(dn(v'_1, v_2) = 2k\). By the induction hypothesis, we have \(d(v'_1, v_2) = \frac{1}{2} \times 2k = k\). Then \(d(v_1, v_2) \leq d(v'_1, v_2) + 1 = k + 1\). On the other hand, \(d(v_1, v_2) > d(v'_1, v_2) = k\) since \(dn(v_1, v_2) > dn(v'_1, v_2)\). Hence, we have:

\[k < d(v_1, v_2) \leq k + 1.\]

Therefore,

\[d(v_1, v_2) = k + 1 = \frac{1}{2}dn(v_1, v_2).\]

The lemma is thus proved. \(\square\)

By Lemma 3.2, we obtain the diameter of \(G(r, n) \ (1 \leq r \leq \frac{n}{2})\) as stated in the following theorem.

**Theorem 3.3.** The diameter of \(G(r, n) \ (r \leq \frac{n}{2})\) is \(\frac{r}{2}\).

**Proof.** The diameter of \(G(r, n) \ (r \leq \frac{n}{2})\)

\[
= \max_{v_1,v_2 \in V(G(r,n))} d(v_1, v_2) = \max_{v_1,v_2 \in V(G(r,n))} \frac{1}{2}dn(v_1, v_2) = \frac{1}{2} \min(r, n-r) = \frac{r}{2}.
\]

Recall that two paths \(P_1\) and \(P_2\) connecting \(v_i\) and \(v_j\) are said to be internally disjoint if \(P_1\) and \(P_2\) have no vertex in common except \(v_i\) and \(v_j\). \(\square\)

**Theorem 3.4.** For any two vertices of \(G(r, n)\) with distance \(k\), there are \(k^2\) internally disjoint paths connecting them.

**Proof.** Let \(v_i\) and \(v_j\) be two vertices of \(G(r, n)\) such that \(d(v_i, v_j) = k\). By Lemma 3.2, we have \(dn(v_i, v_j) = 2k\). Without loss of generality, we may assume that

\[
v_i = (1, \ldots, 1, 0, \ldots, 0, *, \ldots, *)
\]

\[
v_j = (0, \ldots, 0, 1, \ldots, 1, *, \ldots, *)
\]

where \(v_i\) and \(v_j\) have the same element in each of the positions with a "*". Now we find \(k^2\) paths from \(v_i\) to \(v_j\) in \(G(r, n)\), and divide them into \(k\) subsets such that each one contains \(k\) paths. Let the \(bp\)th internal vertex in the \(th\) path of the \(th\) subset be denoted by \(C_{bp}^h\) for \(1 \leq p \leq k - 1, \ 1 \leq t \leq k, \ 1 \leq h \leq k\). Then the first \(k\) elements of \(C_{bp}^h\) are as follows:

\[
\begin{array}{ccc}
1, \ldots, 1, & 0, \ldots, 0, & 1, \ldots, 1 \\
\ h-1 & \ p & \ k-h-p+1 \\
\end{array}
\]

or

\[
\begin{array}{ccc}
0, \ldots, 0, & 1, \ldots, 1, & 0, \ldots, 0 \\
\ k-p+h-1 & \ k-p & \ k-h+1 \\
\end{array}
\]

And the \((k + 1)\)th to \(2k\)th elements of \(C_{bp}^h\) are as follows:

\[
\begin{array}{ccc}
0, \ldots, 0, & 1, \ldots, 1, & 0, \ldots, 0 \\
\ t-1 & \ p & \ k-t-p+1 \\
\end{array}
\]

or

\[
\begin{array}{ccc}
1, \ldots, 1, & 0, \ldots, 0, & 1, \ldots, 1 \\
\ k-p & \ k-p & \ k-t+1 \\
\end{array}
\]

The above mentioned \(k^2\) paths contain \(k^2(k - 1)\) internal vertices. Let \(C_{bp}^h\) and \(C'_{bp'}^h\) be two of them. If they are the same vertices, then \(p = p'\), and one can deduce easily that \(t = t'\) and \(h = h'\). Hence we come to the conclusion that the above \(k^2\) paths connecting \(v_i\) and \(v_j\) are internally disjoint.

