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Dual canonical bases for the quantum general linear supergroup ☆

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Abstract

Dual canonical bases of the quantum general linear supergroup are constructed which are invariant under the multiplication of the quantum Berezinian. By setting the quantum Berezinian to identity, we obtain dual canonical bases of the quantum special linear supergroup $\mathcal{O}_q(SL_{m|n})$. We apply the dual canonical bases to study invariant subalgebras of the quantum supergroups under left and right translations. In the case $n = 1$, it is shown that each invariant subalgebra is spanned by a part of the dual canonical bases. This in turn leads to dual canonical bases for any Kac module constructed by using an analogue of Borel–Weil theorem.

1. Introduction

Crystal bases and canonical bases were introduced by Kashiwara [7,8] and Lusztig [10,11] in the context of quantized universal enveloping algebras of symmetrizable Kac–Moody algebras (including finite-dimensional simple Lie algebras) and the associated quantized function algebras in the early 1990s. Since then their theories have been exten-

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sively developed, leading to many new developments in representation theory. A natural problem is to study similar bases for the quantized enveloping superalgebras and the associated quantized function algebras [24], the cousins of the quantized universal enveloping algebras. Musson and Zou [15] constructed a crystal basis for each finite-dimensional irreducible module over the quantized enveloping superalgebra of $\mathrm{osp}(1|2n)$. In [1], the crystal basis for any highest weight module in a subcategory $\mathcal{O}_{\mathrm{int}}$ of finite-dimensional modules over $U_q(\mathfrak{gl}_{m|n})$ is constructed, also see [26]. Zou [27] also constructed crystal bases for highest weight modules over the quantized universal superalgebra of the simple Lie superalgebra $D(2,1;\alpha)$ (in Kac's notation).

In contrast, little seems to be known about canonical bases or global crystal bases. The purpose of this paper is to study dual canonical bases for the quantum general linear supergroup $GL_q(m \mid n)$ and the quantum special linear supergroup $SL_q(m \mid n)$. There are various ways to approach these quantum supergroups. Manin [13,14] introduced the coordinate algebra of a quantum supermatrix and its inverse supermatrix. This superalgebra has a Hopf superalgebraic structure, thus is regarded as a version of the quantum general linear supergroup (which depends on more than one deformation parameters). The paper [24] studied the Hopf subalgebra of the finite dual of the quantized universal enveloping superalgebra $U_q(\mathfrak{gl}_{m|n})$ generated by the matrix elements of the natural representation and its dual. It was shown that this Hopf subalgebra separate points of $U_q(\mathfrak{gl}_{m|n})$ in the sense that if $x, y \in U_q(\mathfrak{gl}_{m|n})$ are not equal, then there exists f in the Hopf subalgebra such that its evaluations on x and y are different. This Hopf subalgebra was taken as the definition of another version of the quantum general linear supergroup. It is not difficult to show that by specializing the parameters of the Hopf superalgebra of [13,14], one obtains the quantum general linear supergroup of [24]. A third version was defined in [2] by localizing the coordinate algebra of a supermatrix (without the inverse) at the quantum determinants of two sub-matrices. As the quantum determinants of these sub-matrices do not commute with each other, properties of this localization is not immediately transparent. One of the results of this paper is to show that the third version of the quantum general linear supergroup is equivalent to the first two.

Any basis for the quantum general linear supergroup will be very useful for studying its structure. In [4], a basis of the letter-place algebra (which is a generalization of coordinate algebra of a quantum supermatrix) was constructed by introducing quantum minors. In principle one may try to extend this basis to a basis for the coordinate algebra of the quantum supermatrix together with its inverse quantum supermatrix, thus to obtain a basis for the quantum general linear supergroup. However, as far as we are aware, this was not achieved before, presumably because of technical difficulties. In fact, it seems that no basis of any kind is known for the quantum general linear supergroup. In this paper we shall construct bases for the quantum general linear supergroup and the associated quantum special linear supergroup. The results are given in Theorem 4.15.

A notable feature of the bases for the quantum general linear supergroup is that they are invariant under the multiplication of the quantum Berezinian. By setting the quantum Berezinian to 1 we get bases for the quantum special linear supergroup. Also, the basis elements consist of $\mathbb{Z}[q]$ combinations of certain monomials, and are invariant under some bar-involution on the quantum supergroups. Therefore, we may regard the bases constructed in Theorem 4.15 as some dual canonical bases for the quantum supergroups.

The algebra of functions on the quantum general linear supergroup admits two actions of the quantized universal enveloping superalgebra $U_q(\mathfrak{gl}_{m|n})$. For any bi-subalgebra U_S of $U_q(\mathfrak{gl}_{m|n})$, the subspace of invariants under the left or right translation with respect to U_S forms a subalgebra, which may be regarded as the algebra of functions on some quantum homogeneous superspace [25] in the general spirit of noncommutative geometry. We apply the dual canonical bases to study such invariant subalgebras. In the case $n = 1$, we show that any subalgebra of invariants is spanned by a part of the dual canonical bases. This in turn leads to dual canonical bases for any Kac module constructed using a Borel–Weil type of construction [25].

The paper is organized as follows. In Section 2, we collect some results on the quantized enveloping algebra $U_q(\mathfrak{gl}_{m|n})$, and the version of the quantum general linear supergroup defined in [24]. The material of this section will be used throughout the remainder of the paper. In Section 3, the coordinate algebra $\mathcal{O}_q(M_{m|n})$ is presented by exhibiting its generators and defining relations following [13,14]; quantum minors [4] in the context of $\mathcal{O}_q(M_{m|n})$ are discussed; and the equivalence of the various versions of the quantum general linear supergroup is proven. In Section 4, we present the construction of the dual canonical bases for $\mathcal{O}_q(GL_{m|n})$ and $\mathcal{O}_q(SL_{m|n})$, and finally in Section 5, we use the dual canonical bases to study subalgebras of the quantum general linear supergroup which are invariant under right (or left) translations of any bi-subalgebra U_S of $U_q(\mathfrak{gl}_{m|n})$.

2. Quantum general linear supergroup

2.1. The quantized enveloping algebra $U_q(\mathfrak{gl}_{m|n})$

Throughout the paper, we will denote by $\mathfrak{g}$ the complex Lie superalgebra $\mathfrak{gl}_{m|n}$, and by $U_q(\mathfrak{g})$ the quantized enveloping superalgebra of $\mathfrak{g}$. Let $\mathbf{I} = \{1, 2, \ldots, m+n\}$ and $\mathbf{I}' = \mathbf{I}\backslash\{m+n\}$. The quantized enveloping superalgebra $U_q(\mathfrak{g})$ is a $\mathbb{Z}_2$-graded associative algebra (i.e., associative superalgebra) over $\mathbb{C}(q)$, q being an indeterminate, generated by $\{K_a,\ K_a^{-1},\ a \in \mathbf{I};\ E_{b\, b+1},\ E_{b+1,b},\ b \in \mathbf{I}'\}$, subject to the relations [24]

$$
\begin{aligned}
&K_a K_a^{-1} = 1, \qquad K_a^{\pm 1} K_b^{\pm 1} = K_b^{\pm 1} K_a^{\pm 1},\\
&K_a E_{b,b\pm 1} K_a^{-1} = q_a^{2\delta_{ab} - 2\delta_{a,b\pm 1}} E_{b,b\pm 1},\\
&[E_{a,a+1}, E_{b+1,b}\} = \delta_{ab}\big(K_a K_{a+1}^{-1} - K_a^{-1} K_{a+1}\big)\big/\big(q_a^2 - q_a^{-2}\big),\\
&(E_{m,m+1})^2 = (E_{m+1,m})^2 = 0,\\
&E_{a,a+1} E_{b,b+1} = E_{b,b+1} E_{a,a+1},\\
&E_{a+1,a} E_{b+1,b} = E_{b+1,b} E_{a+1,a}, \quad |a-b| \geqslant 2,\\
&\mathcal{S}^{(+)}_{a,a\pm 1} = \mathcal{S}^{(-)}_{a,a\pm 1} = 0, \quad a \neq m,\\
&E_{m-1,m+2} E_{m,m+1} + E_{m,m+1} E_{m-1,m+2} = 0,\\
&E_{m+2,m-1} E_{m+1,m} + E_{m+1,m} E_{m+2,m-1} = 0,
\end{aligned}
$$

where

$$\begin{aligned}\mathcal{S}^{(+)}_{a,a\pm1} &= (E_{a,a+1})^2 E_{a\pm1,a+1\pm1} - \big(q^2+q^{-2}\big) E_{a,a+1} E_{a\pm1,a+1\pm1} E_{a,a+1} \\ &\quad + E_{a\pm1,a+1\pm1}(E_{a,a+1})^2, \\ \mathcal{S}^{(-)}_{a,a\pm1} &= (E_{a+1,a})^2, E_{a+1\pm1,a\pm1} - \big(q^2+q^{-2}\big) E_{a+1,a} E_{a+1\pm1,a\pm1} E_{a+1,a} \\ &\quad + E_{a+1\pm1,a\pm1}(E_{a+1,a})^2,\end{aligned}$$

and $E_{m-1,m+2}$ and $E_{m+2,m-1}$ are the $a=m-1$, $b=m+1$, cases of the following elements

$$\begin{aligned}E_{a,b} &= E_{a,c}E_{c,b} - q_c^{-2} E_{c,b}E_{a,c}, \\ E_{b,a} &= E_{b,c}E_{c,a} - q_c^{2} E_{c,a}E_{b,c}, \quad a<c<b.\end{aligned}$$

Let

$$[a] = \begin{cases} 0, & \text{if } a \leqslant m, \\ 1, & \text{if } a > m. \end{cases}$$

Then $q_a = q^{(-1)^{[a]}}$, and

$$[E_{a,a+1}, E_{b+1,b}\} = E_{a,a+1}E_{b+1,b} - (-1)^{[a]+[a+1]} E_{b+1,b}E_{a,a+1}.$$

The $\mathbb{Z}_2$ grading of the algebra is specified such that the elements $K_a^{\pm1}$, $\forall a \in \mathbf{I}$, and $E_{b,b+1}$, $E_{b+1,b}$, $b \neq m$, are even, while $E_{m,m+1}$ and $E_{m+1,m}$ are odd. We shall denote by n^+ the subalgebra of $U_q(\mathfrak{g})$ generated by $E_{a,a+1}$ for all $a \leqslant m+n-1$ and by n^- the subalgebra generated by $E_{a+1,a}$ for all $a \leqslant m+n-1$.

It is well known that $U_q(\mathfrak{g})$ has the structure of a $\mathbb{Z}_2$ graded Hopf algebra (i.e., Hopf superalgebra), with a co-multiplication

$$\begin{aligned}\Delta(E_{a,a+1}) &= E_{a,a+1} \otimes K_a K_{a+1}^{-1} + 1 \otimes E_{a,a+1}, \\ \Delta(E_{a+1,a}) &= E_{a+1,a} \otimes 1 + K_a^{-1} K_{a+1} \otimes E_{a+1,a}, \\ \Delta\big(K_a^{\pm1}\big) &= K_a^{\pm1} \otimes K_a^{\pm1},\end{aligned}$$

co-unit

$$\begin{aligned}\epsilon(E_{a,a+1}) &= E_{a+1,a} = 0, \quad \forall a \in \mathbf{I}', \\ \epsilon\big(K_b^{\pm1}\big) &= 1, \quad \forall b \in \mathbf{I},\end{aligned}$$

and antipode

$$S(E_{a,a+1}) = -E_{a,a+1}K_a^{-1}K_{a+1},$$
$$S(E_{a+1,a}) = -K_aK_{a+1}^{-1}E_{a+1,a},$$
$$S\big(K_a^{\pm 1}\big) = K_a^{\mp 1}.$$

Sometimes, we also use E_i and F_i to denote $E_{i,i+1}$ and $E_{i+1,i}$, respectively.

Let $\{\epsilon_a \mid a \in \mathbf{I}\}$ be the basis of a vector space with a bilinear form $(\epsilon_a, \epsilon_b) = (-1)^{[a]}\delta_{ab}$. The roots of the classical Lie superalgebra $\mathfrak{gl}_{m|n}$ can be expressed as $\epsilon_a - \epsilon_b$, $a \neq b$, $a, b \in \mathbf{I}$. It is known [23] that every finite-dimensional irreducible $U_q(\mathfrak{g})$ module is of highest weight type and is uniquely characterized by a highest weight. For $\lambda = \sum_a \lambda_a\epsilon_a$, $\lambda_a \in \mathbb{Z}$, we shall use the notation $L(\lambda)$ to denote the irreducible $U_q(\mathfrak{g})$ module with a unique (up to scalar multiples) vector $v_\lambda \neq 0$ such that

$$E_{a,a+1}v_\lambda = 0, \quad a \in \mathbf{I}',$$
$$K_bv_\lambda = q_b^{2\lambda_b}v_\lambda, \quad b \in \mathbf{I}.$$

We shall refer to λ as the highest weight of $L(\lambda)$. Then $L(\lambda)$ is finite dimensional if and only if λ satisfies $\lambda_a - \lambda_{a+1} \in \mathbb{Z}_+$, $a \neq m$. In that case, $L(\lambda)$ has the same weight space decomposition as that of the corresponding irreducible $\mathfrak{gl}_{m|n}$ module with the same highest weight λ.

