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Perturbation of frame sequences and its applications to shift-invariant spaces ☆

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Abstract

We generalize the main result in [O. Christensen, H.O. Kim, R.Y. Kim, J.K. Lim, Perturbation of frame sequences in shift-invariant spaces, J. Geom. Anal. 15 (2005) 181–191] in order to make it comparable with existing results. Then we compare the special cases of the three results in the literature in the setting of the perturbation of the generating sets of finitely generated shift-invariant spaces of $L^2(\mathbb{R}^d)$.

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1. Introduction

There are, at least, three results on the perturbation of frame sequences in a Hilbert space in the literature [9,11,12]. The statements of the main results in [9,12] (cf. Propositions 1.2 and 1.3) involve three parameters and some geometric conditions, whereas the statement of the main result in [11] involves only one parameter and some geometric conditions. In this article, we generalize the main result in [11] in order to make it comparable with other results in the literature. Moreover,

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we improve the Bessel bound in the main results in [9,12] (see (2.1)). Then, we compare the special cases of the three results (all involving only one parameter but widely used in practical applications to wavelet and exponential frames [1,13]) in the setting of the perturbation of the generating sets of finitely generated shift-invariant spaces of $L^2(\mathbb{R}^d)$ [16,21]. In particular, we show that, in this setting, the result in [11] is more general than those in [9,12] (Proposition 3.9).

We first recall some basic facts about frames and frame sequences which will be needed in this article. Throughout this article $\mathscr{H}$ denotes a separable Hilbert space over the complex field $\mathbb{C}$. Let I be a countable index set. A sequence $F := \{f_i\}_{i\in I}$ in $\mathscr{H}$ is said to be a *Bessel sequence* if there exists a positive constant B, called a *Bessel bound*, such that, for each $f \in \mathscr{H}$, $\sum_{i\in I} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2$. The infimum of Bessel bounds, which is known to be a Bessel bound, is called the *optimal Bessel bound*. F is said to be *frame* for $\mathscr{H}$ if there exist positive constants A and B, called a *lower* and an *upper frame bound*, respectively, such that, for each $f \in \mathscr{H}$, $A\|f\|^2 \leqslant \sum_{i\in I} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2$. The supremum of lower frame bounds and the infimum of upper frame bounds, which are known to be a lower frame bound and an upper frame bound, are called the *optimal lower frame bound* and the *optimal upper frame bound*, respectively. If the above equalities hold only for each $f \in \overline{\operatorname{span}}\, F$, then F is called a *frame sequence*. For any sequence $F := \{f_i\}_{i\in I} \subset \mathscr{H}$, its *pre-frame operator* $T_F : \ell^2(I) \to \mathscr{H}$ is defined to be $T_F c := \sum_{i\in I} c(i) f_i$, which is defined, at least, for each finitely supported c. Then F is a Bessel sequence if and only if T_F is bounded. In this case, the optimal Bessel bound is $\|T_F\|^2$. It is direct to see that $T_F^* f = (\langle f, f_i\rangle)_{i\in I}$ for $f \in \mathscr{H}$. Moreover, F is a frame for $\mathscr{H}$ if and only if T_F is bounded and onto, and it is a frame sequence if and only if T_F is bounded and has closed range [8,17]. In this case, the optimal lower frame bound is $\|T_F^\dagger\|^{-2}$ and the optimal upper frame bound is $\|T_F\|^2$, where $T_F^\dagger$ denotes the pseudo-inverse of the bounded operator T_F with closed range [14]. Finally, if there exist positive constants A and B, called *Riesz bounds* such that, for each finitely supported $c \in \ell^2(I)$, $A\|c\|^2 \leqslant \|\sum_{i\in I} c(i) f_i\|^2 \leqslant B\|c\|^2$, then F is said to be a *Riesz sequence*. It is direct to see that F is a Riesz sequence if and only if $T_F : \ell^2(I) \to \overline{\operatorname{span}}\, F$ is bounded and invertible. If F is complete in $\mathscr{H}$, then F is said to be a *Riesz basis* for $\mathscr{H}$. It is known that a Riesz sequence is a frame sequence, and that a Riesz basis is a bounded unconditional basis for $\mathscr{H}$. We refer to [10,15,23] for the theory of frames and Riesz bases and their numerous applications to various branches of Mathematics.

In this article we are interested in the problem of finding conditions under which the perturbation of a frame sequence is also a frame sequence. The following result [5, Theorem 2] is one of the most general and also typical results about frame perturbations for the whole space $\mathscr{H}$ which generalizes the main results in [6,7].