In the following, we give an example. Let \(v_i = (11110000)\) and \(v_j = (00001111)\) be two vertices in \(G(4, 8)\) satisfying \(d(v_i, v_j) = 4\). There are \(4^2\) internally disjoint paths connecting \(v_i\) and \(v_j\) as follows:

\((h = 1)\)

\[
\begin{array}{c}
t = 1 : (11110000) \rightarrow (01111000) \rightarrow (00111100) \rightarrow (00011110) \rightarrow (00001111) \\
t = 2 : (11110000) \rightarrow (01110100) \rightarrow (00110110) \rightarrow (00010111) \rightarrow (00001111) \\
t = 3 : (11110000) \rightarrow (01110010) \rightarrow (00110011) \rightarrow (00011011) \rightarrow (00001111) \\
t = 4 : (11110000) \rightarrow (01110001) \rightarrow (00111001) \rightarrow (00011101) \rightarrow (00001111)
\end{array}
\]
(h = 2)

\[
\begin{align*}
(t = 1 : (11110000) &\rightarrow (10111000) \rightarrow (10011100) \rightarrow (10011110) \rightarrow (00001111)) \\
(t = 2 : (11110000) &\rightarrow (10110100) \rightarrow (10010110) \rightarrow (10000111) \rightarrow (00001111)) \\
(t = 3 : (11110000) &\rightarrow (10110010) \rightarrow (10010011) \rightarrow (10001011) \rightarrow (00001111)) \\
(t = 4 : (11110000) &\rightarrow (10110001) \rightarrow (10011001) \rightarrow (10001101) \rightarrow (00001111))
\end{align*}
\]

(h = 3)

\[
\begin{align*}
(t = 1 : (11110000) &\rightarrow (11011000) \rightarrow (11001100) \rightarrow (01001110) \rightarrow (00001111)) \\
(t = 2 : (11110000) &\rightarrow (11010100) \rightarrow (11000110) \rightarrow (01000111) \rightarrow (00001111)) \\
(t = 3 : (11110000) &\rightarrow (11010010) \rightarrow (11000111) \rightarrow (01001111) \rightarrow (00001111)) \\
(t = 4 : (11110000) &\rightarrow (11010001) \rightarrow (11001011) \rightarrow (01010111) \rightarrow (00001111))
\end{align*}
\]

(h = 4)

\[
\begin{align*}
(t = 1 : (11110000) &\rightarrow (11101000) \rightarrow (01101100) \rightarrow (00101110) \rightarrow (00001111)) \\
(t = 2 : (11110000) &\rightarrow (11100100) \rightarrow (01100111) \rightarrow (00100111) \rightarrow (00001111)) \\
(t = 3 : (11110000) &\rightarrow (11100010) \rightarrow (01100011) \rightarrow (00101101) \rightarrow (00001111)) \\
(t = 4 : (11110000) &\rightarrow (11100001) \rightarrow (01101001) \rightarrow (00110101) \rightarrow (00001111)).
\end{align*}
\]

4. Some properties of cliques

In this section, we consider the properties of cliques in interchange graph \( G(r, n) \). Let \( S \) be a subset of the vertex set of \( G(r, n) \). We use \( G[S] \) to denote the spanning subgraph of \( G(r, n) \) induced by \( S \). Note that \( S \) is a clique of \( G \) if and only if \( G[S] \) is a complete graph.

**Theorem 4.1.** Let \( S \) with \( |S| = s \geq 2 \) be a subset of the vertex set of \( G(r, n) \) \((r \leq s/2)\). Then \( S \) is a clique of \( G(r, n) \) if and only if, the vertices of \( S \) can be arranged in the order \( v_1, v_2, \ldots, v_t \) such that there exist \( s \) positions \( m_1, m_2, \ldots, m_s \) and for each vertex \( v_i \) of \( S \), the \( m_j \)-position \((t = 1, 2, \ldots, s)\) has number 1 in, and the other position \( m_j, j \neq t, 1 \leq j \leq s \) has number 0 (or 1); moreover, the vertices of \( S \) have the same elements in all positions except the positions \( m_t \)th, \( m_{t+1} \)th, \( m_{t+2} \)th, \( m_{t+3} \)th.