The natural $U_q(\mathfrak{g})$-module $\mathbb{E}$ has the standard basis $\{v_a \mid a \in \mathbf{I}\}$, such that

$$K_av_b = q_a^{2\delta_{ab}}v_b, \qquad E_{a,a\pm 1}v_b = \delta_{b,a\pm 1}v_a.$$

The $U_q(\mathfrak{g})$ modules $\mathbb{E}^{\otimes k}$, $k \in \mathbb{Z}_+$ ($\mathbb{E}^0 = \mathbb{C}$) were shown to be completely reducible [24]. The irreducible summands of these $U_q(\mathfrak{g})$-modules, referred to as irreducible contravariant tensor modules, can be characterized in the following way. Let $\mathbb{Z}_+$ be the set of nonnegative integers. Define a subset $\mathcal{P}$ of $\mathbb{Z}_+^{m+n}$ by

$$\mathcal{P} = \big\{p = (p_1, p_2, \ldots, p_{m+n}) \in \mathbb{Z}_+^{m+n} \mid p_{m+1} \leqslant n,\ p_a \geqslant p_{a+1},\ a \in \mathbf{I}'\big\}.$$

We associate with each $p \in \mathcal{P}$ a $\mathfrak{gl}_{m|n}$-weight defined by

$$\lambda^{(p)} = \sum_{i=1}^{m} p_i\epsilon_i + \sum_{\nu=1}^{n}\sum_{\mu=1}^{p_{m+\nu}} \epsilon_{m+\mu},$$

and let

$$\Lambda^{(1)} = \big\{\lambda^{(p)} \mid p \in \mathcal{P}\big\}. \tag{2.1}$$

From results of [3,22] we know that an irreducible $U_q(\mathfrak{g})$-module is a contravariant tensor if and only if its highest weight belongs to $\Lambda^{(1)}$.

Let $L(\lambda)$ be an irreducible contravariant tensor $U_q(\mathfrak{g})$ module with highest weight $\lambda \in \Lambda^{(1)}$. Denote by $\bar{\lambda}$ its lowest weight, and set $\lambda^\dagger = -\bar{\lambda}$. An explicit formula for $\lambda^\dagger$

was given in [3, Section III.B], where a more compact characterization was also given for $\Lambda^{(1)}$ and also the set

$$\Lambda^{(2)} := \{\lambda^\dagger \mid \lambda \in \Lambda^{(1)}\}. \tag{2.2}$$

We refer to that paper for details. Now the dual module $L(\lambda)^\dagger$ of $L(\lambda)$, which we will call a covariant tensor module, has highest weight $\lambda^\dagger$. The most important example is the dual module $\mathbb{E}^\dagger = L(-\epsilon_{m+n})$ of $\mathbb{E}$.

The situation with tensor powers $\mathbb{E}^{\otimes k}$ and $(\mathbb{E}^\dagger)^{\otimes k}$ can be summarized into the following proposition [24].

Proposition 2.1.

(1) *Each $U_q(\mathfrak{g})$-module $\mathbb{E}^{\otimes k}$ (respectively $(\mathbb{E}^\dagger)^{\otimes k}$), $k \in \mathbb{Z}_+$, can be decomposed into a direct sum of irreducible modules with highest weights belonging to $\Lambda^{(1)}$ (respectively $\Lambda^{(2)}$).*
(2) *Every irreducible $U_q(\mathfrak{g})$-module with highest weight belonging to $\Lambda^{(1)}$ (respectively $\Lambda^{(2)}$) is a direct summand of some tensor powers of $\mathbb{E}$ (respectively $\mathbb{E}^\dagger$).*

2.2. The quantum general linear supergroup

Let $(U_q(\mathfrak{g}))^0$ be the finite dual of $U_q(\mathfrak{g})$, which, by standard Hopf algebra theory, is a $\mathbb{Z}_2$-graded Hopf algebra with structure dualizing that of $U_q(\mathfrak{g})$. Let us denote by π the representation of $U_q(\mathfrak{g})$ on $\mathbb{E}$ relative to the standard basis $\{v_a \mid a \in \mathbf{I}\}$:

$$x v_a = \sum_b \pi(x)_{b,a} v_b, \quad x \in U_q(\mathfrak{g}),$$

then we have the elements $t_{a,b} \in (U_q(\mathfrak{g}))^0$, $a, b \in \mathbf{I}$, defined by

$$t_{a,b}(x) = \pi(x)_{a,b}, \quad \forall x \in U_q(\mathfrak{g}).$$

Note that $t_{a,b}$ is even if $[a] + [b] \equiv 0 \pmod 2$, and odd otherwise.

Consider the subalgebra G_q^π of $(U_q(\mathfrak{g}))^0$ generated by $t_{a,b}$, $a, b \in \mathbf{I}$. The multiplication which G_q^π inherits from $(U_q(\mathfrak{g}))^0$ is given by

$$\begin{aligned}\langle tt', x\rangle &= \sum_{(x)} \langle t \otimes t', x_{(1)} \otimes x_{(2)}\rangle \\ &= \sum_{(x)} (-1)^{[t'][x_{(1)}]} \langle t, x_{(1)}\rangle \langle t', x_{(2)}\rangle, \quad \forall t, t' \in G_q^\pi,\ x \in U_q(\mathfrak{g}).\end{aligned}$$

To better understand the algebraic structure of G_q^π, recall that the Drinfeld version of $U_q(\mathfrak{g})$ admits a universal R matrix, which in particular satisfies

$$R\Delta(x) = \Delta'(x)R, \quad \forall x \in U_q(\mathfrak{g}).$$

Applying $\pi \otimes \pi$ to both sides of the equation yields

$$R^{\pi\pi}(\pi \otimes \pi)\Delta(x) = (\pi \otimes \pi)\Delta'(x)R^{\pi\pi}, \tag{2.3}$$

where $R^{\pi\pi} := (\pi \otimes \pi)R$ is given by

$$R^{\pi\pi} = q^{2\sum_{a\in\mathbf{I}} e_{a,a}\otimes e_{a,a}(-1)^{[a]}} + \left(q^2 - q^{-2}\right)\sum_{a<b} e_{a,b} \otimes e_{b,a}(-1)^{[b]}.$$

As we work with the Jimbo version of $U_q(\mathfrak{g})$ in this paper, it is problematic to talk about a universal R matrix. However, it is important to note that Eq. (2.3) makes perfect sense in the present setting.

We can re-interpret Eq. (2.3) in terms of $t_{a,b}$ in the following way. Set $t = \sum_{a,b} e_{a,b} \otimes t_{a,b}$. Then

$$R_{12}^{\pi\pi} t_1 t_2 = t_2 t_1 R_{12}^{\pi\pi}. \tag{2.4}$$

The co-multiplication Δ of G_q^π is also defined in the standard way by

$$\left\langle \Delta(t_{a,b}), x \otimes y \right\rangle = \langle t_{a,b}, xy \rangle = \pi(xy)_{a,b}, \quad \forall x, y \in U_q(\mathfrak{g}).$$

We have

$$\Delta(t_{a,b}) = \sum_{c\in\mathbf{I}} (-1)^{([a]+[c])([c]+[b])} t_{a,c} \otimes t_{c,b}. \tag{2.5}$$

G_q^π also has the unit ϵ, and the co-unit $1_{U_q(\mathfrak{g})}$. Therefore, G_q^π has the structures of a $\mathbb{Z}_2$-graded bi-algebra.

Let $\pi^{(\lambda)}$ be an arbitrary irreducible contravariant tensor representation of $U_q(\mathfrak{g})$. Define the elements $t_{i,j}^{(\lambda)}$, $i, j = 1, 2, \ldots, \dim_{\mathbb{C}} \pi^{(\lambda)}$, of $(U_q(\mathfrak{g}))^0$ by

$$t_{i,j}^{(\lambda)}(x) = \pi^{(\lambda)}(x)_{i,j}, \quad \forall x \in U_q(\mathfrak{g}).$$

These will be called the matrix elements of the irreducible representation $\pi^{(\lambda)}$. It is an immediate consequence of Proposition 2.1 that for every $\lambda \in \Lambda^{(1)}$, the elements $t_{i,j}^{(\lambda)} \in G_q^\pi$, for all i, j, and every $f \in G_q^\pi$ can be expressed as a linear sum of such elements. We have the following result [24].

Proposition 2.2. *As a vector space,*

$$G_q^\pi = \bigoplus_{\lambda\in\Lambda^{(1)}} T^{(\lambda)}, \quad \textit{where } T^{(\lambda)} = \bigoplus_{i,j=1}^{\dim \pi^{(\lambda)}} \mathbb{C}(q) t_{i,j}^{(\lambda)}.$$

Let $\{\bar{v}_a \mid a \in \mathbf{I}\}$ be the basis of $\mathbb{E}^\dagger$ dual to the standard basis of $\mathbb{E}$, i.e.,

$$\bar{v}_a(v_b) = \delta_{a,b}.$$

Denote by $\bar{\pi}$ the covariant vector irreducible representation relative to this basis. Let $\bar{t}_{a,b}$, $a, b \in \mathbf{I}$, be the elements of $(U_q(\mathfrak{g}))^0$ such that

$$\bar{t}_{a,b}(x) = \bar{\pi}(x)_{a,b}, \quad \forall x \in U_q(\mathfrak{g}).$$

Note that $\bar{t}_{a,b}$ is even if $[a] + [b] \equiv 0 \pmod 2$, and odd otherwise. These elements generate a $\mathbb{Z}_2$-graded bi-subalgebra $G_q^{\bar{\pi}}$ of $(U_q(\mathfrak{g}))^0$. We want to point out that the $\bar{t}_{a,b}$ obey the relation

$$R_{12}^{\bar{\pi}\bar{\pi}} \bar{t}_1 \bar{t}_2 = \bar{t}_2 \bar{t}_1 R_{12}^{\bar{\pi}\bar{\pi}}, \tag{2.6}$$

where $\bar{t} = \sum_{a,b} e_{a,b} \otimes \bar{t}_{b,a}$ and $R^{\bar{\pi}\bar{\pi}} = (\bar{\pi} \otimes \bar{\pi})R$ is given by

$$R^{\bar{\pi}\bar{\pi}} = q^{2\sum_{a\in\mathbf{I}} e_{a,a}\otimes e_{a,a}(-1)^{[a]}} + \left(q^2 - q^{-2}\right) \sum_{a>b} e_{a,b} \otimes e_{b,a}(-1)^{[b]}.$$

Also, the co-multiplication is given by

$$\Delta(\bar{t}_{a,b}) = \sum_{c\in\mathbf{I}} (-1)^{([a]+[c])([c]+[b])} \bar{t}_{a,c} \otimes \bar{t}_{c,b}.$$

Similar to the case of G_q^{π}, we let $\bar{T}^{(\lambda)}$ be the subspace of $G_q^{\bar{\pi}}$ spanned by the matrix elements of the irreducible representation with highest weight $\lambda \in \Lambda^{(2)}$. Then it follows from Proposition 2.1 that

Proposition 2.3. *As a vector space,*

$$G_q^{\bar{\pi}} = \bigoplus_{\mu\in\Lambda^{(2)}} \bar{T}^{(\mu)}.$$

Definition 2.4. The algebra G_q of functions on the quantum general linear supergroup $GL_q(m \mid n)$ is the $\mathbb{Z}_2$-graded subalgebra of $U_q(\mathfrak{g})^0$ generated by $\{t_{a,b}, \bar{t}_{a,b} \mid a, b \in \mathbf{I}\}$.

The $t_{a,b}$ and $\bar{t}_{a,b}$, besides obeying the relations (2.4) and (2.6), also satisfy

$$R_{12}^{\bar{\pi}\pi} \bar{t}_1 t_2 = t_2 \bar{t}_1 R_{12}^{\bar{\pi}\pi}, \tag{2.7}$$

where $R^{\bar{\pi}\pi} := (\bar{\pi} \otimes \pi)R$ is given by

$$R^{\bar{\pi}\pi} = q^{-2\sum_{a\in\mathbf{I}} e_{a,a}\otimes e_{a,a}(-1)^{[a]}} - \left(q^2 - q^{-2}\right) \sum_{a<b} e_{b,a} \otimes e_{b,a}(-1)^{[a]+[b]+[a][b]}.$$

As both G_q^{π} and $G_q^{\bar{\pi}}$ are $\mathbb{Z}_2$-graded bi-algebras, G_q inherits a natural bi-algebra structure. It also admits an antipode $S: G_q \to G_q$, which is a linear anti-automorphism given by

$$S(t_{a,b}) = (-1)^{[a][b]+[a]}\bar{t}_{b,a}, \qquad S(\bar{t}_{a,b}) = (-1)^{[a][b]+[b]}q^{2(2\rho,\epsilon_a-\epsilon_b)}t_{b,a}. \tag{2.8}$$

Therefore, G_q has the structures of a $\mathbb{Z}_2$-graded Hopf algebra. It was also shown in [24] that G_q separates points of $U_q(\mathfrak{g})$ in the sense that for any $x, y \in U_q(\mathfrak{g})$ such that $x \neq y$, then there exists $f \in G_q$ such that $\langle f, x\rangle \neq \langle f, y\rangle$.

There are two natural actions of $U_q(\mathfrak{g})$ on the quantized function algebra $\mathcal{O}_q(GL_{m|n})$, which correspond to left and right translations in the classical setting. The two actions are respectively defined, for all $x \in U_q(\mathfrak{g})$, $f \in \mathcal{O}_q(GL_{m|n})$, by

$$R_x(f) = \sum_{(f)} f_{(1)}\big\langle f_{(2)}, \omega(x)\big\rangle,$$

$$L_x(f) = \sum_{(f)}\big\langle f_{(1)}, \omega(x)\big\rangle f_{(2)}(-1)^{[x][f_{(1)}]},$$

where we have used Sweedler's notation $\Delta(f) = \sum f_{(1)} \otimes f_{(2)}$. Note that L is a left action while R is a right action. Furthermore, the two actions commute.