Proposition 1.1 [5]. *Let $F := \{f_i\}_{i\in I}$ be a frame for $\mathscr{H}$ with bounds A and B, and $G := \{g_i\}_{i\in I}$ a sequence in $\mathscr{H}$. Suppose that there exist non-negative $\lambda_1, \lambda_2, \mu$ with $\lambda_2 < 1$ such that*

$$\left\|\sum_{i\in I} c(i)(f_i - g_i)\right\| \leqslant \lambda_1 \left\|\sum_{i\in I} c(i) f_i\right\| + \lambda_2 \left\|\sum_{i\in I} c(i) g_i\right\| + \mu\|c\| \tag{1.1}$$

for each finitely supported $c \in \ell^2(I)$, and

$$\lambda_1 + \frac{\mu}{\sqrt{A}} < 1. \tag{1.2}$$

Then G is a frame for $\mathscr{H}$ with bounds

$$A\left(1-\frac{\lambda_1+\lambda_2+\mu/\sqrt{A}}{1+\lambda_2}\right)^2 \quad \textit{and} \quad B\left(1+\frac{\lambda_1+\lambda_2+\mu/\sqrt{B}}{1-\lambda_2}\right)^2.$$

For the perturbation of frame sequences we need geometric conditions apart from (1.1) and (1.2). Before stating the conditions we review some concepts which play important roles in our discussion. Let X and Y be closed subspaces of $\mathscr{H}$. Define

$$R(X,Y) := \inf_{x\in X\setminus\{0\}} \frac{\|P_Y x\|}{\|x\|}, \quad S(X,Y) := \sup_{x\in X\setminus\{0\}} \frac{\|P_Y x\|}{\|x\|} = \|P_Y|_X\|,$$

where P_Y denotes the orthogonal projection onto Y and $P_Y|_X$ its restriction to X. $R(X,Y)$ and $S(X,Y)$ are called the *infimum* and *supremum cosine angle between X and Y*, respectively [22]. R is not symmetric, whereas S is symmetric [22]. They satisfy the following relations: $S(X,Y) = (1 - R(X,Y^\perp)^2)^{1/2}$. It is known that $R(X,Y) = R(Y^\perp, X^\perp)$ [22]. We use the convention that $R(\{0\},Y) = 1$ and $S(\{0\},Y) = 0$ for any closed subspace Y. We mention only one geometric meaning of the infimum cosine angle. By definition, $\|P_Y x\| \geqslant R(X,Y)\|x\|$ for any $x \in X$. Suppose that $R(X,Y) > 0$. Then, $P_Y|_X$ is bounded below. In particular, $P_Y|_X$ is one-to-one. Moreover, it is direct to see that $(P_Y|_X)^* = P_X|_Y$ if we consider P_Y as an operator from X to Y. Hence, $P_X|_Y$ is onto. The *gap* $\delta(X,Y)$ between non-trivial X and Y is defined to be $\delta(X,Y) := \sup_{x\in X,\|x\|=1} \operatorname{dist}(x,Y)$ [18]. Note that

$$\delta(X,Y) = \sup_{x\in X,\|x\|=1} \inf_{y\in Y} \|x-y\| = \sup_{x\in X,\|x\|=1} \|x - P_Y x\| = \sup_{x\in X,\|x\|=1} \|P_{Y^\perp} x\| = \|P_{Y^\perp}|_X\|.$$

Therefore, $\delta(X,Y) = S(X,Y^\perp) = (1 - R(X,Y)^2)^{1/2}$. These equalities enable us to define gaps between possibly trivial subspaces.

We now state the first known result [9, Theorem 3.2] about perturbation of frame sequences involving the infimum cosine angle between the kernels of the pre-frame operators. It was originally stated in terms of the gap between the kernels of the pre-frame operators.

Proposition 1.2 [9]. *Let $F := \{f_i\}_{i\in I} \subset \mathscr{H}$ be a frame sequence with bounds A and B, and $G := \{g_i\}_{i\in I}$ a sequence in $\mathscr{H}$. Let T_F and T_G be the pre-frame operators of F and G, respectively. Suppose that there exist non-negative $\lambda_1, \lambda_2, \mu$ with $\lambda_2 < 1$ such that (1.1) is satisfied for each finitely supported $c \in \ell^2(I)$. Then G is a Bessel sequence with a Bessel bound*

$$B\left(1+\frac{\lambda_1+\lambda_2+\mu/\sqrt{B}}{1-\lambda_2}\right)^2. \tag{1.3}$$

Moreover, if

$$R(\ker T_F, \ker T_G) > 0, \quad \lambda_1 + \frac{\mu}{\sqrt{A}R(\ker T_F, \ker T_G)} < 1, \tag{1.4}$$

then G is a frame sequence with a lower frame bound

$$AR(\ker T_F, \ker T_G)\left(1-\frac{\lambda_1+\lambda_2+\mu\Big/\Big[\sqrt{A}R(\ker T_F,\ker T_G)\Big]}{1+\lambda_2}\right)^2.$$

The following result [12, Theorem 3.1] involves the infimum cosine angle between the ranges of the pre-frame operators.

Proposition 1.3 [12]. *Let $F := \{f_i\}_{i\in I} \subset \mathscr{H}$ be a frame sequence with bounds A and B, and $G := \{g_i\}_{i\in I}$ a sequence in $\mathscr{H}$. Let $\mathscr{H}_F := \overline{\operatorname{span}}\, F$ and $\mathscr{H}_G := \overline{\operatorname{span}}\, G$. Suppose that there exist non-negative $\lambda_1, \lambda_2, \mu$ with $\lambda_2 < 1$ such that (1.1) is satisfied for each finitely supported $c \in \ell^2(I)$. Then G is a Bessel sequence with a Bessel bound (1.3). If*

$$\lambda_1 + \frac{\mu}{\sqrt{A}} < R(\mathscr{H}_G, \mathscr{H}_F), \tag{1.5}$$

then G is a frame sequence with a lower frame bound

$$A\left(1 - \frac{\lambda_1 + \lambda_2 + \mu/\sqrt{A}}{1+\lambda_2}\right)^2. \tag{1.6}$$

Moreover, $\mathscr{H}_F$ is isomorphic to $\mathscr{H}_G$ and $\mathscr{H}_F^\perp$ is isomorphic to $\mathscr{H}_G^\perp$.

