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Seven-modular lattices and a septic base Jacobi identity

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Abstract

A quadratic Jacobi identity to the septic base is introduced and proved by means of modular lattices and codes over rings. As an application the theta series of all the 6-dimensional 7-modular lattices with an Hermitian structure over $\mathbf{Q}(\sqrt{-7})$ are derived.

1. Introduction

For $|q|<1$, let

$$\vartheta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2},$$

$$\vartheta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

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and

$$\vartheta_4(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}.$$

One of Jacobi's famous identities states that

$$\vartheta_3^4(q) = \vartheta_2^4(q) + \vartheta_4^4(q). \tag{1.1}$$

This identity has been generalized in various ways, in particular in the context of Vertex Operator Algebras [10].

Around 1991, Borwein and Borwein [4] discovered a cubic analogue of (1.1), namely,

$$a^3(q) = b^3(q) + c^3(q),$$

where $a(q), b(q), c(q)$ are the bi-dimensional theta series

$$a(q) = \sum_{m,n \in \mathbf{Z}} q^{m^2+mn+n^2},$$

$$b(q) = \sum_{m,n \in \mathbf{Z}} \omega^{m-n} q^{m^2+mn+n^2}, \quad \omega = e^{2\pi i/3}$$

and

$$c(q) = \sum_{m,n \in \mathbf{Z}} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}.$$

The aim of this note is to introduce another bi-dimensional generalization of Jacobi's identity (1.1). Our main result is

Theorem 1.1. *Let*

$$A(q) = \sum_{m,n \in \mathbf{Z}} q^{2(m^2+mn+2n^2)},$$

$$B(q) = \sum_{m,n \in \mathbf{Z}} (-1)^{m-n} q^{m^2+mn+2n^2}$$

and

$$C(q) = \sum_{m,n \in \mathbf{Z}} q^{2((m+1/2)^2+(m+1/2)n+2n^2)}.$$

Then

$$A^2(q) = B^2(q) + C^2(q).$$

The proof techniques as in [19] are a combination of codes and lattices techniques. In particular, we will follow Bachoc's approach [1] for the construction of 7-modular lattices from codes over the ring $\mathbf{F}_2 \times \mathbf{F}_2$.

Like the Borweins' cubic identity, our base 7 identity was unknown to Ramanujan [2]. However, when calculating the theta series of a famous 7-modular lattice (related to the polarization of the Klein curve) we encounter a formula akin to the ones in [6,17]. It appears that our identity belongs to the septic analogue of Ramanujan's theories of elliptic functions to the alternative bases [3].

2. Notations and definitions

2.1. Codes

Let R_4 denote the ring with 4 elements $\mathbf{F}_2 + v\mathbf{F}_2$ where $v^2 = v$. This ring contains two maximal ideals v and $(v+1)$. Observe that both of $R_4/(v)$ and $R_4/(v+1)$ are $\mathbf{F}_2$. The Chinese Remainder Theorem tells us that

$$R_4 = (v) \oplus (v+1),$$

so that ring $R_4 \cong \mathbf{F}_2 \times \mathbf{F}_2$. An explicit isomorphism is

$$va + (v+1)b \mapsto (a, b), \quad a, b \in \mathbf{F}_2.$$

A *code* over R_4 is an R_4-submodule of R_4^n.

Let $K := \mathbf{Q}(\sqrt{-7})$ be the quadratic number field with ring of integers $\mathcal{O} = \mathbf{Z}[\alpha]$, with $\alpha^2 + \alpha + 2 = 0$. Then we can regard R_4 as $\mathcal{O}/(2)$ when v is the image of α under reduction modulo 2. Denote by a bar the conjugation which fixes $\mathbf{F}_2$ and swaps v and $1+v$. The natural scalar product induced by the hermitian scalar product of $\mathbf{C}^n$ is then given by

$$\sum_i x_i \overline{y_i}.$$

The Bachoc weight as defined in [1, Definition 3.1] is of course $w_B(0) = 0$. But more surprisingly $w_B(v) = w_B(1+v) = 2$ and $w_B(1) = 1$. Define the Bachoc composition of x say, $n_i(x)$, $i = 0, 1, 2$, as the number of entries in x of Bachoc weight i. In terms of Bachoc composition, we have

$$w_B = n_1 + 2n_2.$$

The symmetrized weight enumerator (swe) is then the polynomial of three variables X, Y, Z, defined by