For convenience, we shall use the algebra anti-automorphism $\omega: U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ given by

$$\omega(K_i) = K_i, \qquad \omega(E_i) = F_i, \qquad \omega(F_i) = E_i,$$

to twist the two actions. This way we obtain a left action and a right action of $U_q(\mathfrak{g})$ on $\mathcal{O}_q(GL_{m|n})$, which will be denoted by $x.f$ and $f.x$ respectively for any $x \in U_q(\mathfrak{g})$ and $f \in \mathcal{O}_q(GL_{m|n})$. The actions can be written down explicitly in terms of generators:

$$E_i.x_{kl} = \delta_{i,k-1}\delta_{il}, \qquad F_i.x_{kl} = \delta ikx_{i+1,l}, \qquad K_i.x_{kl} = q^{2\delta_{ik}}x_{kl}, \tag{2.9}$$

$$x_{kl}.E_i = \delta_{i+1,l}x_{ki}, \qquad x_{kl}.F_i = \delta_{li}x_{k,i-1}, \qquad x_{kl}.K_i = q^{2\delta_{li}}x_{kl}. \tag{2.10}$$

These actions give the Hopf superalgebra G_q a natural $U_q(\mathfrak{g})$-bi-module structure.

2.3. *Bar-involution on quantized function algebra*

Denote by $-$ the involution of the base field $\mathbb{Q}(q)$ which is given as follows:

$$-: \mathbb{Q}(q) \to \mathbb{Q}(q), \quad q \mapsto q^{-1}.$$

There is an automorphism of the quantized enveloping algebra $U_q(\mathfrak{g})$ denoted by $^-$ which is given by:

$$\bar{E}_i = E_i, \qquad \bar{F}_i = F_i, \qquad \bar{K}_j = K_j^{-1}, \quad \bar{q} = q^{-1} \text{ for all } i, j.$$

We define a linear map $\dagger$ on G_q as follows:

$$\langle f^\dagger, x\rangle = \langle f, \bar{x}\rangle^-,$$

for any $f \in G_q$ and $x \in U_q(\mathfrak{g})$.

We have the notions of a left weight and right weight of elements of G_q with respect to the actions (2.10) and (2.9). An element of G_q is called homogeneous if it is both a left weight vector and right weight vector. For a homogeneous element f, we denote by $w_l(f)$ its left weight, and by $w_r(f)$ its right weight.

Proposition 2.5. *For any homogeneous elements $f, g \in G_q$ with weights $w_l(f), w_r(f)$ and $w_l(g), w_r(g)$, respectively,*

$$(fg)^\dagger = (-1)^{[f][g]} q^{2(w_l(f), w_l(g)) - 2(w_r(f), w_r(g))} g^\dagger f^\dagger.$$

Proof. Denote by Θ the quasi R-matrix which satisfies the following identities in the completion of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$:

$$\Theta\bar{\Theta} = \bar{\Theta}\Theta = 1 \otimes 1,$$
$$\Theta(- \otimes -)\Delta(x) = \Delta(\bar{x})\Theta.$$

Let Φ be the algebra automorphism of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ defined by

$$\begin{aligned} &E_i \otimes 1 \mapsto E_i \otimes K_i, && 1 \otimes E_i \mapsto K_i \otimes E_i, \\ &F_i \otimes 1 \mapsto F_i \otimes K_i^{-1}, && 1 \otimes F_i \mapsto K_i^{-1} \otimes F_i, \\ &K_i \otimes 1 \mapsto K_i \otimes 1, && 1 \otimes K_i \mapsto 1 \otimes K_i, \end{aligned}$$

for all $i = 1, 2, \ldots, m+n-1$. One can check easily that

$$\Delta(\bar{x}) = \Phi \circ (- \otimes -) \circ \Delta'(x),$$

where Δ' is the opposite co-multiplication.

Now, for any homogeneous elements f and g, we have

$$\begin{aligned} \langle (fg)^\dagger, x\rangle &= \langle fg, \bar{x}\rangle^- = \langle f \otimes g, \Delta(\bar{x})\rangle^- \\ &= \langle f \otimes g, \Phi \circ (- \otimes -) \circ \Delta'(x)\rangle^- \\ &= q^{2(w_r(f), w_r(g)) - 2(w_l(f), w_l(g))} \langle f \otimes g, (- \otimes -)\Delta'(x)\rangle^- \\ &= q^{2(w_r(f), w_r(g)) - 2(w_l(f), w_l(g))} \langle f^\dagger \otimes g^\dagger, \Delta'(x)\rangle \\ &= (-1)^{[f][g]} q^{2(w_r(f), w_r(g)) - 2(w_l(f), w_l(g))} \langle g^\dagger f^\dagger, x\rangle. \qquad \square \end{aligned}$$

The linear map $\dagger$ can be furnished into an anti automorphism in the following way:

Lemma 2.6. *The mapping* $\bar{}: G_q \to G_q$ *defined, for any homogeneous element* $f \in G_q$ *with weights* $(w_l(f), w_r(f))$*, by*

$$f \mapsto q^{(w_l(f),w_l(f))-(w_r(f),w_r(f))} f^\dagger, \quad q \mapsto q^{-1}$$

is an anti-automorphism of the superalgebra G_q.

Proof. For any homogeneous elements $f, g \in G_q$ with weights $(w_l(f), w_r(f))$ and $(w_l(g), w_r(g))$, respectively, we have

$$\overline{fg} = q^{A(f,g)}(fg)^\dagger = (-1)^{[f][g]} q^{B(f,g)} g^\dagger f^\dagger, \tag{2.11}$$

where

$$A(f,g) = \big(w_l(f) + w_l(g), w_l(f) + w_l(g)\big) - \big(w_r(f) + w_r(g), w_r(f) + w_r(g)\big),$$
$$B(f,g) = A(f,g) + 2\big(w_r(f), w_r(g)\big) - 2\big(w_l(f), w_l(g)\big).$$

The far right-hand side of Eq. (2.11) can be easily shown to be equal to $(-1)^{[f][g]}\bar{g}\bar{f}$, thus completing the proof. □

3. Coordinate algebra of quantum supermatrix

Let X be an $(m+n)\times(m+n)$ quantum supermatrix. We shall always write it in block form

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A and D are respectively $m\times m$ and $n\times n$ sub-matrices of even entries, while B and C are respectively $m\times n$ and $n\times m$ sub-matrices of odd entries. The coordinate algebra $\mathcal{O}_q(M_{m|n})$ [13] of X is a $\mathbb{Z}_2$-graded algebra (i.e., superalgebra) generated by the entries of the quantum supermatrix X satisfying the relation

$$R^{\pi,\pi} X_1 X_2 = X_2 X_1 R^{\pi,\pi},$$

where $X_1 = X \otimes 1$ and $X_2 = 1 \otimes X$. The defining relations can be written explicitly as follows:

$$x_{ij}x_{ik} = (-1)^{([i]+[j])([i]+[k])} q^{2(-1)^{[i]}} x_{ik}x_{ij}, \quad j<k,$$
$$x_{ij}x_{kj} = (-1)^{([i]+[j])([k]+[j])} q^{2(-1)^{[j]}} x_{kj}x_{ij}, \quad i<k,$$
$$x_{ij}x_{kl} = (-1)^{([i]+[j])([k]+[l])} x_{kl}x_{ij}, \quad i<k,\ j>l,$$
$$x_{ij}x_{kl} = (-1)^{([i]+[j])([k]+[l])} x_{kl}x_{ij} + (-1)^{[k][j]+[k][l]+[j][l]}\big(q^2 - q^{-2}\big) x_{il}x_{kj},$$
$$i<k,\ j<l.$$

Note that the matrix A is a quantum matrix with deformation parameter q while D is also a quantum matrix with deformation parameter q^{-1}. We shall refer A as a q-matrix and D as a q^{-1}-matrix. The superalgebra $\mathcal{O}_q(M_{m|n})$ has a bi-superalgebra structure, and it is customary to take the following coproduct and counit:

$$\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}, \qquad \epsilon(x_{ij}) = \delta_{ij}.$$

Let us recall the definition of quantum minors in the present context. We shall largely follow [4], besides some slight change in conventions. Denote

$$(\ ,\ ) : \mathbb{Z}^{m+n} \times \mathbb{Z}^{m+n} \to \mathbb{Z}, \qquad (\underline{a}, \underline{b}) = \sum_{i>j} a_i b_j,$$

where $\underline{a} = (a_1, a_2, \ldots, a_{m+n})$, $\underline{b} = (b_1, b_2, \ldots, b_{m+n}) \in \mathbb{Z}^{m+n}$. Also, denote by

$$[\underline{a}] = \sum_i a_i [i] \in \mathbb{Z}_2 \quad \text{and} \quad \{\underline{a}\} = \sum_i a_i \big([i] + \bar{1}\big) \in \mathbb{Z}_2.$$

The quantum superspace (or rather its coordinate algebra) A_q is a superalgebra generated by $x_1, x_2, \ldots, x_m, \ldots, x_{m+n}$ with parity assignment $[x_i] = [i]$ and defining relations

$$\begin{aligned} &x_i^2 = 0, \quad \text{if } [i] = \bar{1}, \\ &x_i x_j = (-1)^{[i][j]} q^2 x_j x_i, \quad \text{for } i < j. \end{aligned}$$

We also introduce the superalgebra A_q^* generated by $\xi_1, \xi_2, \ldots, \xi_m, \ldots, \xi_{m+n}$ with parity assignment $[\xi_i] = \bar{1} - [i]$ and defining relations

$$\begin{aligned} &\xi_i^2 = 0, \quad \text{if } [i] = \bar{0}, \\ &\xi_i \xi_j = (-1)^{([i]+\bar{1})([j]+\bar{1})} q^2 \xi_j \xi_i, \quad \text{for } i > j. \end{aligned}$$

The ordered monomials

$$x^{\underline{a}} := x_1^{a_1} x_2^{a_2} \cdots x_{m+n}^{a_{m+n}}$$

for all $\underline{a} = (a_1, a_2, \ldots, a_{m+n}) \in \mathbb{Z}_+^m \times \mathbb{Z}_2^n$, form a basis of A_q. Similarly, the ordered monomials

$$\xi^{\underline{b}} := \xi_1^{b_1} \xi_2^{b_2} \cdots \xi_{m+n}^{b_{m+n}},$$

for all $\underline{b} = (b_1, b_2, \ldots, b_{m+n}) \in \mathbb{Z}_2^m \times \mathbb{Z}_+^n$, form a basis of A_q^*.

The superalgebras A_q and A_q^* are comodule superalgebras of $\mathcal{O}_q(M_{m|n})$ with the coactions given below:

Theorem 3.1. [13] *There exist superalgebra morphisms*

$$\delta : A_q \to \mathcal{O}_q(M_{m|n}) \otimes A_q, \qquad x_i \mapsto \sum_j x_{ij} \otimes x_j,$$

$$\delta^* : A_q^* \to \mathcal{O}_q(M_{m|n}) \otimes A_q^*, \qquad \xi_i \mapsto \sum_j x_{ij} \otimes \xi_j,$$

which give $\mathcal{O}_q(M_{m|n})$-comodule structures to A_q and A_q^*.

For any $\underline{a} = (a_1, a_2, \ldots, a_{m+n}), \underline{b} = (b_1, b_2, \ldots, b_{m+n}) \in \mathbb{Z}^{m+n}$, define elements $\Delta(\underline{a}, \underline{b})$ and $\Delta(\underline{a}, \underline{b})^*$ of $\mathcal{O}_q(M_{m|n})$ by

$$\delta\big(x^{\underline{a}}\big) = \sum_{\underline{b}} (-1)^{\sum_{i>j} a_i b_j [i][j] + \sum_{i>j} b_i b_j [i][j]} \Delta(\underline{a}, \underline{b}) \otimes x^{\underline{b}},$$

$$\delta^*\big(\xi^{\underline{a}}\big) = \sum_{\underline{b}} (-1)^{\sum_{i>j} a_i b_j ([i]+\bar{1})([j]+\bar{1}) + \sum_{i>j} b_i b_j ([i]+\bar{1})([j]+\bar{1})} \Delta(\underline{a}, \underline{b})^* \otimes \xi^{\underline{b}},$$

which will be referred to as quantum minors. Note that our definition differs from that of [4] by a scalar. See also [16,17] for discussions of quantum minors.

The following quantum Laplace expansion can be derived directly from the definition of the quantum minors.