In Section 2 we improve the Bessel (upper frame) bound in previous propositions.

2. Main result

We state and prove another result about the perturbation of frame sequences involving the infimum cosine angle between the ranges of the pre-frame operators which generalizes [11, Theorem 2.1].

Theorem 2.1. *Let $F := \{f_i\}_{i\in I} \subset \mathscr{H}$ be a frame sequence with bounds A and B, and $G := \{g_i\}_{i\in I}$ a sequence in $\mathscr{H}$. Let $\mathscr{H}_F := \overline{\operatorname{span}}\, F$ and $\mathscr{H}_G := \overline{\operatorname{span}}\, G$. Suppose that there exist non-negative $\lambda_1, \lambda_2, \mu$ with $\lambda_2 < 1$ such that (1.1) is satisfied for each finitely supported $c \in \ell^2(I)$. Then G is a Bessel sequence with a Bessel bound*

$$B\left(S(\mathscr{H}_G, \mathscr{H}_F) + \frac{\lambda_1 + S(\mathscr{H}_G, \mathscr{H}_F)\lambda_2 + \mu/\sqrt{B}}{1-\lambda_2}\right)^2. \tag{2.1}$$

If

$$\sqrt{B}\left(\frac{\lambda_1}{1-\lambda_2} + S(\mathscr{H}_G, \mathscr{H}_F)\frac{\lambda_2}{1-\lambda_2}\right) + \frac{\mu}{1-\lambda_2} < \sqrt{A}, \tag{2.2}$$

then $R(\mathscr{H}_F, \mathscr{H}_G) > 0$. If, in addition to (2.2),

$$R(\mathscr{H}_G, \mathscr{H}_F) > 0, \tag{2.3}$$

then G is a frame sequence with a lower frame bound

$$A\left\{1 - \left[\sqrt{\frac{B}{A}}\left(\frac{\lambda_1}{1-\lambda_2} + S(\mathscr{H}_G, \mathscr{H}_F)\frac{\lambda_2}{1-\lambda_2}\right) + \frac{1}{\sqrt{A}}\frac{\mu}{1-\lambda_2}\right]\right\}^2. \tag{2.4}$$

Moreover, $P_{\mathscr{H}_G}|_{\mathscr{H}_F}$ is an isomorphism from $\mathscr{H}_F$ onto $\mathscr{H}_G$.

Proof. Note that (1.1) implies that $\|T_F c - T_G c\| \leqslant \lambda_1\|T_F c\| + \lambda_2\|T_G c\| + \mu\|c\|$ for each finitely supported $c \in \ell^2(I)$. Since $\lambda_2 < 1$ and $\|T_F\| \leqslant \sqrt{B}$,

$$\|T_G\| \leqslant \sqrt{B}\left(1+\frac{\lambda_1+\lambda_2+\mu/\sqrt{B}}{1-\lambda_2}\right).$$

We give a sharper estimate of $\|T_G\|$. First, note that (1.1) implies that $\{f_i - g_i\}_{i\in I}$ is a Bessel sequence in $\mathscr{H}$ with a Bessel bound less than or equal to

$$\left(\lambda_1\sqrt{B}+\lambda_2\|T_G\|+\mu\right)^2. \tag{2.5}$$

For $g \in \mathscr{H}_G$ we have

$$\begin{aligned}
\sum_{i\in I}|\langle g, g_i\rangle|^2 &= \sum_{i\in I}|\langle g, f_i\rangle - \langle g, f_i - g_i\rangle|^2\\
&= \sum_{i\in I}|\langle g, f_i\rangle|^2 + \sum_{i\in I}|\langle g, f_i - g_i\rangle|^2 - 2\Re\sum_{i\in I}\langle g, f_i\rangle\overline{\langle g, f_i - g_i\rangle}\\
&\leqslant \sum_{i\in I}|\langle g, f_i\rangle|^2 + \sum_{i\in I}|\langle g, f_i - g_i\rangle|^2 + 2\sqrt{\sum_{i\in I}|\langle g, f_i\rangle|^2}\sqrt{\sum_{i\in I}|\langle g, f_i - g_i\rangle|^2}\\
&= \left(\sqrt{\sum_{i\in I}|\langle g, f_i\rangle|^2} + \sqrt{\sum_{i\in I}|\langle g, f_i - g_i\rangle|^2}\right)^2\\
&= \left(\sqrt{\sum_{i\in I}|\langle P_{\mathscr{H}_F}g, f_i\rangle|^2} + \sqrt{\sum_{i\in I}|\langle g, f_i - g_i\rangle|^2}\right)^2\\
&\leqslant \left(\sqrt{B}\|P_{\mathscr{H}_F}g\| + (\lambda_1\sqrt{B}+\lambda_2\|T_G\|+\mu)\|g\|\right)^2\\
&\leqslant \left(\sqrt{B}S(\mathscr{H}_G,\mathscr{H}_F) + \lambda_1\sqrt{B}+\lambda_2\|T_G\|+\mu\right)^2\|g\|^2.
\end{aligned}$$