$$swe_C(X, Y, Z) = \sum_{x \in C} X^{n_0(x)} Y^{n_1(x)} Z^{n_2(x)}.$$

2.2. Lattices

An n-dimensional *lattice* L is a discrete subgroup of $\mathbf{R}^n$. Its theta series is

$$\theta_L(q) = \sum_{x \in L} q^{x \cdot x},$$

where $x \cdot x = \sum_{i=1}^{n} x_i^2$, and $q = \exp(\pi i z)$, $z \in \mathscr{H}$, where

$$\mathscr{H} := \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}.$$

The lattice L is called *integral* if it is contained in its dual L^* defined as

$$L^* := \{y \in \mathbf{R}^n, \forall x \in L, x \cdot y \in \mathbf{Z}\}.$$

A topic of current interest in research is the study of *modular* lattices. The salient property of these lattices introduced by Quebbemann [16] is that their theta series is a modular form for a suitable subgroup of the modular group. Specifically, an integral lattice L is said to be ℓ-modular [1,13] for some prime ℓ if L is isometric to $\sqrt{\ell} L^*$.

Theorem 2.1 (Quebbemann [16, Theorem 7]). *Let*

$$\Delta_6(q) := q^2 \prod_{n=1}^{\infty} (1 - q^{2n})^3 (1 - q^{14n})^3.$$

The theta series of an even 7-modular lattice is an isobaric polynomial in the two variables $A(\sqrt{q})$ and Δ_6.

The special cusp form Δ_6 is called a CM-form in [15, (3.b)] where an expansion as a twisted theta series attached to the quadratic form $[1, 0, 7]$ is given.

3. Preliminaries

Define the construction $A_K(C)$ as the preimage in $\mathcal{O}^n$ of $C \subseteq R_4^n$ under reduction modulo 2. Specifically,

$$A_K(C) := \{y \in \mathcal{O}^n \mid y \pmod 2 \in C\}.$$

Theorem 3.1. *If $C \subseteq R_4^n$ is a self-dual code then the lattice $A_K(C)/\sqrt{2}$ is even 7-modular.*

Proof. The assertion follows by Bachoc [1, Proposition 3.6] and can alternatively be derived directly by checking that $\mathcal{O}$ is 7-modular for the bilinear form

$$(x, y) \mapsto Tr_K(x\bar{y}).$$

To compute the theta series of a lattice $A_K(C)$ as a function of swe_C we need to define some auxiliary theta series. Following [1], we introduce

$$\theta_0(q) = \sum_{x \in 2\mathcal{O}} q^{x\bar{x}},$$

$$\theta_1(q) = \sum_{x \in 1+2\mathcal{O}} q^{x\bar{x}}$$

and

$$\theta_2(q) = \sum_{x \in \alpha+2\mathcal{O}} q^{x\bar{x}}.$$

We quote [1, Proposition 4.2] in the case at hand. □

Theorem 3.2 (Bachoc). *If $C \subseteq R_4^n$ is a code of length n, then the theta series of the lattice $A_K(C)$ satisfies*

$$\theta_{A_K(C)} = swe_C(\theta_0(q), \theta_1(q), \theta_2(q)).$$

4. Proof of Theorem 1.1

We first express $A(q)$, $B(q)$ and $C(q)$ in term of Jacobi's functions $\vartheta_i(q)$, $i = 1, 2, 3$.

Lemma 4.1. *Let $A(q)$, $B(q)$ and $C(q)$ be given as in Theorem 1.1. Then*

$$A(q) = \vartheta_3(q^2)\vartheta_3(q^{14}) + \vartheta_2(q^2)\vartheta_2(q^{14}), \tag{4.1}$$

$$B(q) = \vartheta_4(q)\vartheta_4(q^7) \tag{4.2}$$

and

$$C(q) = \vartheta_2(q^2)\vartheta_3(q^{14}) + \vartheta_3(q^2)\vartheta_2(q^{14}). \tag{4.3}$$

Proof. Identity (4.1) can be found, for example, in [5, p. 1738]. For the proofs of the subsequent identities, we will need the following simple identity, namely, for any odd integer n,

$$\sum_{m=-\infty}^{\infty} (-1)^m q^{(m+\frac{n}{2})^2} = 0. \tag{4.4}$$

This identity follows immediately from [20, p. 464].