**Proof.** Sufficiency.

Suppose that the vertices of \( S \) can be arranged in the order \( v_1, v_2, \ldots, v_t \) such that there exist \( s \) positions \( m_1, m_2, \ldots, m_s \), and for each vertex \( v_i \) of \( S \), the \( m_j \)-position \((t = 1, 2, \ldots, s)\) has number 1 in, and the other position \( m_j, j \neq t, 1 \leq j \leq s \) has number 0 (note that the case that for each vertex \( v_i \) of \( S \), the \( m_j \)-position \((t = 1, 2, \ldots, s)\) has number 0 in, and the other position \( m_j, j \neq t, 1 \leq j \leq s \) has number 1 can be dealt with in a similar way); moreover, the vertices of \( S \) have the same elements in all positions except the positions \( m_t \)th, \( m_{t+1} \)th, \( m_{t+2} \)th, \( m_{t+3} \)th. It is not difficult to see that the \( s \) vertices of \( S \) can be expressed as follows:

\[
\begin{align*}
&\begin{array}{cccccccc}
& v_1 &=& 1 & 0 & 0 & \cdots & 1 & 0 \\
& v_2 &=& 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
& v_t &=& 0 & \cdots & 0 & 1 & \cdots & 0 & \cdots \\
& v_s &=& 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\
\end{array}
\end{align*}
\]

Note that in each of the positions with a dot, the vertices of \( S \) have the same elements. We can check that \( v_i \) is adjacent to \( v_j \), \((i \neq j, 1 \leq i, j \leq s)\). Therefore, \( S \) is a clique of \( G(r, n) \).

Necessary.

By induction on \( s \). When \( s = 2, 3 \), it holds clearly. Assume that it holds for \( s = k \geq 3 \). Let \( T \) be a clique in \( G(r, n) \) with \( k+1 \) vertices. In \( T \), we take a subset \( T' \) with \( k \) vertices. Clearly, \( T' \) is also a clique of \( G(r, n) \). By induction hypothesis, the vertices of \( T' \) can be arranged in the order \( v_1, v_2, \ldots, v_k \) such that there exist \( k \) positions \( m_1, m_2, \ldots, m_k \), and for each vertex \( v_i \) of \( T' \), the \( m_j \)-position \((t = 1, 2, \ldots, k)\) has number 1, and the other position \( m_j, j \neq t, 1 \leq j \leq k \) has number 0; moreover, the vertices of \( T' \) have the same elements in all positions except the positions \( m_1 \)th, \( m_2 \)th, \( \ldots, m_k \)th. (As above, the case that for each vertex \( v_i \) of \( T' \), the \( m_j \)-position \((t = 1, 2, \ldots, k)\) has number 0 in, and the other position \( m_j, j \neq t, 1 \leq j \leq k \) has number 1 can be dealt with in a similar way. We omit the details.)

Now we consider the last vertex of \( T \), denoted by \( v_{k+1} \). First, we assume that for \( v_{k+1} \), there exist at least one 1 and one 0 simultaneously among the positions \( m_1, m_2, \ldots, m_k \). That is, there exist \( p \) and \( q \), \( p \neq q, 1 \leq p, q \leq k \) such that \( v_{k+1} \) presents 1 at position \( m_p \) and presents 0 at position \( m_q \). By the induction hypothesis, we know that vertex \( v_{k+1} \) presents 0 at
The number of vertices contained in the maximum clique of $G$, for an arbitrary edge $e$, for an arbitrary vertex $v$, and for an arbitrary 0, 1 or $v_i$ at position $m_i$. Thus $v_{k+1} = v_p$, a contradiction. The vertex $v_{k+1}$ presents the same 0 or the same 1 at positions $m_1, m_2, \ldots, m_k$. For $k \geq 3$, if $v_{k+1}$ presents the same 1 at positions $m_1, m_2, \ldots, m_k$, it is not difficult to check that $v_{k+1}$ cannot be adjacent to $v_1, v_2, \ldots, v_k$, a contradiction. Therefore, $v_{k+1}$ presents 0 at each of the positions $m_1, m_2, \ldots, m_k$. Since $v_{k+1}$ and $v_1$ are adjacent, $v_{k+1}$ presents at least a 1 at positions except $m_1, m_2, \ldots, m_k$. Without loss of generality, we assume that the $v_{k+1}$ presents 1 at position $m_{k+1}$. Hence, $v_1$ presents 0 at $m_{k+1}$. By the induction hypothesis, $v_1, v_2, \ldots, v_{k+1}$ present 0 at position $m_{k+1}$.