This shows that

$$\|T_G\| = \|T_G^*\| \leqslant \sqrt{B}S(\mathscr{H}_G,\mathscr{H}_F) + \lambda_1\sqrt{B}+\lambda_2\|T_G\|+\mu.$$

Therefore,

$$\begin{aligned}
\|T_G\| &\leqslant \frac{\sqrt{B}S(\mathscr{H}_G,\mathscr{H}_F)+\lambda_1\sqrt{B}+\mu}{1-\lambda_2}\\
&= \sqrt{B}\,\frac{S(\mathscr{H}_G,\mathscr{H}_F)+\lambda_1+\mu/\sqrt{B}}{1-\lambda_2}\\
&= \sqrt{B}\left(S(\mathscr{H}_G,\mathscr{H}_F)+\frac{\lambda_1+S(\mathscr{H}_G,\mathscr{H}_F)\lambda_2+\mu/\sqrt{B}}{1-\lambda_2}\right),
\end{aligned}$$

which shows that (2.1) is a Bessel bound. For notational convenience we let β to be (2.1), which is the square of the last term.

(2.5) implies that a Bessel bound of $\{f_i - g_i\}_{i\in I}$ is $(\lambda_1\sqrt{B} + \lambda_2\sqrt{\beta} + \mu)^2$. Let $f \in \mathscr{H}_F \setminus \{0\}$.

$$\begin{aligned}\beta\|P_{\mathscr{H}_G} f\|^2 &\geqslant \sum_{i\in I} |\langle P_{\mathscr{H}_G} f, g_i\rangle|^2 = \sum_{i\in I} |\langle f, g_i\rangle|^2 \qquad (2.6)\\ &\geqslant \left(\sqrt{\sum_{i\in I} |\langle f, f_i\rangle|^2} - \sqrt{\sum_{i\in I} |\langle f, f_i - g_i\rangle|^2}\right)^2\\ &\geqslant \left(\sqrt{A} - (\lambda_1\sqrt{B} + \lambda_2\sqrt{\beta} + \mu)\right)^2 \|f\|^2. \qquad (2.7)\end{aligned}$$

This shows that

$$R(\mathscr{H}_F, \mathscr{H}_G) \geqslant \frac{\sqrt{A} - (\lambda_1\sqrt{B} + \lambda_2\sqrt{\beta} + \mu)}{\sqrt{\beta}},$$

which is strictly positive if

$$\begin{aligned}\sqrt{A} &> \lambda_1\sqrt{B} + \lambda_2\sqrt{\beta} + \mu\\ &= \lambda_1\sqrt{B} + \lambda_2\sqrt{B}\left(S(\mathscr{H}_G, \mathscr{H}_F) + \frac{\lambda_1 + S(\mathscr{H}_G, \mathscr{H}_F)\lambda_2 + \mu/\sqrt{B}}{1-\lambda_2}\right) + \mu\\ &= \lambda_1\sqrt{B} + \lambda_2\sqrt{B}\left(S(\mathscr{H}_G, \mathscr{H}_F) + \frac{\lambda_1 + S(\mathscr{H}_G, \mathscr{H}_F)\lambda_2}{1-\lambda_2}\right) + \frac{\mu\lambda_2}{1-\lambda_2} + \mu\\ &= \sqrt{B}\left(\lambda_1 + \lambda_2 S(\mathscr{H}_G, \mathscr{H}_F) + \lambda_2\frac{\lambda_1 + S(\mathscr{H}_G, \mathscr{H}_F)\lambda_2}{1-\lambda_2}\right) + \frac{\mu}{1-\lambda_2}\\ &= \sqrt{B}\left(\lambda_1 + \lambda_2 S(\mathscr{H}_G, \mathscr{H}_F) + \frac{\lambda_1\lambda_2}{1-\lambda_2} + S(\mathscr{H}_G, \mathscr{H}_F)\frac{\lambda_2^2}{1-\lambda_2}\right) + \frac{\mu}{1-\lambda_2}\\ &= \sqrt{B}\left(\frac{\lambda_1}{1-\lambda_2} + S(\mathscr{H}_G, \mathscr{H}_F)\frac{\lambda_2}{1-\lambda_2}\right) + \frac{\mu}{1-\lambda_2}.\end{aligned}$$

Hence if (2.2) is satisfied, then $R(\mathscr{H}_F, \mathscr{H}_G) > 0$. Moreover, our calculation shows that $\sqrt{A} - (\lambda_1\sqrt{B} + \lambda_2\sqrt{\beta} + \mu) > 0$ if (2.2) is satisfied.