Next, note that

$$\begin{aligned} B(q) &= \sum_{m,n=-\infty}^{\infty} (-1)^{m-n} q^{m^2+mn+2n^2} \\ &= \sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{(m+\frac{n}{2})^2+\frac{D}{4}n^2}, \end{aligned}$$

with $D = 7$, and that

$$\begin{aligned} B(q) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{Dn^2/4} \sum_{m=-\infty}^{\infty} (-1)^m q^{(m+\frac{n}{2})^2} \\ &= \sum_{n\in 2\mathbf{Z}} (-1)^n q^{Dn^2/4} \sum_{m=-\infty}^{\infty} (-1)^m q^{(m+\frac{n}{2})^2} \\ &\quad + \sum_{n\in 2\mathbf{Z}+1} (-1)^n q^{Dn^2/4} \sum_{m=-\infty}^{\infty} (-1)^m q^{(m+\frac{n}{2})^2} \\ &= \sum_{n\in 2\mathbf{Z}} (-1)^n q^{Dn^2/4} \sum_{m=-\infty}^{\infty} (-1)^m q^{(m+\frac{n}{2})^2} = \vartheta_4(q^D)\vartheta_4(q) \end{aligned}$$

by (4.4).

Finally, rewrite $C(q)$ as

$$\begin{aligned} C(q) &= \sum_{m,n=-\infty}^{\infty} q^{2((m+\frac{n}{2})^2+\frac{D}{4}n^2)} \\ &= \sum_{m\in\mathbf{Z},n\in 2\mathbf{Z}} q^{2((m+\frac{n}{2})^2+\frac{D}{4}n^2)} + \sum_{m\in\mathbf{Z},n\in 2\mathbf{Z}+1} q^{2((m+\frac{n}{2})^2+\frac{D}{4}n^2)} \\ &= \vartheta_2(q^2)\vartheta_3(q^{14}) + \vartheta_3(q^2)\vartheta_2(q^{14}). \qquad \square \end{aligned}$$

We proceed to express $\theta_0(q), \theta_1(q)$, and $\theta_2(q)$ as functions of $A(q)$, $B(q)$, and $C(q)$.

Proposition 4.2. *For all $z \in \mathscr{H}$ we have*

$$\theta_0(q) = A(q^2)$$

and

$$\theta_1(q) = C(q^2).$$

Proof. If we set $x = m - n\alpha$, then the norm form becomes

$$x\bar{x} = m^2 + mn + 2n^2 =: NF(m,n). \tag{4.5}$$

If we set $x = 1 + 2(m - n\alpha)$, then the norm form becomes

$$x\bar{x} = 4[(m+1/2)^2 + (m+1/2)n + 2n^2]. \qquad \square$$

In view of these expressions it is natural to look for an expression for θ_2 involving q^2. To that end, we shall require the following *duplication formulas*:

Lemma 4.3. *For all $z \in \mathscr{H}$ we have*

$$A(q) = 2A(q^2) - B(q^2)$$

and

$$B(q) = A(q^2) - C(q^2).$$

Proof. From (4.1) and (4.3), we find, after some simplification, that

$$\begin{aligned} A(q^2) - C(q^2) &= (\vartheta_3(q^4) - \vartheta_2(q^4))(\vartheta_3(q^{28}) - \vartheta_2(q^{28})) \\ &= \vartheta_4(q)\vartheta_4(q^7) = B(q). \end{aligned}$$

This completes the proof of the second assertion. To derive the first assertion, we claim that

$$A(q) + B(q^2) = 2 \sum_{m,n=-\infty}^{\infty} q^{2(2m^2+2mn+4n^2)}. \tag{4.6}$$

Indeed

$$A(q) + B(q^2) = \sum_{m,n=-\infty}^{\infty} (1 + (-1)^{m-n})q^{2(m^2+mn+2n^2)}$$
$$= 2\left[\sum_{m,n\in 2\mathbf{Z}} + \sum_{m,n\in 2\mathbf{Z}+1}\right] q^{2(m^2+mn+2n^2)}$$
$$= 2\left[\sum_{m\in 2\mathbf{Z},n\in \mathbf{Z}} + \sum_{m,n\in 2\mathbf{Z}+1} + \sum_{m\in 2\mathbf{Z},n\in 2\mathbf{Z}+1}\right] q^{2(m^2+mn+2n^2)}.$$