In the above, we have proved that the $k + 1$ vertices of $T$ can be expressed as:

$$
\begin{align*}
v_1 &= (\cdots 1 \cdots m_p \cdots m_q \cdots m_k \cdots m_{k+1} \\
v_2 &= (\cdots 0 \cdots 0 \cdots 0 \cdots 0 \cdots 0 \cdots 0 \cdots 0 \cdots 0) \\
v_q &= (\cdots 0 \cdots 0 \cdots 0 \cdots 0 \cdots 1 \cdots 0 \cdots 0 \cdots 0) \\
v_k &= (\cdots 0 \cdots 0 \cdots 0 \cdots 0 \cdots 0 \cdots 0 \cdots 0 \cdots 0) \\
v_{k+1} &= (\cdots 0 \cdots 0 \cdots 0 \cdots 0 \cdots 0 \cdots 0 \cdots 0 \cdots 0)
\end{align*}
$$

We can see that the vertices of clique $T$ have the same elements at any positions except $k + 1$ positions: $m_1, m_2, \ldots, m_k, m_{k+1}$. Otherwise, they cannot be obtained from the other by a single interchange. □

We obtain the size of the maximum clique of $G(r, n)$ ($r \leq \frac{n}{2}$) from Theorem 3.1.

**Corollary 4.2.** The number of vertices contained in the maximum clique of $G(r, n)$ ($r \leq \frac{n}{2}$) is $n - r + 1$.

In the following, we call a clique $k$-clique if it contains $k$ vertices. We know from Theorem 3.1 that for a $k$-clique, there are $n - k$ positions in which each of the vertices in the $k$-clique presents the same 1 or 0 in the $n - k$ locations. At the same time, except the $k$ positions, there are $k$ positions among which each vertex of the $k$-clique has exactly one 1 and $k - 1$ 0's, or exactly one 0 and $k - 1$ 1's. We call the former $k$-clique as an 1-type $k$-clique, the latter $k$-clique as an 0-type $k$-clique.

**Theorem 4.3.** For an arbitrary vertex $v$ of an interchange graph $G(r, n)$, the number of $k(\geq 3)$-clique containing $v$ is

$$
\binom{r}{1} \binom{n - r}{k - 1} + \binom{r}{1} \binom{n - r}{k - 1},
$$

including $\binom{r}{1} \binom{n - r}{k - 1}$ 1-type $k$-cliques and $\binom{r}{1} \binom{n - r}{k - 1}$ 0-type $k$-cliques.

**Proof.** By Theorem 4.1, it is not difficult to see that for an arbitrary vertex $v$ of $G(r, n)$, if $v$ possesses $k(\geq 3)$ positions in which exactly one 1 and $k - 1$ 0's appear, then vertex $v$ must be contained in a 1-type $k$-clique; similarly, if $v$ possesses $k(\geq 3)$ positions in which exactly one 0 and $k - 1$ 1's appear, then vertex $v$ must be contained in a 0-type $k$-clique. One can check that there are $\binom{r}{1} \binom{n - r}{k - 1}$ ways for vertex $v$ to find $k(\geq 3)$ positions in which exactly one 1 and $k - 1$ 0's appear, and $\binom{r}{1} \binom{n - r}{k - 1}$ ways for vertex $v$ to find $k(\geq 3)$ positions in which exactly one 0 and $k - 1$ 1's appear. So, for an arbitrary vertex $v$ of $G(r, n)$, the number of $k(\geq 3)$-cliques containing $v$ is $\binom{r}{1} \binom{n - r}{k - 1} + \binom{r}{1} \binom{n - r}{k - 1}$, including $\binom{r}{1} \binom{n - r}{k - 1}$ 1-type $k$-cliques, $\binom{r}{1} \binom{n - r}{k - 1}$ 0-type $k$-cliques. □