Now, suppose that (2.3), in addition to (1.1) and (2.2), is satisfied. Then $R(\mathscr{H}_F, \mathscr{H}_G)$ and $R(\mathscr{H}_G, \mathscr{H}_F)$ are greater than 0. Hence $P_{\mathscr{H}_G}|_{\mathscr{H}_F}$ and $P_{\mathscr{H}_F}|_{\mathscr{H}_G}$ are bounded below (see the discussion following Proposition 1.1). Since $(P_{\mathscr{H}_F}|_{\mathscr{H}_G})^* = P_{\mathscr{H}_G}|_{\mathscr{H}_F}$ if we consider $P_{\mathscr{H}_G}|_{\mathscr{H}_F}$ as an operator from $\mathscr{H}_F$ to $\mathscr{H}_G$, $P_{\mathscr{H}_G}|_{\mathscr{H}_F}$ is onto. Therefore, $P_{\mathscr{H}_G}|_{\mathscr{H}_F}$ is an isomorphism from $\mathscr{H}_F$ onto $\mathscr{H}_G$.

Let $f \in \mathscr{H}_F$. Since $\|f\| \geqslant \|P_{\mathscr{H}_G} f\|$, the calculations (2.6) and (2.7) show that

$$\begin{aligned}\sum_{i\in I} |\langle P_{\mathscr{H}_G} f, g_i\rangle|^2 &\geqslant \left(\sqrt{A} - (\lambda_1\sqrt{B} + \lambda_2\sqrt{\beta} + \mu)\right)^2 \|f\|^2\\ &\geqslant \left(\sqrt{A} - (\lambda_1\sqrt{B} + \lambda_2\sqrt{\beta} + \mu)\right)^2 \|P_{\mathscr{H}_G} f\|^2. \qquad (2.8)\end{aligned}$$

Now, for any $g \in \mathscr{H}_G$, there exists unique $f \in \mathscr{H}_F$ such that $P_{\mathscr{H}_G}|_{\mathscr{H}_F} f = g$. (2.8) implies that

$$\sum_{i\in I} |\langle g, g_i\rangle|^2 \geqslant \left(\sqrt{A} - (\lambda_1\sqrt{B} + \lambda_2\sqrt{\beta} + \mu)\right)^2 \|g\|^2.$$

This shows that G satisfies the lower frame condition with a lower frame bound

$$\left(\sqrt{A}-(\lambda_1\sqrt{B}+\lambda_2\sqrt{\beta}+\mu)\right)^2 = A\left(1-\frac{\lambda_1\sqrt{B}+\lambda_2\sqrt{\beta}+\mu}{\sqrt{A}}\right)^2.$$

A routine calculation shows that the above quantity equals (2.4). □

Since $S(\mathscr{H}_G,\mathscr{H}_F)\leqslant 1$, (2.2) improves the Bessel bound (1.3) in Propositions 1.2 and 1.3. Hence, we may replace (1.3) with (2.2) in the statement of Propositions 1.2 and 1.3. If we let $\lambda_1=\lambda_2=0$ in Theorem 2.1, then we recover [11, Theorem 1.2].

3. Applications to finitely generated shift-invariant spaces

In this section we apply Propositions 1.2 and 1.3 and Theorem 2.1 to the perturbation of the generating sets of a finitely generated shift-invariant subspace of $L^2(\mathbb{R}^d)$. Since in most of the applications of the perturbation results to exponential frames and wavelet frames [1,13] the parameters λ_1 and λ_2 are assumed to be 0, we also assume that $\lambda_1=\lambda_2=0$. We now rephrase the perturbation results in Sections 1 and 2 in this setting.

We first review those parts of the theory of (finitely generated) shift-invariant subspaces of $L^2(\mathbb{R}^d)$ [16] which will be used in our discussion. Every material we review is contained in [2–4,16,19–21]. A closed subspace S of $L^2(\mathbb{R}^d)$ is said to be a *shift-invariant* (*sub*)*space* if $T_kS\subset S$ for each $k\in\mathbb{Z}^d$, where $T_kf(x):=f(x-k)$. For $x\in\mathbb{T}^d:=\mathbb{R}^d/\mathbb{Z}^d\simeq[0,1]^d$ and $f\in L^2(\mathbb{R}^d)$ we define $\hat{f}_{\|x}:=(\hat{f}(x+k))_{k\in\mathbb{Z}^d}$, which is a member of $\ell^2(\mathbb{Z}^d)$ a.e.; and for $S\subset L^2(\mathbb{R}^d)$, we define $\hat{S}_{\|x}:=\{\hat{f}_{\|x}:f\in S\}$, where we use the following form of the Fourier transform for $f\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$: $\hat{f}(x):=\int_{\mathbb{R}^d}f(t)\mathrm{e}^{-2\pi\mathrm{i}x\cdot t}\,\mathrm{d}t$. Of course, the Fourier transform extends to be a unitary operator on $L^2(\mathbb{R}^d)$ by a theorem of Plancherel. It is known that a closed subspace S is shift-invariant if and only if $\hat{S}_{\|x}$ is a closed subspace of $\ell^2(\mathbb{Z}^d)$ for a.e. $x\in\mathbb{T}^d$. In this case $\hat{S}_{\|x}$ is said to be the *fiber space* of S at $x\in\mathbb{T}^d$. For $\Phi\subset L^2(\mathbb{R}^d)$ we define $\mathscr{S}(\Phi):=\overline{\operatorname{span}}\,\{T_k\varphi:k\in\mathbb{Z}^d,\varphi\in\Phi\}$, which is obviously a shift-invariant subspace. $\mathscr{S}(\Phi)$ is said to be a *shift-invariant* (*sub*)*space generated by* Φ, and Φ a *generating set*. If S is a shift-invariant space, we define its *spectrum* as follows: $\sigma(S):=\{x\in\mathbb{T}^d:\hat{S}_{\|x}\neq\{0\}\}$. $\sigma(S)$ is defined modulo sets of Lebesgue measure zero. The set equality and containment of subsets of $\mathbb{R}^d$ in this section are assumed to hold modulo sets of Lebesgue measure zero with occasional exceptions which are clear from the context. This convention follows from the nature of the theory of shift-invariant spaces [16]. It is known that $(\mathscr{S}(\Phi))^{\wedge}_{\|x}=\overline{\operatorname{span}}\,\hat{\Phi}_{\|x}$ for a.e. $x\in\mathbb{T}^d$ [2,3,16]. The following proposition gives the angles between two shift-invariant spaces via those between the fiber spaces [4, Proposition 2.10; 20, Lemma 3.1].