Proposition 3.2.

$$\Delta\big(\underline{a} + \underline{a}', \underline{b}\big) = \sum_{\underline{b} = \underline{c} + \underline{c}'} (-1)^{[\underline{c}]([\underline{a}'] + [\underline{c}'])} (-1)^{\sum_{i>j} (a_i a_j' + a_i c_j' + a_i' c_j + c_i c_j' + c_i c_j')[i]j]}$$
$$\times q^{2(\underline{a}, \underline{a}') - 2(\underline{c}, \underline{c}')} \Delta(\underline{a}, \underline{c}) \Delta\big(\underline{a}', \underline{c}'\big);$$

$$\Delta\big(\underline{a} + \underline{a}', \underline{b}\big)^* = \sum_{\underline{b} = \underline{c} + \underline{c}'} (-1)^{\{\underline{c}\}(\{\underline{a}'\} + \{\underline{c}'\})} (-1)^{\sum_{i>j} (a_i a_j' + a_i c_j' + a_i' c_j + c_i c_j' + c_i c_j')([i]+\bar{1}]([j]+\bar{1})}$$
$$\times q^{2(\underline{a}, \underline{a}') - 2(\underline{c}, \underline{c}')} \Delta(\underline{a}, \underline{c})^* \Delta\big(\underline{a}', \underline{c}'\big)^*.$$

Proof. Consider $\Delta(\underline{a}, \underline{b})$. We have

$$\begin{aligned}
\delta\big(x^{\underline{a}+\underline{a}'}\big) &= (-1)^{\sum_{i>j} a_i a_j' [i][j]} q^{2(\underline{a}, \underline{a}')} \delta\big(x^{\underline{a}}\big) \delta\big(x^{\underline{a}'}\big) \\
&= (-1)^{\sum_{i>j} a_i a_j' [i][j]} q^{2(\underline{a}, \underline{a}')} \left(\sum_{\underline{c}} (-1)^{\sum_{i>j} a_i c_j [i][j] + \sum_{i>j} c_i c_j [i][j]} \Delta(\underline{a}, \underline{c}) \otimes x^{\underline{c}} \right) \\
&\quad \times \left(\sum_{\underline{c}'} (-1)^{\sum_{i>j} a_i' c_j' [i][j] + \sum_{i>j} c_i' c_j' [i][j]} \Delta\big(\underline{a}', \underline{c}'\big) \otimes x^{\underline{c}'} \right)
\end{aligned}$$

$$= \sum_{\underline{b}} \sum_{\underline{c}+\underline{c}'=\underline{b}} (-1)^{[\underline{c}]([\underline{a}']+[\underline{c}'])} (-1)^{\sum_{i>j}(a_i a'_j + a_i c_j + c_i c_j + a'_i c'_j + c'_i c'_j)[i][j]}$$
$$\times q^{2(\underline{a},\underline{a}')-2(\underline{c},\underline{c}')} \Delta(\underline{a},\underline{c}) \Delta(\underline{a}',\underline{c}') \otimes x^{\underline{b}}.$$

The quantum Laplace expansion for $\Delta(\underline{a},\underline{b})^*$ can be proved similarly. □

For any $r, s \in \mathbb{Z}$, $r < s$, denote by $[r,s] = \{r, r+1, \ldots, s\}$. The following quantum minors will play important roles later. Let

$$\det_q A := \Delta\big([1,m],[1,m]\big)^*,$$
$$\det_{q^{-1}} D := \Delta\big([m+1,m+n],[m+1,m+n]\big).$$

Then

$$\det_q A = \sum_{\sigma\in S_m} \big(-q^2\big)^{l(\sigma)} x_{1\sigma(1)} x_{2\sigma(2)} \cdots x_{m\sigma(m)},$$
$$\det_{q^{-1}} D = \sum_{\tau\in S_n} \big(-q^{-2}\big)^{l(\tau)} x_{m+1,m+\tau(1)} x_{m+2,m+\tau(2)} \cdots x_{m+n,m+\tau(n)}.$$

For $r \leqslant \min\{m,n\}$, we have

$$\Delta\big([1,r],[m+1,m+r]\big)^* = \sum_{\sigma\in S_r} \big(-q^2\big)^{l(\sigma)} x_{1,m+\sigma(1)} x_{2,m+\sigma(2)} \cdots x_{r,m+\sigma(r)},$$
$$\Delta\big([m+1,m+r],[1,r]\big) = \sum_{\sigma\in S_r} \big(-q^{-2}\big)^{l(\sigma)} x_{m+1,\sigma(1)} x_{m+2,\sigma(2)} \cdots x_{m+r,\sigma(r)}.$$

As we shall see later, the quantum minors $\det_q A$ and $\Delta([1,r],[m+1,m+r])^*$ are annihilated by every E_i under left translation, while $\det_{q^{-1}} D$ and $\Delta([m+1,m+r],[1,r])$ are annihilated by every F_i under left translation. However, the following minors

$$\Delta\big([1,m],[1,m]\big) = \sum_{\sigma\in S_m} q^{-2l(\sigma)} x_{1\sigma(1)} x_{2\sigma(2)} \cdots x_{m\sigma(m)},$$
$$\Delta\big([m+1,m+n],[m+1,m+n]\big)^*$$
$$= \sum_{\tau\in S_n} q^{-2l(\tau)} x_{m+1,m+\tau(1)} x_{m+2,m+\tau(2)} \cdots x_{m+n,m+\tau(n)}$$

do not behave well under the left and right translations.

Let $\mathcal{S}$ be the multiplicative set of products of powers of $\det_q A$ and $\det_{q^{-1}} D$. Denote by $\mathcal{O}_q(GL_{m|n})$ the localization of $\mathcal{O}_q(M_{m|n})$ at $\mathcal{S}$. Then the inverse matrix X^{-1} of X lies in

$\mathcal{O}_q(GL_{m|n})$, as can be shown by an explicit calculation [18]. We shall always write X^{-1} in the block form

$$X^{-1} = \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix},$$

where $\bar{A}$ is an $m \times m$ q^{-1}-matrix while $\bar{D}$ is a $n \times n$ q-matrix [24]. In [12], the explicit formula for the quantum Berezinian was given:

$$Ber_q = \det_q A \det_q \bar{D},$$

which is also known to be central in $\mathcal{O}_q(GL_{m|n})$.

Remark 3.3. The commutation relations among all of the entries in the matrices X and X^{-1} were given in [24] by using R matrices.

Let us define the following quantum q^{-1}-matrix,

$$D' := (y_{\mu,\nu})_{\mu,\nu=m+1}^{m+n} = D - CA^{-1}B,$$

the entries of which all belong to $\mathcal{O}_q(GL_{m|n})$. Also note that D' is in fact the inverse matrix of $\bar{D}$.

Proposition 3.4. *The following commutation relations hold*:

$$\begin{aligned}
&\det_q Ax_{ij} = x_{ij} \det_q A, \qquad \det_q Ax_{\mu,j} = q^2 x_{\mu,j} \det_q A,\\
&\det_q Ax_{i,\nu} = q^2 x_{i,\nu} \det_q A, \qquad \det_q Ay_{\mu,\nu} = y_{\mu,\nu} \det_q A;\\
&\det_{q^{-1}} D' x_{ij} = x_{ij} \det_{q^{-1}} D', \qquad \det_{q^{-1}} D' x_{\mu,j} = q^2 x_{\mu,j} \det_{q^{-1}} D',\\
&\det_{q^{-1}} D' x_{i,\nu} = q^2 x_{i,\nu} \det_{q^{-1}} D', \qquad \det_{q^{-1}} D' y_{\mu,\nu} = y_{\mu,\nu} \det_{q^{-1}} D'\\
&\quad \text{for } i, j = 1, 2, \ldots, m, \ \mu, \nu = m+1, m+2, \ldots, m+n.
\end{aligned}$$

Moreover, $\det_q A(\det_{q^{-1}} D')^{-1}$ *is central in* $\mathcal{O}_q(GL_{m|n})$.

Proof. The inverse matrix of the quantum matrix A is

$$A^{-1} = \left(\left(-q^2\right)^{j-i} A_{ji} (\det_q A)^{-1}\right),$$

where A_{ji} is the determinant of the quantum matrix obtained from A by deleting the jth row and ith column. Now the claim that $\det_q A$ commutes with $y_{\mu,\nu}$ is equivalent to the relation

$$\det_q Ax_{\mu,\nu} = x_{\mu,\nu} \det_q A + \left(q^2 - q^{-2}\right) \sum_{k,l} \left(-q^2\right)^{l-k} x_{\mu,k} A_{lk} x_{l,\nu}.$$

We use induction on m to prove it. If $m = 1$, we have

$$x_{11}x_{i+1,j+1} = x_{i+1,j+1}x_{11} + \left(q^2 - q^{-2}\right)x_{i+1,1}x_{1,j+1}$$

which is one of the defining relations. In general, by quantum Laplace expansion,

$$\begin{aligned}\det_q A x_{\mu,\nu} &= \sum_s \left(-q^2\right)^{m-s} A_{ms} x_{ms} x_{\mu,\nu} \\ &= \sum_s \left(-q^2\right)^{m-s} A_{ms}\left[x_{\mu,\nu}x_{ms} + \left(q^2 - q^{-2}\right)x_{\mu,s}x_{m,\nu}\right] \\ &= \sum_s \left(-q^2\right)^{m-s} A_{ms} x_{\mu,\nu} x_{ms} \\ &\quad + \left(q^2 - q^{-2}\right)\sum_s \left(-q^2\right)^{s-m} x_{\mu,s} A_{ms} x_{m,\nu}.\end{aligned}$$

Denote by $A_{m,l;s,k}$ the determinant of the quantum matrix obtained from the quantum matrix A by deleting the m, lth rows and s, kth columns. It follows from the induction hypothesis that

$$\begin{aligned}&\sum_s \left(-q^2\right)^{m-s} A_{ms} x_{\mu,\nu} x_{ms} \\ &\quad = \sum_s \left(-q^2\right)^{m-s}\left[A_{ms}x_{\mu,\nu}x_{ms} + \left(q^2 - q^{-2}\right)\sum_{k,l} x_{\mu,k}A_{m,l;s,k}x_{l,\nu}x_{ms}\right] \\ &\quad = x_{\mu,\nu}\det_q A + \left(q^2 - q^{-2}\right)\sum_{k,l} x_{\mu,k}A_{lk}x_{l,\nu}.\end{aligned}$$

Combining this with the last equation, we get the desired formulae.

Note that $\bar{D}$ is the inverse matrix of D'. Hence, $\det_q A$ commutes with all the entries of $\bar{D}$. By the relation given in [24], we can also see that $\det_{q^{-1}} \bar{D}$ commutes with all of the entries in A, and therefore, $\det_{q^{-1}} D' = (\det_{q^{-1}} \bar{D})^{-1}$ commutes with all of the entries in A. The other formulas can be proved similarly. Consequently, $\det_q A(\det_{q^{-1}} D')^{-1}$ is a central element. □

The following result shows that the constructions of the quantized function algebras of the quantum general supergroup in [2,13,24] are in fact equivalent.

Theorem 3.5. *As Hopf algebras,*

$$G_q \cong \mathcal{O}_q(GL_{m|n}).$$

Proof. Identifying $T^{(\lambda)}$ with $L(\lambda)\otimes L(\lambda^{\dagger})$ as $U_q(\mathfrak{g})$-bimodules, we obtain

$$G_q^{\pi} \cong \bigoplus_{\lambda\in\Lambda^{(1)}} L(\lambda)\otimes L\big(\lambda^{\dagger}\big),$$

upon using quantum Peter–Weyl theorem, Proposition 2.2.

By the universal property of $\mathcal{O}_q(M_{m|n})$ [13, Theorem 1.6], there is a surjective homomorphism

$$\Phi:\mathcal{O}_q(M_{m|n})\to G_q^{\pi},\quad x_{ij}\mapsto(-1)^{[i][j]+[j]}q^{(\epsilon_i,\epsilon_i)-(\epsilon_j,\epsilon_j)}t_{ij},\quad i,j=1,2,\ldots,m+n.$$

The map also preserves the co-product and co-unit, as can be easily seen by inspection.

The algebra $\mathcal{O}_q(M_{m|n})$ is $\mathbb{Z}_+$-graded with gradation assignment $\deg x_{ij}=1$. Denote by $\mathcal{O}_q(M_{m|n})_k$ the homogeneous component of degree k. Also note that $G_q^{\pi}=\bigoplus_k(G_q^{\pi})_k$, with

$$\big(G_q^{\pi}\big)_k \cong \bigoplus_{\lambda:|\lambda|=k} L(\lambda)\otimes L\big(\lambda^{\dagger}\big),$$

where $|\lambda|=\sum_a\lambda_a$ for $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_m|\lambda_{m+1},\ldots,\lambda_{m+n})\in\Lambda^{(1)}$. By Proposition 3 of [23], each $L(\lambda)$ has the same dimension as its classical counter part, thus by Proposition 3.3 of [19],

$$\dim\big(G_q^{\pi}\big)_k=\sum_{r=0}^{k}\binom{m^2+n^2+r-1}{r}\binom{2mn}{k-r}$$

which equals to $\dim\mathcal{O}_q(M_{m|n})_k$. Hence, as bialgebras,

$$G_q^{\pi}\cong\mathcal{O}_q(M_{m|n}).$$

The natural embedding of G_q^{π} in G_q leads to an embedding of $\mathcal{O}_q(M_{m|n})$ in G_q.

As an intermediate step to the proof of the theorem, we introduce the localization of $\mathcal{O}_q(M_{m|n})[(\det_q A)^{-1}]$ of $\mathcal{O}_q(M_{m|n})$ at $\det_q A$. Then A is invertible in $\mathcal{O}_q(M_{m|n})[(\det_q A)^{-1}]$, thus the entries of the matrix $D-CA^{-1}B$ all belong to this superalgebra. We shall still denote this matrix by D' by an abuse of notation. Now we localize $\mathcal{O}_q(M_{m|n})[(\det_q A)^{-1}]$ at $\det_{q^{-1}} D'$, and denote the resulting superalgebra by $\mathcal{O}'_q(GL_{m|n})$. Obviously $\mathcal{O}'_q(GL_{m|n})$ is isomorphic to $\mathcal{O}_q(GL_{m|n})$.

Now we want to show that G_q and $\mathcal{O}'_q(GL_{m|n})$ are isomorphic.