**Theorem 4.4.** For an arbitrary edge $e = v_1v_2$ of an interchange graph $G(r, n)$, the number of $k(\geq 3)$-clique containing $e$ is

$$
\binom{n - r - 1}{k - 2} + \binom{r - 1}{k - 2},
$$

including $\binom{n - r - 1}{k - 2}$ 1-type $k$-cliques and $\binom{r - 1}{k - 2}$ 0-type $k$-cliques.

**Proof.** For the two end vertices $v_1$ and $v_2$ of $e$, there exist exactly two positions, say $m_i$ and $m_j$, such that $v_1$ presents 1, 0, while $v_2$ presents 0, 1 or vice versa. To ensure that the cliques containing vertex $v_1$ simultaneously contains $v_2$, one can check that there are $\binom{n - r - 1}{k - 2}$ different ways for vertex $v_1$ to find $k(\geq 3)$ positions in which exactly one 1 and $k - 1$ 0's appear, at the same time, there are $\binom{r - 1}{k - 2}$ different ways for vertex $v_1$ to find $k(\geq 3)$ positions in which exactly one 0 and $k - 1$ 1's appear. So, for an arbitrary edge $v_1v_2$ of $G(r, n)$, the number of $k(\geq 3)$-cliques containing $v_1v_2$ is $\binom{n - r - 1}{k - 2} + \binom{r - 1}{k - 2}$, including $\binom{n - r - 1}{k - 2}$ 1-type $k$-cliques, $\binom{r - 1}{k - 2}$ 0-type $k$-cliques. □
By Corollary 4.2, Theorems 4.3 and 4.4, we have the following result.

**Corollary 4.5.** For interchange graph \( G(r, n) \), if \( r < \frac{n}{2} \), then each vertex \( v \) of \( G(r, n) \) is contained in exactly \( r \) maximum cliques, and each edge \( e \) is contained in exactly one maximum clique. If \( r = \frac{n}{2} \), each vertex \( v \) of \( G(r, n) \) is contained in \( 2r \) maximum cliques and each edge is contained in exactly 2 maximum cliques.

**Theorem 4.6.** The number of \( k \)-cliques of \( G(r, n) \) \( (r \leq \frac{n}{2}) \) is \( N_k = \binom{n}{k} \left( \binom{n-k}{r-1} + \binom{n-k}{r+k+1} \right) \) \((3 \leq k \leq (n - r + 1))\), including \( \binom{n}{k} \binom{n-k}{r-1} \) 1-type \( k \)-cliques and \( \binom{n}{k} \binom{n-k}{r+k+1} \) 0-type \( k \)-cliques.

**Proof.** By Theorem 4.1, it is immediately obtained. \( \square \)

By Corollary 4.2, the maximum cliques of \( G(r, n) \) \( (r \leq \frac{n}{2}) \) are \((n-r+1)\)-clique. Together with Theorem 4.6, we have:

**Corollary 4.7.** If \( r < \frac{n}{2} \), there are \( \binom{n-r}{2r} \) maximum cliques of 1-type in \( G(r, n) \); if \( r = \frac{n}{2} \), there are \( 2 \binom{n-r}{2r} \) maximum cliques in \( G(r, n) \), including \( \binom{n-r}{2r} \) ones of 1-type and \( \binom{n-r}{2r} \) ones of 0-type. \( \square \)