From (4.4), we deduce that

$$\left(\sum_{m,n\in 2\mathbf{Z}+1} - \sum_{m\in 2\mathbf{Z},n\in 2\mathbf{Z}+1}\right) q^{2(m^2+mn+2n^2)}$$
$$= -\sum_{m\in \mathbf{Z},n\in 2\mathbf{Z}+1} q^{2(m^2+mn+2n^2)}$$
$$= -\sum_{m\in \mathbf{Z},n\in 2\mathbf{Z}+1} q^{2((m+\frac{n}{2})^2+\frac{D}{4}n^2)}. \qquad \square$$

Proposition 4.4. *For all $z\in\mathscr{H}$, we have*

$$\theta_2(q) = A(q^2) - B(q^2).$$

Proof. Writing $x = \alpha + 2(m - n\alpha)$ we see that

$$x\bar{x} = 4[m^2 + 2(n - 1/2)^2 + m(n - 1/2)]$$

or in other words,

$$\theta_2(q) = \sum_{\substack{m,n\in\mathbf{Z} \\ m \text{ even}, n \text{ odd}}} q^{NF(m,n)},$$

where $NF(m,n)$ is given by (4.5). Introduce for convenience

$$\theta_3(q) = \sum_{\substack{m,n\in\mathbf{Z} \\ m \text{ odd}, n \text{ odd}}} q^{NF(m,n)}.$$

Since

$$\mathbf{Z}\times\mathbf{Z} = \{m,n\in\mathbf{Z} \mid m \text{ even}, n \text{ even}\} \cup \{m,n\in\mathbf{Z} \mid m \text{ odd}, n \text{ odd}\}$$
$$\cup \{m,n\in\mathbf{Z} \mid m \text{ odd}, n \text{ even}\} \cup \{m,n\in\mathbf{Z} \mid m \text{ even}, n \text{ odd}\},$$

we may split the sum over $(m, n) \in \mathbf{Z} \times \mathbf{Z}$ in A and B into four sums and obtain the following system of two equations in $\theta_2(q)$ and $\theta_3(q)$:

$$A(q^{1/2}) = \theta_2(q) + A(q^2) + C(q^2) + \theta_3(q)$$

and

$$B(q) = -\theta_2(q) + A(q^2) - C(q^2) + \theta_3(q).$$

Solving for θ_2 we find that

$$2\theta_2(q) = A(\sqrt{q}) - B(q) - 2C(q^2).$$

The result follows by Lemma 4.3. □

We shall require the following lemma.

Lemma 4.5. *For all* $z \in \mathscr{H}$ *we have*

$$\theta_{(\sqrt{2}\mathcal{O})^2}(q) = (2A(q^2) - B(q^2))^2.$$

Proof. By definition

$$\theta_{(\sqrt{2}\mathcal{O})^2}(q) = A^2(q).$$

Applying Lemma 4.3 for $A(q)$, we complete the proof of the lemma.

We now complete the proof of Theorem 1.1.

Proof of Theorem 1.1. By Scharlau [18] there exists—up to isometry—a unique 7-modular lattice of dimension 4 over $\mathbf{Z}$. An immediate candidate is $\mathcal{O}^2$. By Theorem 3.1 another candidate is $A_K(C_2)/\sqrt{2}$ where C_2 is the length 2 self-dual code with generator matrix $[1, 1]$.

Now the *swe* of that code is computed in [1, p. 102] and evaluated as

$$swe_{C_2}(X, Y, Z) = X^2 + Y^2 + 2Z^2.$$

The theta series of $A_K(C_2)$ can then be computed on applying Theorem 3.2. By the preceding discussion it should equal $\theta_{(\sqrt{2}\mathcal{O})^2}(q)$ computed in Lemma 4.5 as a function of $A(q)$, $B(q)$, and $C(q)$. This yields

$$(2A(q^2) - B(q^2))^2 = A^2(q^2) + C^2(q^2) + 2(A(q^2) - B(q^2))^2,$$

which reduces to the desired identity. □

5. Seven-modular lattices in dimension 6

5.1. Extremal case

The lattice called

- A_6^2 in Craig's notation [8, Chapter 8, p. 223],
- P_6 in Barnes notation [13, p. 131],
- J in Cohen's notation [7]