In the superalgebra G_q, the quantum supermatrix (t_{ij}) is invertible with the inverse matrix $(\bar{t}_{ij})$. The antipode maps the quantum supermatrix to its inverse matrix. Hence, $\bar{t}_{ij}$'s can be obtained from t_{ij}'s together with $(\det_q A)^{-1}(\det_{q^{-1}} D)^{-1}$. Therefore, we have a surjection from $\mathcal{O}_q(GL_{m|n})$ to G_q, which induces a surjective map $\phi:\mathcal{O}'_q(GL_{m|n})\to G_q$.

Let k belong to the kernel of the map ϕ. Then by using Proposition 3.4 and the fact that $\det_q A$ and $\det_{q^{-1}} D'$ commute, we see that for some positive integer i and a sufficiently large j,

$$(\det_q A)^i \big(\det_{q^{-1}} D'\big)^j k \in \mathcal{O}_q(M_{m|n}).$$

This element, belonging to $\ker\phi$, must vanish. However, the quantum determinants $\det q A$ and $\det_{q^{-1}} D'$ are invertible in $\mathcal{O}'_q(GL_{m|n})$, thus $k = 0$. This proves the injectivity of ϕ, thus establishing that $\mathcal{O}_q(GL_{m|n})$ and G_q are isomorphic as associative superalgebras.

The fact that the co-algebraic structures of $\mathcal{O}_q(GL_{m|n})$ and G_q also coincide follows from computations in [2], which showed that $\Delta(\det_q A)$ and $\Delta(\det_{q^{-1}} D)$ are invertible elements of $G_q \otimes G_q$. $\square$

4. Construction of bases

We shall follow [9] to construct bases for the Hopf superalgebras $\mathcal{O}_q(GL_{m|n})$ and $\mathcal{O}_q(SL_{m|n})$. It was shown in [13] that the ordered monomials form a basis for the superalgebra $\mathcal{O}_q(M_{m|n})$. However, no results seem to be available in the literature on bases for $\mathcal{O}_q(GL_{m|n})$ and $\mathcal{O}_q(SL_{m|n})$. As we have pointed out in the previous section, some quantum minors behave very well under the left and right actions of the quantum enveloping superalgebra $U_q(\mathfrak{g})$. Hence, it is natural to expect that any nice basis of $\mathcal{O}_q(GL_{m|n})$ should contain these quantum minors. Also, if a basis of $\mathcal{O}_q(GL_{m|n})$ is invariant under the multiplication of the quantum Berezinian, then we can get from it a basis for $\mathcal{O}_q(SL_{m|n})$ by setting the quantum Berezinian to 1.

The map defined in the lemma below is inspired by the definition of the anti-automorphism $^-$ in the previous section and hence will be denoted by the same notation. It is a main ingredient for the construction of the dual canonical bases.

Lemma 4.1.

(1) *The mapping*

$$^- : x_{ij} \mapsto x_{ij}, \quad q \mapsto q^{-1}$$

extends to a superalgebra anti-automorphism of $\mathcal{O}_q(M_{m|n})$ regarded as a superalgebra over $\mathbb{Q}$.

(2) *The anti-automorphism $^-$ of $\mathcal{O}_q(M_{m|n})$ extends uniquely to $\mathcal{O}_q(GL_{m|n})$ by requiring*

$$\overline{(\det_q A)^{-1}} = (\det_q A)^{-1}, \qquad \overline{(\det_{q^{-1}} D)^{-1}} = (\det_{q^{-1}} D)^{-1}.$$

The lemma can be proved easily by inspecting the defining relations.

Remark 4.2. The anti-automorphism $^-$ commutes with the isomorphism ϕ in the proof of Theorem 3.5. Indeed, one can directly check that the elements $(-1)^{[i][j]+[j]}q^{(\epsilon_i,\epsilon_i)-(\epsilon_j,\epsilon_j)}t_{ij}$ are bar invariant by using Proposition 2.5 and Lemma 2.6.

Arrange the generators according to the lexicographic order, namely

$$x_{11} < x_{12} < \cdots x_{1,m+n} < x_{21} < \cdots$$
$$< x_{m,1} < x_{m,2} < \cdots < x_{m,m} < \cdots < x_{m+n,m+n}.$$

For any matrix $M = (m_{ij}) \in M_{m+n}(\mathbb{Z}_+)$, $m_{ij} = 0, 1$ if $[i] + [j] = \bar{1}$, we define a monomial x^M by

$$x^M = x_{11}^{m_{11}} x_{12}^{m_{12}} \cdots x_{1,m+n}^{m_{1,m+n}} x_{21}^{m_{21}} \cdots x_{2m+n}^{m_{2,m+n}} \cdots x_{m+n,m+n}^{m_{m+n,m+n}}. \tag{4.1}$$

Observe that the factors are arranged in the lexicographic order.

To construct a basis using Lusztig's method [9], we need to modify the monomials. Define the *normalized monomials*

$$x(M) = q^{-\sum_{i,j<k}(-1)^{[i]}m_{ij}m_{ik} - \sum_{l,s<t}(-1)^{[l]}m_{sl}m_{tl}} x^M.$$

We shall impose a partial order on the set of the normalized monomials by given a partial order to the matrices M in the following way. Let $M = (m_{ij}) \in M_{m+n}(\mathbb{Z}_+)$. If $m_{ij}m_{st} \geqslant 1$ for two pairs of indices i, j and s, t satisfying $i < s$, $j < t$, we define a new matrix $M' = (m'_{uv}) \in M_{m+n}(\mathbb{Z}_+)$ with

$$m'_{ij} = m_{ij} - 1, \qquad m'_{st} = m_{st} - 1,$$
$$m'_{it} = m_{it} + 1, \qquad m'_{sj} = m_{sj} + 1,$$
$$m'_{uv} = m_{uv}, \quad \text{for all other entries.}$$

We say that the matrix M' is obtained from the matrix M by a 2×2 sub-matrix transformation. Using this we may define a partial order on the set $M_{m+n}(\mathbb{Z}_+)$ such that $M < N$ if M can be obtained from N by a sequence of 2×2 sub-matrix transformations.

Given $M = (m_{ij}) \in M_{m+n}(\mathbb{Z}_+)$, we define the row sums $ro(M)$ and the column sums $co(M)$ of the matrix, respectively, by

$$ro(M) = \left(\sum_j m_{1j}, \ldots, \sum_j m_{m+n,j}\right) = \big(r_1(M), r_2(M), \ldots, r_{m+n}(M)\big),$$
$$co(M) = \left(\sum_j m_{j1}, \ldots, \sum_j m_{j,m+n}\right) = \big(c_1(M), c_2(M), \ldots, c_{m+n}(M)\big).$$

Note that the 2×2 sub-matrix transformations keep the row sums and column sums unchanged. Let us also introduce the following notation, which will be frequently used below:

$$\mathbb{M} = \left\{ \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \,\middle|\, \begin{array}{l} M_1 \in M_m(\mathbb{Z}_+),\ M_2 \in M_{m\times n}(\mathbb{Z}_2), \\ M_3 \in M_{n\times m}(\mathbb{Z}_2),\ M_4 \in M_n(\mathbb{Z}_+) \end{array} \right\}.$$

Whenever an element of $\mathbb{M}$ is considered, we assume that it is in this block form. The set $\mathbb{M}$ is viewed as a subset of $M_{m+n}(\mathbb{Z}_+)$, and the partial order $\leqslant$ induces a partial order on $\mathbb{M}$ which is denoted also by $\leqslant$.

The following lemma will be needed when constructing the dual canonical basis. It follows directly from the defining relations of the algebra $\mathcal{O}_q(M_{m|n})$.

Lemma 4.3. *For any $M \in \mathbb{M}$,*

$$\overline{x(M)} = x(M) + \sum_{T<M} c_{M,T}\, x(T),$$

where the coefficients $c_{M,T} \in \mathbb{Z}[q, q^{-1}]$.

Let H the subalgebra of $\mathcal{O}_q(M_{m|n})$ generated by the entries of the matrices A and B, namely, the entries $x_{ij}, i \leqslant m$ of the quantum supermatrix. We construct a basis for the subalgebra.

Theorem 4.4.

(1) *For any $\left(\begin{smallmatrix} M_1 & M_2 \\ 0 & 0 \end{smallmatrix}\right) \in \mathbb{M}$, there exists a unique element $\Omega_q\left(\begin{smallmatrix} M_1 & M_2 \\ 0 & 0 \end{smallmatrix}\right) \in H$ determined by the following conditions*:
  (a) $\overline{\Omega_q\left(\begin{smallmatrix} M_1 & M_2 \\ 0 & 0 \end{smallmatrix}\right)} = \Omega_q\left(\begin{smallmatrix} M_1 & M_2 \\ 0 & 0 \end{smallmatrix}\right)$.
  (b) $\Omega_q\left(\begin{smallmatrix} M_1 & M_2 \\ 0 & 0 \end{smallmatrix}\right) = x\left(\begin{smallmatrix} M_1 & M_2 \\ 0 & 0 \end{smallmatrix}\right) + \sum h^{M_2 M_2'}_{M_1 M_1'} x\left(\begin{smallmatrix} M_1' & M_2' \\ 0 & 0 \end{smallmatrix}\right)$, *where the sum is over* $\left(\begin{smallmatrix} M_1' & M_2' \\ 0 & 0 \end{smallmatrix}\right) \in \mathbb{M}$ *satisfying the condition* $\left(\begin{smallmatrix} M_1' & M_2' \\ 0 & 0 \end{smallmatrix}\right) < \left(\begin{smallmatrix} M_1 & M_2 \\ 0 & 0 \end{smallmatrix}\right)$, *and* $h^{M_2 M_2'}_{M_1 M_1'} \in q\mathbb{Z}[q]$.

(2) *The elements $\Omega_q\left(\begin{smallmatrix} M_1 & M_2 \\ 0 & 0 \end{smallmatrix}\right)$ form a basis of H.*

(3) *The quantum minors $\Delta((i_1, i_2, \ldots, i_r), (j_1, j_2, \ldots, j_r))^*$ with $1 \leqslant i_1 < i_2 < \cdots < i_r$, $1 \leqslant j_1 < j_2 \cdots < j_r \leqslant m+n$ are basis elements.*

Proof. We only need to prove the last statement. For simplicity, we shall only consider the statement for $\Delta([1, r], [s, s+r])^*$. From the definition we can see that $\Delta([1, r], [s, s+r])^*$ are indeed of the form as that in (1)(b). Now, we show the bar invariance of these quantum minors. It is clear that

$$\delta^* \circ \psi = (- \otimes \psi) \circ \delta^*,$$

where ψ is the anti-automorphism of A_q^* fixing all generators and sending q to q^{-1}. Note that both sides are algebra anti-automorphisms so we only need to check the generators. Another fact we need is

$$\psi(\xi_s\xi_{s+1}\cdots\xi_{s+r}) = q^{r(r+1)}\xi_s\xi_{s+1}\cdots\xi_{s+r}.$$

Hence,

$$\begin{aligned}\delta^*\big(\psi(\xi_1\xi_2\cdots\xi_r)\big) &= \overline{\Delta\big([1,r],[s,s+r]\big)^*}\otimes\psi(\xi_s\xi_{s+1}\cdots\xi_{s+r})+\cdots\\ &= q^{r(r+1)}\overline{\Delta\big([1,r],[s,s+r]\big)^*}\otimes\xi_s\xi_{s+1}\cdots\xi_{s+r}+\cdots,\end{aligned}$$

which implies $\overline{\Delta([1,r],[s,s+r])^*} = \Delta([1,r],[s,s+r])^*$. □

For any matrix M, denote by $S(M)$ the sum of all entries in M. The quantum determinant $\det_q A$ of the quantum matrix A is a special quantum minor $\Delta([1,m],[1,m])^*$ which has the following property.

Lemma 4.5.

$$\det_q A\,\Omega_q\begin{pmatrix} M_1 & M_2\\ 0 & 0\end{pmatrix} = q^{S(M_2)}\Omega_q\begin{pmatrix} M_1+I_m & M_2\\ 0 & 0\end{pmatrix}.$$

Proof. It is known that $\det_q A x_{ij} = x_{ij}\det_q A$ for $i, j = 1,2,\ldots,m$. The same argument as Lemma 3.3 in [6] shows that $\det_q A x_{i,\mu} = q^2 x_{i,\mu}\det_q A$, for $\mu = m+1, m+2,\ldots,m+n$. The relations together with part (1) of Theorem 4.4 imply that $q^{-S(M_2)}\det_q A\,\Omega_q\left(\begin{smallmatrix} M_1 & M_2\\ 0 & 0\end{smallmatrix}\right)$ is bar invariant. By Theorem 5.2 of [20], $q^{-S(M_2)}\det_q A\,\Omega_q\left(\begin{smallmatrix} M_1 & M_2\\ 0 & 0\end{smallmatrix}\right)$ is of the form $\Omega_q\left(\begin{smallmatrix} M_1' & M_2'\\ 0 & 0\end{smallmatrix}\right)$ which must be equal to $\Omega_q\left(\begin{smallmatrix} M_1+I_m & M_2\\ 0 & 0\end{smallmatrix}\right)$. □

We can perform similarly analysis for the subalgebra generated by the entries of C to prove the following result.