Proposition 3.1 [4,20]. *For two shift-invariant spaces U and V of $L^2(\mathbb{R}^d)$ the angles are given by the following formulas*:

$$R(U,V)=\begin{cases}\operatorname*{ess-inf}_{x\in\sigma(U)} R(\hat{U}_{\|x},\hat{V}_{\|x}), & \text{if } U\neq\{0\},\\ 1, & \text{if } U=\{0\},\end{cases}$$

$$S(U,V)=\text{ess-sup}\{S(\hat{U}_{\|x},\hat{V}_{\|x}):x\in\sigma(U)\cap\sigma(V)\}.$$

The following proposition gives characterizations of shift-invariant frame sequences and Riesz sequences [21, Theorem 2.3.6; 3, Theorem 2.3] in terms of the eigenvalues of certain collection of matrices. For $\Phi := \{\varphi_i\}_{i=1}^n \subset L^2(\mathbb{R}^d)$ and $x \in \mathbb{T}^d$, we let $E(\Phi) := \{T_k\varphi : k \in \mathbb{Z}^d, \varphi \in \Phi\}$ and $G_\Phi(x) := (\langle \hat{\varphi}_{j\,\|x}, \hat{\varphi}_{i\,\|x}\rangle_{\ell^2(\mathbb{Z}^d)})_{1\leqslant i,j\leqslant n}$, which is an $n \times n$ matrix for a.e. $x \in \mathbb{T}^d$. $G_\Phi(x)$ is said to be the *Gramian* of Φ at x. Note that the pre-frame operator $T_\Phi : \ell^2(\mathbb{Z}^d)^n \to \mathscr{S}(\Phi)$ of $E(\Phi)$ is $T_\Phi c := \sum_{j=1}^n \sum_{k\in\mathbb{Z}^d} c_j(k) T_k \varphi_j$, which is defined, at least, for each finitely supported $c := (c_j)_{j=1}^n \in \ell^2(\mathbb{Z}^d)^n$.

Proposition 3.2 [3,21]. *For $\Phi := \{\varphi_1, \varphi_2, \ldots, \varphi_n\} \subset L^2(\mathbb{R}^d)$ $E(\Phi)$ is a Bessel sequence with a Bessel bound B if and only if*

the eigenvalues of $G_\Phi(x) \leqslant B$ *for a.e.* $x \in \mathbb{T}^d$;

$E(\Phi)$ is a frame sequence with frame bounds A and B if and only if

$A \leqslant$ *the non-zero eigenvalues of* $G_\Phi(x) \leqslant B$ *for a.e.* $x \in \sigma(S)$;

$E(\Phi)$ is a Riesz sequence with Riesz bounds A and B if and only if

$A \leqslant$ *the eigenvalues of* $G_\Phi(x) \leqslant B$ *for a.e.* $x \in \mathbb{T}^d$.

Suppose that $\Phi \subset L^2(\mathbb{R}^d)$ is finite. Recall that the maximum eigenvalue of $G_\Phi(x)$ is $\|G_\Phi(x)\|$ since $G_\Phi(x)$ is Hermitian by definition (we use the operator norm of the matrix $G_\Phi(x)$). Recall also that $(\mathscr{S}(\Phi))^\wedge_{\|x} = \operatorname{span} \hat{\Phi}_{\|x}$ a.e. since Φ is finite. Hence $\dim(\mathscr{S}(\Phi))^\wedge_{\|x} = \dim \operatorname{span} \hat{\Phi}_{\|x} = \operatorname{rank} G_\Phi(x)$ a.e. If $E(\Phi)$ is a Riesz sequence, then $\sigma(\mathscr{S}(\Phi)) = \mathbb{T}^d$; whereas if $E(\Phi)$ is a frame sequence, then $\sigma(\mathscr{S}(\Phi))$ can be a proper subset of $\mathbb{T}^d$.