is registered as a perfect 7-modular lattice in dimension 3 over $\mathcal{O}$ of determinant 7^3, kissing number 42 and norm 4 in Nebe–Sloane Catalogue of lattices [11]. According to [18] an extremal (i.e. norm 4) 7-modular lattice in dimension 3 over $\mathcal{O}$ is unique. The construction of [14, Section 3, p. 237] attributed to Serre shows that $A_6^2 = A_K(C_3)$. Here $C_3 = vR_3 + (1+v)R_3^{\perp}$ where R_3 stands for the binary linear code of generator $[1,1,1]$. It occurred [14] in relation to the Jacobian of the Klein curve. Another geometric construction, using Mordell–Weil lattices can be found in [9] where the theta series is computed (using Quebbemann's theorem) as

$$\theta_{A_6^2}(q) = A(q)^3 - 6\Delta_6(q).$$

Combining this information with [1] where it is shown that $swe_{C_3}(X,Y,Z) = X^3 + Z^3 + 3XZ^2 + 3ZY^2$, we obtain

Theorem 5.1. *The theta series of* A_6^2 *is*

$$\theta_{A_6^2}(q) = 5A^3(q^2) - 9A^2(q^2)B(q^2) + 6A(q^2)B^2(q^2) + 3C^2(q^2)A(q^2) - 3C^2(q^2)B(q^2) - B^3(q^2).$$

A third evaluation of $\theta_{A_6^2}$ can be obtained by using Ramanujan modular equations to the base 7. Let $f(-q) := \prod_{j=1}^{\infty} (1-q^j)$. On applying Lemma 2.2 of [5] we get

$$\theta_{A_6^2}(q) = \frac{f^7(-q^2)}{f(-q^{14})} + 7\Delta_6(q) + 49q^2\frac{f^7(-q^{14})}{f(-q^2)}.$$

On the other hand, according to Ranghachari [17, p. 370] the theta series of the lattice A_6^* admits the similar expression

$$\theta_{A_6^*}(q) = \frac{f^7(-q^2)}{f(-q^{14})} + 7\Delta_6(q) + 7q^2\frac{f^7(-q^{14})}{f(-q^2)}.$$

This comes from the fact that both theta series are invariant under $\Gamma_0(7)$ with the same quadratic character [6]. The corresponding space of weight 3 modular forms is three dimensional and spanned by the three functions

$$\frac{f^7(-q^2)}{f(-q^{14})}, \quad \Delta_6(q) \quad \text{and} \quad q^2\frac{f^7(-q^{14})}{f(-q^2)}.$$

5.2. Norm 2

There are two other self-dual R_4-codes in length 3, namely $C_{3,2} := v\langle[0,1,1]\rangle + (v+1)\langle[0,1,1]\rangle^{\perp}$ with weight enumerator

$$X^3 + X^2Z + XY^2 + 2XZ^2 + Y^2Z + 2Z^3$$

and $C_{3,3} := v\langle[0,0,1]\rangle + (v+1)\langle[0,0,1]\rangle^{\perp}$ with weight enumerator

$$X^3 + 3X^2Z + 3XZ^2 + Z^3.$$

Since $A_K(C_{3,2})$ and $A_K(C_{3,3})$ seem to have the same theta series

$$(1 + 6q^2 + 24q^4 + 56q^6 + 114q^8 + 168q^{10} + 280q^{12} + 294q^{14} + 444q^{16} + O(q^{18}))$$

we are led to conjecture the cubic relation

$$-2A^3(q^2) + A^2(q^2)B(q^2) + 2A(q^2)C^2(q^2) + 2A(q^2)B^2(q^2) - C^2(q^2)B(q^2) - B^3(q^2) = 0$$

which is equivalent to

$$(2A(q^2) - B(q^2))(A^2(q^2) - B^2(q^2) - C^2(q^2)) = 0,$$

which is certainly true.

Proposition 5.2. *The lattices $A_K(C_{3,2})$ and $A_K(C_{3,3})$ are isometric. Their theta series is*

$$8A^3(q^2) - 12A^2(q^2)B(q^2) + 6A(q^2)B^2(q^2) - B^3(q^2).$$

Proof. The first assertion follows by inspection of Schulze–Pillot's database of Hermitian lattices [12]. The second assertion follows on applying Theorem 2.2 to $swe_{C_{3,3}}$.

To conclude there are exactly two classes of $\mathcal{O}$ lattices in (real) dimension 6 and they can both be constructed by using codes over $\mathbf{F}_2 \times \mathbf{F}_2$. □

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