Theorem 4.6. *For any $(m+n)\times(m+n)$ matrix $\left(\begin{smallmatrix} 0 & 0\\ M_3 & 0\end{smallmatrix}\right)\in\mathbb{M}$, there exists a unique element $\Omega_{q^{-1}}\left(\begin{smallmatrix} 0 & 0\\ M_3 & 0\end{smallmatrix}\right)$ with properties*

(1) $\overline{\Omega_{q^{-1}}\left(\begin{smallmatrix} 0 & 0\\ M_3 & 0\end{smallmatrix}\right)} = \Omega_{q^{-1}}\left(\begin{smallmatrix} 0 & 0\\ M_3 & 0\end{smallmatrix}\right)$.
(2) $\Omega_{q^{-1}}\left(\begin{smallmatrix} 0 & 0\\ M_3 & 0\end{smallmatrix}\right) = x\left(\begin{smallmatrix} 0 & 0\\ M_3 & 0\end{smallmatrix}\right) + \sum_{T_3<M_3} h_{T_3M_3}x\left(\begin{smallmatrix} 0 & 0\\ T_3 & 0\end{smallmatrix}\right)$, *where $h_{T_3M_3}\in q^{-1}\mathbb{Z}[q^{-1}]$.*

The elements $\Omega_{q^{-1}}\left(\begin{smallmatrix} 0 & 0\\ M_3 & 0\end{smallmatrix}\right)$ form a basis of the subalgebra generated by the entries of C. The quantum minors $\Delta((i_1,i_2,\ldots,i_r),(j_1,j_2,\ldots,j_r))$ with $m+1\leqslant i_1<i_2<\cdots<i_r\leqslant m+n$, $1\leqslant j_1<j_2<\cdots<j_r\leqslant n$ are basis elements for $r\leqslant\min\{m,n\}$.

Now we consider the subalgebra generated by the entries of A, B, C. It has a basis

$$\left\{N\begin{pmatrix} M_1 & M_2 \\ M_3 & 0 \end{pmatrix} := q^{-\sum_i c_i(M_1)c_i(M_3)}\Omega_q\begin{pmatrix} M_1 & M_2 \\ 0 & 0 \end{pmatrix}\Omega_{q^{-1}}\begin{pmatrix} 0 & 0 \\ M_3 & 0 \end{pmatrix}\right\},$$

where $c_i(M_1)$ and $c_i(M_3)$ are the ith column sums of M_1 and M_3, respectively. The basis is ordered in accordance with the order of the matrices $\left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & 0 \end{smallmatrix}\right)$.

From the construction, it is clear that

$$N\begin{pmatrix} M_1 & M_2 \\ M_3 & 0 \end{pmatrix} := x\begin{pmatrix} M_1 & M_2 \\ M_3 & 0 \end{pmatrix} + \text{lower terms}.$$

Hence, by Lemma 4.3, we have

Lemma 4.7.

$$\overline{N\begin{pmatrix} M_1 & M_2 \\ M_3 & 0 \end{pmatrix}} = N\begin{pmatrix} M_1 & M_2 \\ M_3 & 0 \end{pmatrix} + \textit{lower terms}.$$

This leads to the following result.

Theorem 4.8. *For any given* $\left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & 0 \end{smallmatrix}\right) \in \mathbb{M}$, *there exists a unique element* $\Omega_q\left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & 0 \end{smallmatrix}\right)$ *determined by the following conditions*:

(1) $\overline{\Omega_q\left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & 0 \end{smallmatrix}\right)} = \Omega_q\left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & 0 \end{smallmatrix}\right)$.

(2) $\Omega_q\left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & 0 \end{smallmatrix}\right) = N\left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & 0 \end{smallmatrix}\right) + \sum h_{M_1M_2M_3}^{M_1'M_2'M_3'} N\left(\begin{smallmatrix} M_1' & M_2' \\ M_3' & 0 \end{smallmatrix}\right)$ *where the summation is over all the matrices* $\left(\begin{smallmatrix} M_1' & M_2' \\ M_3' & 0 \end{smallmatrix}\right) < \left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & 0 \end{smallmatrix}\right)$, *and* $h_{M_1M_2M_3}^{M_1'M_2'M_3'} \in q\mathbb{Z}[q]$.

The elements $\Omega_q\left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & 0 \end{smallmatrix}\right)$ *form a basis of the subalgebra generated by the entries of* A, B, C.

Arguments analogous to [6, Theorem 4.3] show that $\det_q A$ q-commutes with all entries in C. Hence, in a way similar to the proof of Lemma 4.5, we can show that

Lemma 4.9.

$$\det_q A\Omega_q\begin{pmatrix} M_1 & M_2 \\ M_3 & 0 \end{pmatrix} = q^{S(M_2)+S(M_3)}\Omega_q\begin{pmatrix} M_1 + I_m & M_2 \\ M_3 & 0 \end{pmatrix}.$$

Recall the definition of the matrix D'. We have

Proposition 4.10. *The entries of the matrix* $D' = (y_{\mu,\nu}) := D - CA^{-1}B$ *are bar invariant.*

Proof. The entries of D' are of the form:

$$x_{\mu,\nu} - \sum_{k,l=1}^{m} (-q^2)^{l-k} x_{\mu,k} A_{lk} (\det_q A)^{-1} x_{l,\nu},$$

where A_{lk} is the quantum minor obtained from A by deleting the lth row and kth column which is bar invariant by [20] Lemma 3.3.

Since $x_{\mu,\nu}$ and $(\det_q A)^{-1}$ are bar invariant and $(\det_q A)^{-1}$ q-commutes with $x_{\mu,k}$ and $x_{l,\nu}$, we only need to show that

$$d_{ij} := \sum_{k,l=1}^{m} (-q^2)^{l-k} x_{\mu,k} A_{lk} x_{l,\nu}$$

is bar invariant. By repeatedly using quantum Laplace expansion, we have

$$\begin{aligned}\overline{d_{ij}} &= -\sum_{k,l=1}^{m} (-q^2)^{k-l} x_{l,\nu} A_{lk} x_{\mu,k} = -\sum_{k,l=1}^{m} (-q^2)^{m-l} (-q^2)^{m-k} x_{l,\nu} x_{\mu,k} A_{lk} \\ &= \sum_{k,l=1}^{m} (-q^2)^{m-k} (-q^2)^{m-l} x_{\mu,k} x_{l,\nu} A_{lk} = \sum_{k,l=1}^{m} (-q^2)^{m-k} (-q^2)^{l-m} x_{\mu,k} A_{lk} x_{l,\nu} \\ &= d_{ij}. \qquad \square\end{aligned}$$

For any matrix $M = \begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix} \in \mathbb{M}$, we define the ordered monomials $y(M)$ in the same way as $x(M)$. The monomials $y(M)$ form a basis of the subalgebra generated by the entries of the quantum q^{-1}-matrix D'. Using the same method as that in [20], we get a basis of the subalgebra generated by the entries of D'.

Theorem 4.11. *For any matrix* $\begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix} \in \mathbb{M}$*, there exists a unique element* $\Omega_{q^{-1}}\begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix}$ *with the properties*:

(1) $\overline{\Omega_{q^{-1}}\begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix}} = \Omega_{q^{-1}}\begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix}$,

(2) $\Omega_{q^{-1}}\begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix} = y\begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix} + \sum_{T_4 < M_4} h_{T_4 M_4} y\begin{pmatrix} 0 & 0 \\ 0 & T_4 \end{pmatrix}$ *where the coefficients* $h_{T_4 M 4} \in q^{-1}\mathbb{Z}[q^{-1}]$,

the elements $\Omega_{q^{-1}}\begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix}$ *form a basis of the subalgebra generated by the entries of* D'. *In particular, all the quantum minors of the* q^{-1}*-matrix* D' *are basis elements.*

By [20, Proposition 3.6], we have

Proposition 4.12.

$$\det_{q^{-1}} D' \Omega_{q^{-1}} \begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix} = \Omega_{q^{-1}} \begin{pmatrix} 0 & 0 \\ 0 & M_4 + I_n \end{pmatrix}.$$

Now we proceed to the construction of a basis of $\mathcal{O}_q(GL_{m|n})$. For any $a, d \in \mathbb{Z}$, and $M = \left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & M_4 \end{smallmatrix}\right) \in \mathbb{M}$, let

$$\Psi(M; a, d) = \sum_j c_j(M_2)c_j(M_4) - \sum_j r_j(M_3)r_j(M_4) - (a + d)\big(S(M_2) + S(M_3)\big),$$

and define

$$N_{a,d}(M) := q^{\Psi(M;a,d)}(\det_q A)^a \Omega_q \begin{pmatrix} M_1 & M_2 \\ M_3 & 0 \end{pmatrix} \Omega_{q^{-1}} \begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix} \big(\det_q D'\big)^d.$$

Proposition 4.13. *The elements*

$$N_{a,d} \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

form a basis of the algebra $\mathcal{O}_q(GL_{m|n})$, where $a, d \in \mathbb{Z}$, and $\left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & M_4 \end{smallmatrix}\right) \in \mathbb{M}$ satisfies the condition that M_1 and M_4 have at least one zero diagonal entry each.

Proof. It was proved in [13] that the ordered monomials form a basis of the algebra $\mathcal{O}_q(M_{m|n})$. Thus the following set of elements

$$P(M; a, d) := q^{\Psi(M;a,d)}(\det_q A)^a x \begin{pmatrix} M_1 & M_2 \\ M_3 & 0 \end{pmatrix} x \begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix} \big(\det_q D'\big)^d$$

form a basis of $\mathcal{O}_q(GL_{m|n})$, where $a, d \in \mathbb{Z}$, and $M = \left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & M_4 \end{smallmatrix}\right) \in \mathbb{M}$ satisfies the condition that M_1 and M_4 must have at least one zero diagonal entry each. The order of the monomials $x(M)$ induces an order on the above basis, where $P(M; a, d) \geqslant P(M'; a', d')$ if and only if $a < a'$ or $a = a'$ and $d < d'$ or $a = a', d = d'$ but $M \geqslant M'$. The element $N_{a,d}\left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & M_4 \end{smallmatrix}\right)$ can be written as

$$N_{a,d} \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = P(M; a, d) + \text{lower terms}.$$

Hence, the statement follows. □

For the construction of a basis, we shall need the following lemma which is derived directly from the defining relations of the algebra $\mathcal{O}_q(M_{m|n})$.

Lemma 4.14. *For any $M \in \mathbb{M}$ and $a, d \in \mathbb{Z}$,*

$$\overline{N_{a,d}(M)} = N_{a,d}(M) + \sum_{N_{a',d'}(T) < N_{a,d}(M)} c_{a,d,a',d',M,T} N_{a',d'}(T),$$

where $c_{a,d,a',d',M,T} \in \mathbb{Z}[q, q^{-1}]$.

Let $\mathbb{M}^0 := \{M = (m_{ij}) \in \mathbb{M} \mid m_{ii} = m_{\mu\mu} = 0 \text{ for some } i \leqslant m,\ \mu \geqslant m+1\}$. By using the lemma, we can prove the following theorem, which is one of the main results of this paper.

Theorem 4.15. *There is a unique basis B_q^* of $\mathcal{O}_q(GL_{m|n})$ consisting of elements $\Omega_{q,a,d}(M)$ with $M \in \mathbb{M}^0$, and $a, d \in \mathbb{Z}$, which is determined by the following conditions*:

(1) $\overline{\Omega_{q,a,d}(M)} = \Omega_{q,a,d}(M)$ *for all* M.
(2) $\Omega_{q,a,d}(M) = N_{a,d}(M) + \sum_{N_{a',d'}(T)<N_{a,d}(M)} h_{a,a',d,d'}(T,M) N_{a',d'}(T)$, *where* $h_{a,a',d,d'}(T,M) \in q\mathbb{Z}[q]$.

Similarly, there is a unique basis of $\mathcal{O}_q(GL_{m|n})$

$$B_{q^{-1}}^* = \left\{\Omega_{q^{-1},a,d}(M) \mid M \in \mathbb{M}^0,\ a, d \in \mathbb{Z}\right\}$$

determined by the following conditions:

(1) $\overline{\Omega_{q^{-1},a,d}(M)} = \Omega_{q^{-1},a,d}(M)$ *for all* M.
(2) $\Omega_{q^{-1},a,d}(M) = N_{a,d}(M) + \sum_{N_{a',d'}(T)<N_{a,d}(M)} h_{a,a',d,d'}(T,M) N_{a',d'}(T)$, *where* $h_{a,a',d,d'}(T,M) \in q^{-1}\mathbb{Z}[q^{-1}]$.

We shall refer to both B_q^* and $B_{q^{-1}}^*$ as dual canonical bases of $\mathcal{O}_q(GL_{m|n})$. These bases contain the quantum minors in Theorems 4.4, 4.6 and 4.11 by construction. Furthermore, we have the following result.