**Lemma 4.8.** There is no common edge for any two different maximum cliques of the same type.

**Proof.** By contradiction. By symmetry, it suffices to consider the case that the maximum cliques are of 1-type. Clearly, there are \( n - r + 1 \) positions among which each vertex of a 1-type maximum clique presents one 1 and \( n - r \) 0’s. We call those \( n - r + 1 \) positions as the non-consistency positions for the maximum clique. Assume that there is a common edge \( v_1v_2 \) between 1-type maximum cliques \( C_1 \) and \( C_2 \). Let \( i_0, i_1, i_2, \ldots, i_{n-r} \) be the non-consistency positions for \( C_1 \). Without loss of generality, we assume that there is one 1 on position \( i_0 \) in \( v_1 \), and there is one 0 on location \( i_1 \) in \( v_2 \). Since \( C_1 \) and \( C_2 \) are different maximum cliques, which is a non-consistency position of \( C_2 \), but is not a non-consistency position of \( C_1 \). Note that maximum clique \( C_2 \) which contains \( v_1, v_2 \) is of 1-type. Thus the positions in which 0’s exist in \( v_1 \) or \( v_2 \) are \( i_0, i_1, i_2, \ldots, i_{n-r} \), and these positions are also the non-consistency positions of \( C_2 \). Hence \( i_0, i_1, i_2, \ldots, i_{n-r}, i_{n-r+1} \) are all non-consistency positions of \( C_2 \), which contradicts the fact that \( C_2 \) has exactly \( n - r + 1 \) non-consistency locations. Therefore, the assumption that \( v_1v_2 \) is a common edge between \( C_1 \) and \( C_2 \) is false. The lemma is thus proved. \( \square \)

**Theorem 4.9.** For \( r \leq \frac{n}{2} \), \( n \geq 3 \), \( t = \binom{n-r}{2r} \), the edge set \( E(G(r, n)) \) of \( G(r, n) \) has \( j \) different partitions: \( E_{j1}, E_{j2}, \ldots, E_{jt} \), where \( j = 1 \) if \( r < \frac{n}{2} \), \( j = 2 \) if \( r = \frac{n}{2} \), such that:

1. \( E_{j1} \cup E_{j2} \cup \cdots \cup E_{jt} = E(G(r, n)) \).
2. \( E_{ip} \cap E_{iq} = \emptyset \), \( p \neq q \), \( 1 \leq p, q \leq t \).
3. The edge-induced subgraph \( G[E_{ip}] \) of \( E_{ip} \) is one of the \( 1 \)-type maximum cliques of \( G(r, n) \).

**Proof.** If \( r < \frac{n}{2} \), \( j = 1 \). By Corollary 4.7, \( G(r, n) \) contains exactly \( \binom{n-r}{2r} \) maximum cliques, all of which are of 1-type, and each edge of \( G(r, n) \) is contained in exactly one maximum clique of \( G(r, n) \). So, the edge set \( E(G(r, n)) \) of \( G(r, n) \) has one partition: \( E_1, E_2, \ldots, E_t \), such that

1. \( E_1 \cup E_2 \cup \cdots \cup E_t = E(G(r, n)) \).
2. \( E_p \cap E_q = \emptyset \), \( p \neq q \), \( 1 \leq p, q \leq t \).
3. The induced subgraph \( G[E_1] \) of \( E_1 \) is an 1-type maximum clique of \( G(r, n) \).

If \( r = \frac{n}{2} \), \( j = 2 \). \( G(r, n) \) has \( \binom{n-r}{2r} \) 1-type maximum cliques and \( \binom{n-r}{2r} \) 0-type maximum cliques. Note that the number of edges of \( \binom{n-r}{2r} \) maximum cliques is \( \binom{n-r+1}{2} \), which equals the edge number of \( G(r, n) \) \( \frac{1}{2} \left( \binom{n}{r} + \binom{n}{r+1} \right) \left( \binom{r}{2} \right) \). By Lemma 4.8, there is no common edge between any two maximum cliques of the same type. So, all of the 1-type maximum cliques and all of the 0-type maximum cliques here are precisely the two kinds of partition into maximum cliques for \( G(r, n) \). \( \square \)

By Corollary 4.5 and Lemma 4.8, we have the following theorem.

**Theorem 4.10.** For any 1-type (or 0-type) maximum clique \( C \), we have: in the case of \( r \leq \frac{n}{2} \), the number of 1-type (0-type) maximum cliques which have common vertices with \( C \) is \( (r-1)(n-r+1) \) \((C\ is\ not\ taken\ account)\); in the case of \( r = \frac{n}{2} \), the number of 1-type (0-type) maximum cliques which have common edges with \( C \) is \( \binom{r+1}{2} \).

**References**