Proposition 3.3. *Suppose that $\Phi := \{\varphi_i\}_{i=1}^n$, $\Psi := \{\psi_i\}_{i=1}^n \subset L^2(\mathbb{R}^d)$, and that $E(\Phi)$ is a frame sequence with bounds A and B. Let T_Φ and T_Ψ denote the pre-frame operators of the sequences $E(\Phi)$ and $E(\Psi)$, respectively. Suppose also that there exist non-negative μ such that*

$$\operatorname*{ess\text{-}sup}_{x\in\mathbb{T}^d} \|G_\Xi(x)\| \leqslant \mu^2, \tag{3.1}$$

where $\Xi := \{\varphi_i - \psi_i\}_{i=1}^n$. Then $E(\Psi)$ is a Bessel sequence with a Bessel bound

$$B\left(S(\mathscr{S}(\Psi), \mathscr{S}(\Phi)) + \frac{\mu}{\sqrt{B}}\right)^2.$$

Moreover, if any one of the following conditions are satisfied, then $E(\Psi)$ is also a frame sequence:

(i) $R(\ker T_\Phi, \ker T_\Psi) > 0$, *and* $\mu < \sqrt{A}R(\ker T_\Phi, \ker T_\Psi)$;
(ii) $\mu < \sqrt{A}R(\mathscr{S}(\Psi), \mathscr{S}(\Phi))$;
(iii) $\mu < \sqrt{A}$, and $R(\mathscr{S}(\Psi), \mathscr{S}(\Phi)) > 0$.

Proof. If $F = E(\Phi)$, $G = E(\Psi)$ and $\lambda_1 = \lambda_2 = 0$ in (1.1), then (1.1) is nothing but the condition that $E(\Xi)$ is a Bessel sequence with Bessel bounds μ^2. Hence (1.1) and (3.1) are equivalent by Proposition 3.2. The facts that (i) or (ii) imply the lower frame bound are special cases of Propositions 1.2 and 1.3; and the fact that (iii) implies the lower frame bound is a special case of Theorem 2.1 (cf. [11, Theorem 3.2]). □

Since $R(\ker T_\Phi, \ker T_\Psi)$ and $R(\mathscr{S}(\Psi), \mathscr{S}(\Phi))$ are less than or equal to 1 by definition, (i) or (ii) or (iii) implies that $\mu < \sqrt{A}$. Theorem 2.1 now implies that $R(\mathscr{S}(\Phi), \mathscr{S}(\Psi)) > 0$. We need the following lemma which is [19, Corollary 4.5].

Lemma 3.4. *Let Φ and Ψ be finite subsets of $L^2(\mathbb{R}^d)$. Suppose that $R(\mathscr{S}(\Phi), \mathscr{S}(\Psi)) > 0$. Then the following conditions are equivalent*:

- $R(\mathscr{S}(\Psi), \mathscr{S}(\Phi)) > 0$;
- $R(\mathscr{S}(\Phi), \mathscr{S}(\Psi)) = R(\mathscr{S}(\Psi), \mathscr{S}(\Phi))$;
- $\dim \operatorname{span} \hat{\Phi}_{\|x} = \dim \operatorname{span} \hat{\Psi}_{\|x}$ *a.e.*;
- $\operatorname{rank} G_\Phi(x) = \operatorname{rank} G_\Psi(x)$ *a.e.*

Proposition 3.5. *Suppose that* (3.1) *is satisfied. Then* (ii) *implies* (iii); *but not vice versa. Moreover*, (i) *does not imply* (ii).

Proof. Suppose that (ii) holds. Then, clearly, $\mu < \sqrt{A}$ since $R(\mathscr{S}(\Psi), \mathscr{S}(\Phi)) \leqslant 1$ by definition. Moreover, $R(\mathscr{S}(\Psi), \mathscr{S}(\Phi)) > 0$ if (ii) holds. Hence (iii) is satisfied.

We now construct an example satisfying (3.1), (i) and (iii) but not satisfying (ii). For notational convenience we let the spatial dimension $d = 1$. The proof is exactly the same for $d > 1$. Let $\{e_n\}_{n\in\mathbb{Z}}$ be the standard orthonormal basis for $\ell^2(\mathbb{Z})$. We first define $\Phi := \{\varphi_1, \varphi_2\}$ via $\hat{\varphi}_1 := \chi_{\mathbb{T}}$ and $\hat{\varphi}_2 := \chi_{(\mathbb{T}+1)}$, where χ denotes a characteristic function. Then, $\hat{\varphi}_{1\,\|x} = e_0$ and $\hat{\varphi}_{2\,\|x} = e_1$ for each $x \in \mathbb{T}$. Hence

$$G_\Phi(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for each $x \in \mathbb{T}$. This shows that $E(\Phi)$ is a Riesz sequence by Proposition 3.2 (actually, $E(\Phi)$ is an orthonormal basis for $\mathscr{S}(\Phi)$). In particular, its frame bounds A and B are all 1. We now define $\Psi := \{\psi_1, \psi_2\}$. For $\varepsilon > 0$, define $\hat{\psi}_1 := \chi_{\mathbb{T}} + \varepsilon\chi_{(\mathbb{T}+2)}$ and $\hat{\psi}_2 := \chi_{(\mathbb{T}+1)} + \varepsilon\chi_{(\mathbb{T}+2)}$. Then, $\hat{\psi}_{1\,\|x} = e_0 + \varepsilon e_2$ and $\hat{\psi}_{2\,\|x} = e_1 + \varepsilon e_2$ for each $x \in \mathbb{T}$. Hence