Theorem 4.16. *The bases B_q^* and $B_{q^{-1}}^*$ are invariant under the multiplication of the quantum Berezinian.*

Proof. Actually, we can show that

$$N_{a,d}(M) Ber_q = N_{a+1,d-1}(M).$$

Write $M = \left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & M_4 \end{smallmatrix}\right)$. Since Ber_q is central, we have

$$\begin{aligned} N_{a,d}\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} Ber_q &= q^{\Psi(M;a,d)} (\det_q A)^a \Omega_q \begin{pmatrix} M_1 & M_2 \\ M_3 & 0 \end{pmatrix} \\ &\quad \times \Omega_{q^{-1}} \begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix} \left(\det_{q^{-1}} D'\right)^d Ber_q \\ &= q^{\Psi(M;a,d)} (\det_q A)^a \Omega_q \begin{pmatrix} M_1 & M_2 \\ M_3 & 0 \end{pmatrix} \\ &\quad \times \det_q A \left(\det_{q^{-1}} D'\right)^{-1} \Omega_{q^{-1}} \begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix} \left(\det_{q^{-1}} D'\right)^d. \end{aligned}$$

Now, the theorem follows from Lemma 4.9 and Proposition 4.12 and an elementary computation of the power of q. $\square$

Setting the quantum Berezinian to 1, we get the superalgebra $\mathcal{O}_q(SL_{m|n})$ of functions on the quantum special linear supergroup $SL_{m|n}$, i.e.

$$\mathcal{O}_q(SL_{m|n}) = \mathcal{O}_q(GL_{m|n})/\langle Ber_q - 1\rangle,$$

where $\langle Ber_q - 1\rangle$ is the ideal of $\mathcal{O}_q(GL_{m|n})$ generated by the central element $Ber_q - 1$. Clearly, we get a basis of $\mathcal{O}_q(SL_{m|n})$ indexed by

$$\left(\mathbb{Z}\begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} + \mathbb{M}^0\right) \cup \left(\mathbb{M}^0 + \mathbb{Z}\begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix}\right).$$

The resulting bases are called dual canonical bases of $\mathcal{O}_q(SL_{m|n})$.

We have proved that $\Delta([1,r],[m+n-r+1,m+n])^*$ and $\Delta([m+n-s+1, m+n],[1,s])$ are dual canonical basis elements for $r, s \leqslant \min\{m,n\}$. We call these quantum minors covariant quantum minors. The same argument as in [5, Theorem 4.3] shows that these covariant quantum minors q-commute with all of the generators x_{ij}. Furthermore, similar to the proof as in [20, Theorem 5.2], we can show that

Theorem 4.17. *The dual canonical basis B_q^* is "invariant" under the multiplication of the covariant minors in the following sense.*

(1) *For any dual canonical basis element $\Omega_{q,a,d}(M)$ (respectively $\Omega_{q^{-1},a,d}(M)$) corresponding to a matrix M such that the $(i, m+n-i)$ entries are zero for all $i = 1, 2, \ldots, r$,*

$$\Omega_{q,a,d}(M)\Delta\big([1,r],[m+n-r+1,m+n]\big)^*$$
$$\Big(\text{respectively } \Omega_{q^{-1},a,d}(M)\Delta\big([1,r],[m+n-r+1,m+n]\big)^*\Big)$$

is also a dual canonical basis element up to a power of q.

(2) *For any dual canonical basis element $\Omega_{q,a,d}(M)$ (respectively $\Omega_{q^{-1},a,d}(M)$) corresponding to a matrix M such that the $(m+n-j, j)$ entries are all zero for all $j = 1, 2, \ldots, s$,*

$$\Omega_{q,a,d}(M)\Delta\big([m+n-s+1,m+n],[1,s]\big)$$
$$\Big(\text{respectively } \Omega_{q^{-1},a,d}(M)\Delta\big([m+n-s+1,m+n],[1,s]\big)\Big)$$

is a dual canonical basis element up to a power of q.

5. Invariant subalgebras

Under the left and right actions of $U_q(\mathfrak{g})$ respectively defined by (2.10) and (2.9), the entries of the matrix D' have the following property.

Lemma 5.1. *For any $\mu, \nu = m+1, m+2, \dots, m+n$ and $i = 1, 2, \dots, m+n-1$,*

$$E_i . y_{\mu+1,\nu} = \delta_{i\mu} y_{\mu,\nu}, \qquad F_i . y_{\mu,\nu} = \delta_{i\mu} y_{\mu+1,\nu},$$
$$y_{\mu,\nu} . F_j = \delta_{j,\nu+1} y_{\mu,\nu-1}, \qquad y_{\mu,\nu} . E_j = \delta_{j\nu} y_{\mu,\nu+1}.$$

Proof. For any $\mu, \nu = m+1, m+2, \dots, m+n$,

$$y_{\mu,\nu} = x_{\mu,\nu} - \sum_{k,l=1}^{m} x_{\mu,k} \left(-q^2\right)^{l-k} A_{lk} (\det_q A)^{-1} x_{l,\nu}.$$

Using the formula for the coproduct,

$$\Delta(E_i) = E_i \otimes K_i K_{i+1}^{-1} + 1 \otimes E_i,$$
$$\Delta(F_i) = F_i \otimes 1 + K_i^{-1} K_{i+1} \otimes F_i,$$

we have

$$E_i . \left(x_{\mu+1,\nu} - \sum_{k,l} x_{\mu+1,k} \left(-q^2\right)^{l-k} A_{lk} (\det_q A)^{-1} x_{l,\nu} \right)$$
$$= \delta_{i\mu} \left(x_{\mu,\nu} - \sum_{k,l} x_{\mu,k} \left(-q^2\right)^{l-k} A_{lk} (\det_q A)^{-1} x_{l,\nu} \right),$$
$$F_i . \left(x_{\mu,\nu} - \sum_{k,l} x_{\mu,k} \left(-q^2\right)^{l-k} A_{lk} (\det_q A)^{-1} x_{l,\nu} \right)$$
$$= \delta_{i\mu} \left(x_{\mu+1,\nu} - \sum_{k,l} x_{\mu+1,k} \left(-q^2\right)^{l-k} A_{lk} (\det_q A)^{-1} x_{l,\nu} \right).$$

The other formulae can be proved similarly. □

Lemma 5.2. *For all i,*

$$E_i . \det_{q^{-1}} D' = \det_{q^{-1}} D' . F_i = 0, \qquad E_i . Ber_q = Ber_q . F_i = 0.$$

Proof. It is easy to check that $E_m . y_{m+1,\nu} = 0$ for all $\nu = m+1, m+2, \dots, m+n$. Indeed,

$$E_m \cdot y_{m+1,\nu} = x_{m,\nu} - \sum_{k,l=1}^{m} x_{m,k}\left(-q^2\right)^{l-k} A_{lk} (\det_q A)^{-1} x_{l,\nu}$$
$$= x_{m,\nu} - \delta_{ml} x_{l,\nu} = 0,$$

which implies that $E_m . \det_{q^{-1}} D' = 0$.

Similarly, we can show that

$$y_{\mu,m+1} . F_{m+1,m} = 0.$$

Clearly, $E_i . \det_{q^{-1}} D' = 0$ for $i \leqslant m$. If $i \geqslant m$, $E_i . \det_{q^{-1}} D' = 0$ is due to the quantum Laplace expansion. It is known that $E_i . \det_q A = 0$ for all i. This together with the formula $Ber_q = \det_q A (\det_{q^{-1}} D')^{-1}$ imply that $E_i . Ber_q = 0$ for all i.

Similarly, we can show that $\det_q A . F_i = 0$, $\det_{q^{-1}} D' . F_i = 0$, $Ber_q . F_i = 0$, for all i. □

We shall employ the dual canonical bases constructed to study invariant subalgebras of $\mathcal{O}_q(GL_{m|n})$ and $\mathcal{O}_q(SL_{m|n})$ under left and right translations. Any subset S of the generators $\{E_i, F_i, K_i^{\pm 1} \mid i = 1, 2, \ldots, m+n-1\}$ generates a subalgebra U_S of $U_q(\mathfrak{g})$.

Definition 5.3. ${}^{L_{U_S}}\mathcal{O}_q(GL_{m|n}) := \{f \in \mathcal{O}_q(SL(n)) \mid x.f = \epsilon(x) f,\ \forall x \in U_S\}$.

It can be easily shown that this is a subalgebra of $\mathcal{O}_q(GL_{m|n})$. It consists of the elements which are invariant under the left action L of U_S. As left and right translations commute, ${}^{L_{U_S}}\mathcal{O}_q(GL_{m|n})$ forms a right $U_q(\mathfrak{g})$-module under R. Thus if T is another subset of the Chevalley generators and the $K_i^{\pm 1}$, we can also consider

$${}^{L_{U_S}}\mathcal{O}_q(GL_{m|n})^{R_{U_T}} := \left\{ f \in {}^{L_S}\mathcal{O}_q(GL_{m|n}) \mid f.x = \epsilon(x) f,\ \forall x \in U_T \right\}.$$

Needless to say, this is a subalgebra of ${}^{L_{U_S}}\mathcal{O}_q(GL_{m|n})$. Below we shall consider in some detail the subalgebras $U_q(n^+)$ and $U_q(n^-)$ (recall that $U_q(n^+)$ (respectively $U_q(n^-)$) is generated by all the E_i's (respectively F_i's)).

Denote by $U_q(\mathfrak{g})_0$ the subalgebra of $U_q(\mathfrak{g})$ generated by all even Chevalley generators and the K_i's. Then

$$U_q(\mathfrak{g})_0 \cong U_q(\mathfrak{gl}_m) \otimes U_q(\mathfrak{gl}_n).$$

Let $U_q(\mathfrak{p})$ be the subalgebra generated by all elements of $U_q(\mathfrak{g})_0$ and E_m. For any integral dominant weight λ, denote by $L_\lambda^{(0)}$ the irreducible right $U_q(\mathfrak{g})_0$ module with highest weight λ. $L_\lambda^{(0)}$ can be extended to a right $U_q(\mathfrak{p})$-module by requiring E_m to act trivially. The Kac module is the induced module

$$K(\lambda) = \mathrm{Ind}_{U_q(\mathfrak{p})}^{U_q(\mathfrak{g})} L_\lambda^{(0)}.$$

Since D' is a q^{-1}-matrix, we can talk about its quantum minors. In particular, we use $\det_{q^{-1}} D'_s$ to denote the quantum minor of the $s \times s$ principal sub-matrix of D', for $s \leqslant n$. We have the following observation.

Proposition 5.4. *The subalgebra of invariants* ${}^{L_{U_q(n^+)}}\mathcal{O}_q(GL_{m|n})^{R_{U_q(n^-)}}$ *is generated by* $Ber_q^{\pm 1}$, *the quantum minors* $\Delta([1,r],[1,r])^*$ *for all* $r = 1, 2, \ldots, m$, *and* $\det_{q^{-1}} D'_s$ *for all* $s = 1, 2, \ldots, n$.

Proof. We can deduce from [25, Theorem 5.2] that

$$^{L_{U_q(n^+)}}\mathcal{O}_q(GL_{m|n}) \cong \bigoplus_{\lambda} K(\lambda),$$

where λ ranges over all integral dominant weights. Hence, the subalgebra ${}^{L_{U_q(n^+)}}\mathcal{O}_q \times (GL_{m|n})^{R_{U_q(n^-)}}$ is spanned by the lowest weight vectors of all of the Kac modules. It is known that the lowest weight of each Kac module is of multiplicity one.

Analogue to the proof of the above lemma, we can see that all these quantum minors $\Delta([1,r],[1,r])^*$ and $\det_{q^{-1}} D'_s$ are $L_{U_q(n^+)} \times R_{U_q(n^-)}$ invariants. In the previous section it was shown that the monomials in the quantum minors $\Delta([1,r],[1,r])^*$, $\det_{q^{-1}} D'_s$, and $Ber_q^{\pm 1}$ are linearly independent. Thus in order to prove our claim, we only need to show that the left weights of these monomials exhaust all of the integral dominant weights.

Note that the left weights of $\Delta([1,r],[1,r])^*$, $\det_{q^{-1}} D'_s$, and Ber_q are respectively given by

$$(\underbrace{1,\ldots,1}_{r},0,\ldots,0 \mid 0,\ldots,0), \qquad (0,\ldots,0 \mid \underbrace{1,\ldots,1}_{s},0,\ldots,0),$$
$$(1,\ldots,1 \mid -1,\ldots,-1).$$

Thus the left weights of their monomials indeed exhaust all the integral dominant weights. □

In the remainder of the paper, we specialize to $n = 1$ to study invariant subalgebras. To this end, we need to have more detailed information on 2×2 quantum minors. For $s \in \mathbb{Z}_+$, we define

$$[s]_{q^2} = \frac{q^{2s}-1}{q^2-1}, \qquad \binom{s}{r}_{q^2} = \frac{[s]_{q^2}[s-1]_{q^2}\cdots[s-r+1]_{q^2}}{[r]_{q^2}[r-1]_{q^2}\cdots[1]_{q^2}}.$$

For any indices $1 \leqslant j < k \leqslant m+1$, denote

$$M_{jk} := x_{1j}x_{2k} - q^2 x_{1k}x_{2j}.$$

The following lemma will be needed to define Kashiwara operators which can be proved similarly as the proof of [21, Lemma 2.7].

Lemma 5.5. *Assume that $i < k \leqslant m$, $j < l \leqslant m$.*

$$\left(x_{ij}x_{kl} - q^2 x_{il}x_{kj}\right)^s = \sum_{m=0}^{s} \left(-q^2\right)^m \binom{s}{m}_{q^4} q^{4m(m-s)} x_{ij}^{s-m} x_{il}^m x_{kj}^m x_{kl}^{s-m},$$

$$\left(x_{ij}x_{m+1,m+1} - q^2 x_{i,m+1}x_{m+1,j}\right)^s = \sum_{m=0}^{s} \left(-q^2\right)^m \binom{s}{m}_{q^4} q^{4m(m-s)} x_{ij}^{s-m} x_{i,m+1}^m x_{m+1,j}^m x_{m+1,m+1}^{s-m}. \tag{5.1}$$

To define the Kashiwara operators $\tilde{E}_1$ and $\tilde{F}_1$, we need an appropriate basis on which the actions of the Kashiwara operators are easy to describe.

Proposition 5.6.

(1) *There exists a basis of the algebra $\mathcal{O}_q(GL_{m|n})$ consisting of the elements of the form*

$$q^l x \begin{pmatrix} 0 & \cdots & 0 & a_{1r} & a_{1,r+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,r-1} & a_{2r} & 0 & \cdots & 0 \end{pmatrix} \prod M_{ij} \prod_{i\geqslant 3, j} x_{ij}^{a_{ij}},$$

where $a_{ij} \in \mathbb{Z}_+$ for all i, j, and the 2×2 quantum minors M_{ij} are of the form $\det_q(\{1,2\}, \{i,j\})$. The product $\prod M_{ij}$ of quantum minors is arranged according to the lexicographic order, namely, $M_{ij} \geqslant M_{st}$ if $j > t$ or $j = t$ and $i \geqslant s$. For given a_{ij}'s and the 2×2 minors M_{ij}, there is a unique choice of integer l redering the following property satisfied.

(2) *The transition matrix between this new basis and the PBW basis consisting of the modified monomials is of the form*:

$$\begin{pmatrix} 1 & \cdots & q\mathbb{Z}[q] \\ 0 & 1 & \cdots \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 1 \end{pmatrix}. \tag{5.2}$$

Note that when examining the actions of $\tilde{E}_1$ and $\tilde{F}_1$ on the new basis, we can ignore the 2×2 minors and those x_{ij} for $i \geqslant 3$ in the expression of the new basis elements. Now the Kashiwara operators $\tilde{E}_1$ and $\tilde{F}_1$ for the left action are defined as follows:

$$\tilde{E}_1 \left(x \begin{pmatrix} 0 & 0 & \cdots & a_{1r} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2r} & \cdots & 0 \end{pmatrix} \right) = \sum_k q^{\sum_{t=1}^{k-1} 2a_{2t}} x \begin{pmatrix} 0 & \cdots & 1 & \cdots & a_{1r} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2k}-1 & \cdots & a_{2r} & \cdots & 0 \end{pmatrix},$$

where the summation is over k such that $a_{2k} \geqslant 1$. Also,

$$\tilde{F}_1\left(x\begin{pmatrix} 0 & \cdots & a_{1r} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2r} & \cdots & 0 \end{pmatrix}\right)$$
$$= \sum_k q^{\sum_{t>k} 2a_{1t}} x\begin{pmatrix} 0 & \cdots & a_{1r} & \cdots & a_{1k}-1 & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2r} & \cdots & 1 & \cdots & 0 \end{pmatrix},$$

where the summation is over k such that $a_{1k} \geqslant 1$. Similarly, we can define the Kashiwara operators $\tilde{E}_i, \tilde{F}_i$ for all $i = 1, 2, \ldots, m+n-1$.

From the definition of Kashiwara operators and the definition of ${}^{L_S}\mathcal{O}_q(G)$, we can show easily that

Lemma 5.7. *If $E_i, F_j \in S$, then*

$$\tilde{E}_i(f) = 0, \qquad \tilde{F}_j(f) = 0, \quad \forall f \in {}^{L_S}\mathcal{O}_q(GL_{m|n}),$$

where $\tilde{E}_i$ and $\tilde{F}_j$ are the Kashiwara operators associated with E_i and F_j.

In the following, we let S be any subset of

$$\left\{E_i, F_i, K_i^{\pm 1} \mid i = 1, 2, \ldots, m+n-1\right\} \backslash \{F_m\},$$

and consider the subalgebra of invariants with respect to S. We have the following result.

Theorem 5.8. *The subalgebra of invariants ${}^{L_S}\mathcal{O}_q(GL_{m|1})$ is spanned by a part of the dual canonical basis B_q^*.*

Proof. When $n = 1$, the entries of the matrix C q-commute with each other and so the elements $\Omega_{q^{-1}}\begin{pmatrix} 0 & 0 \\ M_3 & 0 \end{pmatrix} = x\begin{pmatrix} 0 & 0 \\ M_3 & 0 \end{pmatrix}$, for all row vectors M_3 form a basis for the subalgebra generated by entries of C. Furthermore, the basis

$$\left\{N\begin{pmatrix} M_1 & M_2 \\ M_3 & 0 \end{pmatrix} := q^{-\sum_i c_i(M_1)c_i(M_3)} \Omega_q\begin{pmatrix} M_1 & M_2 \\ 0 & 0 \end{pmatrix} \Omega_{q^{-1}}\begin{pmatrix} 0 & 0 \\ M_3 & 0 \end{pmatrix}\right\},$$

of the subalgebra generated by the entries of the matrices A, B and C is related to the basis $\left\{x\begin{pmatrix} M_1 & M_2 \\ M_3 & 0 \end{pmatrix}\right\}$ by a transition matrix of the form (5.2). Therefore, the basis elements $\Omega_q\begin{pmatrix} M_1 & M_2 \\ M_3 & 0 \end{pmatrix}$ can be expressed as $\mathbb{Z}[q]$ combinations of the monomials $x\begin{pmatrix} M_1' & M_2' \\ M_3' & 0 \end{pmatrix}$.

The matrix D has only one element. Thus the basis $\left\{\Omega_{q,a,d}\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}\right\}$ can be constructed as $\mathbb{Z}[q]$ combinations of elements of the monomial basis

$$q^{-(a+d)(S(M_2)+S(M_3))}(\det_q A)^a x\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}\left(\det_{q^{-1}} D'\right)^d,$$
$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \in \mathbb{M}, \ a, d \in \mathbb{Z}.$$

Now, the same argument as in [21] shows that the subalgebra of invariants is spanned by a part of the dual canonical basis. □

Now we consider the invariants with respect to the subalgebra $U_q(n_+)$ generated by all $E_1, E_2, \dots, E_m$, we deduce that

Theorem 5.9. *Any Kac module is spanned by a part of the dual canonical basis.*

Proof. By [25, Theorem 5.2], ${}^{L_{U_q(n^+)}}\mathcal{O}_q(GL_{m|n}) \cong \oplus_\lambda K(\lambda)$, where λ range over all integral dominant weights. Using Theorem 5.8 and consider the left weight space of weight λ, we get a basis of $K(\lambda)$. □

In the case of $GL_{1|1}$, the basis elements are given as:

$$q^{(d-a)(b+c)} x_{11}^a x_{12}^b x_{21}^c \left(x_{22} + q^2 x_{12} x_{11}^{-1} x_{21}\right)^d,$$

where $a, d \in \mathbb{Z}$ and $b, c \in \mathbb{Z}_+$.

In case of $GL_{2|1}$, by Theorems 4.16, 4.17 and the computation for $GL_{1|1}$, we only need to consider the basis elements parametrized by matrices

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where $a \geqslant 1$ which can be computed easily. Applying the left action of $U_q(n^+)$, we get the subalgebra ${}^{L_{U_q(n^+)}}\mathcal{O}_q(GL_{2|1})$ which is spanned by

$$\begin{aligned}
&q^{-aa_{21}-ba_{12}-a_{11}a_{12}-a_{11}a_{21}-a\alpha-b\alpha+a\beta+b\beta} (\det_q A)^\alpha x_{11}^{a_{11}} x_{12}^{a_{12}} w^a x_{13}^b {x'_{33}}^\beta,\\
&q^{-2\alpha-a_{11}a_{12}-a_{11}-a_{12}+1+2\beta} (\det_q A)^\alpha x_{11}^{a_{11}} x_{12}^{a_{12}} x_{13} x_{23} {x'_{33}}^\beta,\\
&q^{-a_{11}a_{12}-a_{12}-2\alpha+2\beta+1} (\det_q A)^\alpha x_{13} w^a {x'_{33}}^\beta,\\
&q^{-a\alpha-\alpha+\beta} x_{11}^{a_{11}} \left(x_{11}x_{23} - q^2 x_{13}x_{21}\right)^a x_{13}^b,
\end{aligned}$$

where $a, b = 0, 1$, a_{ij} are nonnegative integers and α, β are integers. Apply the right actions of $R_{U_q(n^-)}$, we see that the subalgebra ${}^{L_{U_q(n^+)}}\mathcal{O}_q(GL_{2|1})^{R_{U_q(n^-)}}$ is spanned by the following elements:

$$\begin{aligned}
&q^{\beta-l} x_{12}^l w^a x_{13} {x'_{33}}^\beta, \qquad q^{b\beta} w^a x_{13}^b {x'_{33}}^\beta, \quad a, b = 0, 1,\ l \geqslant 1,\\
&q^{2\beta+1-l} x_{12}^l x_{13} x_{23} {x'_{33}}^\beta, \qquad q^{-l} x_{12}^l x_{13} {x'_{33}}^\beta w^a.
\end{aligned}$$

Remark 5.10. The algebra ${}^{L_{U_q(n^+)}}\mathcal{O}_q(GL_{2|1})^{R_{U_q(n^-)}}$ is not finitely generated. Indeed, any generating set of the algebra should contain the elements $x_{12}^l x_{13}$ for all $l \geqslant 1$.

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References

[1] G. Benkart, S.-J. Kang, M. Kashiwara, Crystal bases for the quantum superalgebra $U^q(\mathfrak{gl}(m,n))$, J. Amer. Math. Soc. 13 (2) (2000) 295–331.
[2] R. Fioresi, On algebraic supergroups and quantum deformations, J. Algebra Appl. 2 (4) (2003) 403–423.
[3] M.D. Gould, R.B. Zhang, Classification of all star irreps of $\mathfrak{gl}(m \mid n)$, J. Math. Phys. 31 (1990) 2552–2559.
[4] R.Q. Huang, J.J. Zhang, Standard basis theorem for quantum linear groups, Adv. Math. 102 (2) (1993) 202–229.
[5] H.P. Jakobsen, H. Zhang, The center of the quantized matrix algebra, J. Algebra 196 (2) (1997) 458–474.
[6] H.P. Jakobsen, H. Zhang, The exponential nature and positivity, preprint, 2005.
[7] M. Kashiwara, On crystal bases of the Q-analogue of universal enveloping algebras, Duke Math. J. 63 (2) (1991) 465–516.
[8] M. Kashiwara, Global crystal bases of quantum groups, Duke Math. J. 69 (2) (1993) 455–485.
[9] G. Lusztig, Left cells in Weyl groups, in: Lie Group Representations, I, College Park, MD, 1982/1983, in: Lecture Notes in Math., vol. 1024, Springer-Verlag, Berlin, 1983, pp. 99–111.
[10] G. Lusztig, Canonical bases arising from quantized enveloping algebras, J. Amer. Math. Soc. 3 (2) (1990) 447–498.
[11] G. Lusztig, Introduction to Quantum Groups, Progr. Math., vol. 110, Birkhäuser Boston, Boston, MA, 1993.
[12] V. Lyubashenko, A. Sudbery, Quantum supergroups of $GL(n \mid m)$ type: Differential forms, Koszul complexes, and Berezinians, Duke Math. J. 90 (1) (1997) 1–62.
[13] I.Yu. Manin, Multiparametric quantum deformation of the general linear supergroup, Comm. Math. Phys. 123 (1) (1989) 163–175.
[14] I.Yu. Manin, Topics in Noncommutative Geometry, M.B. Porter Lectures, Princeton Univ. Press, Princeton, NJ, 1991.
[15] I.M. Musson, Y.M. Zou, Crystal bases for $U_q(\mathrm{osp}(1,2r))$, J. Algebra 210 (1998) 514–534.
[16] M. Noumi, H. Yamada, K. Mimachi, Finite-dimensional representations of the quantum group $GL_q(n;C)$ and the zonal spherical functions on $U_q(n-1)\backslash U_q(n)$, Japan. J. Math. (N.S.) 19 (1) (1993) 31–80.
[17] B. Parshall, J.P. Wang, Quantum linear groups, Mem. Amer. Math. Soc. 89 (1991) 439, vi+157 pp.
[18] H.H. Phung, On the structure of quantum super groups $GL_q(m \mid n)$, J. Algebra 211 (2) (1999) 363–383.
[19] M. Scheunert, R.B. Zhang, The general linear supergroup and its Hopf superalgebra of regular functions, J. Algebra 254 (1) (2002) 44–83.
[20] H. Zhang, On dual canonical bases, J. Phys. A 37 (32) (2004) 7879–7893.
[21] H. Zhang, R.B. Zhang, Dual canonical bases for the quantum special linear group and invariant subalgebras, Lett. Math. Phys. 73 (2005) 165–181.
[22] R.B. Zhang, Universal L-operator and invariants of the quantum supergroup $U_q(\mathfrak{gl}(m \mid n))$, J. Math. Phys. 33 (1992) 1970–1979.
[23] R.B. Zhang, Finite-dimensional irreducible representations of the quantum supergroup $U_q(\mathfrak{gl}(m \mid n))$, J. Math. Phys. 34 (3) (1993) 1236–1254.
[24] R.B. Zhang, Structure and representations of the quantum general linear supergroup, Comm. Math. Phys. 195 (3) (1998) 525–547.
[25] R.B. Zhang, Quantum superalgebra representations on cohomology groups of non-commutative bundles, J. Pure Appl. Algebra 191 (3) (2004) 285–314.
[26] Y.M. Zou, On the structure of $U_q(sl(m,1))$: Crystal bases, J. Phys. A 32 (46) (1999) 8197–8207.
[27] Y.M. Zou, Crystal bases for $U_q(\Gamma(\sigma_1,\sigma_2,\sigma_3))$, Trans. Amer. Math. Soc. 353 (2001) 3789–3802.