$$G_\Psi(x) = \begin{pmatrix} 1+\varepsilon^2 & \varepsilon^2 \\ \varepsilon^2 & 1+\varepsilon^2 \end{pmatrix}$$

for each $x \in \mathbb{T}$. Note that the eigenvalues of $G_\Psi(x)$ are 1 and $1 + 2\varepsilon^2$. Hence $E(\Psi)$ is a Riesz sequence with Riesz bounds 1 and $1 + 2\varepsilon^2$. If we let $\xi_1 := \varphi_1 - \psi_1$ and $\xi_2 := \varphi_2 - \psi_2$, then $\hat{\xi}_{1\,\|x} = \hat{\xi}_{2\,\|x} = -\varepsilon e_2$ for each $x \in \mathbb{T}$. Hence

$$G_\Xi(x) = \begin{pmatrix} \varepsilon^2 & \varepsilon^2 \\ \varepsilon^2 & \varepsilon^2 \end{pmatrix}$$

for each $x \in \mathbb{T}$. Since the eigenvalues of the $G_\Xi(x)$ are 0 and $2\varepsilon^2$, $\|G_\Xi(x)\| = 2\varepsilon^2$ for each $x \in \mathbb{T}$. Hence we may take $\mu = \sqrt{2}\varepsilon$ in (3.1).

If $\varepsilon < 1/\sqrt{2}$, then $\mu < \sqrt{A} = 1$. Then Theorem 2.1 implies that $R(\mathscr{S}(\Phi), \mathscr{S}(\Psi)) > 0$. Now, Lemma 3.4 implies that $R(\mathscr{S}(\Psi), \mathscr{S}(\Phi)) > 0$ since $\operatorname{rank} G_\Phi(x) = \operatorname{rank} G_\Psi(x) = 2$ for each $x \in \mathbb{T}$. This shows that (iii) is satisfied. On the other hand, both $E(\Phi)$ and $E(\Psi)$ are Riesz sequences. Therefore, T_Φ and T_Ψ are isomorphisms from $\ell^2(\mathbb{Z})^n$ onto $\mathscr{S}(\Phi)$ and $\mathscr{S}(\Psi)$, respectively. In particular, $\ker T_\Phi$ and $\ker T_\Psi$ are trivial. Hence $R(\ker T_\Phi, \ker T_\Psi) = 1$ by definition. This shows that (i) is satisfied.

We now show that (ii) is not satisfied for certain ε with $0 < \varepsilon < 1/\sqrt{2}$ by computing $R(\mathscr{S}(\Psi), \mathscr{S}(\Phi))$. Note that, for each $x \in \mathbb{T}$, $(\mathscr{S}(\Psi))^\wedge_{\|x} = \operatorname{span} \hat{\Psi}_{\|x} = \operatorname{span}\{e_0 + \varepsilon e_2, e_1 + \varepsilon e_2\}$, and

$(\mathscr{S}(\Phi))^{\wedge}_{\|x} = \hat{\Phi}_{\|x} = \text{span}\,\{e_0, e_1\}$. Hence for any $\alpha, \beta \in \mathbb{C}$, the orthogonal projection of $\alpha(e_0 + \varepsilon e_2) + \beta(e_1 + \varepsilon e_2)$ onto $(\mathscr{S}(\Phi))^{\wedge}_{\|x}$ is $\alpha e_0 + \beta e_1$. This shows that

$$\begin{aligned}R((\mathscr{S}(\Psi))^{\wedge}_{\|x}, (\mathscr{S}(\Phi))^{\wedge}_{\|x}) &= \inf_{(\alpha,\beta)\neq(0,0)} \frac{\|\alpha e_0 + \beta e_1\|}{\|\alpha(e_0 + \varepsilon e_2) + \beta(e_1 + \varepsilon e_2)\|}\\ &= \inf_{(\alpha,\beta)\neq(0,0)} \left(\frac{|\alpha|^2 + |\beta|^2}{|\alpha|^2 + |\beta|^2 + \varepsilon^2|\alpha+\beta|^2}\right)^{1/2}\\ &= \inf_{(\alpha,\beta)\neq(0,0)} \left(\frac{1}{1 + \varepsilon^2\frac{|\alpha+\beta|^2}{|\alpha|^2+|\beta|^2}}\right)^{1/2} = \frac{1}{\sqrt{1+2\varepsilon^2}}\end{aligned}$$

since $|\alpha + \beta|^2/(|\alpha|^2 + |\beta|^2) \leqslant 2$ and $|1 + 1|^2/(1^2 + 1^2) = 2$. Hence, for any $\varepsilon > 0$, $R(\mathscr{S}(\Psi), \mathscr{S}(\Phi)) = 1/\sqrt{1+2\varepsilon^2} > 0$ by Proposition 3.1. Therefore, (ii) is dissatisfied if $\mu = \sqrt{2}\varepsilon \geqslant 1/\sqrt{1+2\varepsilon^2} = \sqrt{A}R(\mathscr{S}(\Psi), \mathscr{S}(\Phi))$. In particular, Condition (ii) is dissatisfied for $(\sqrt{5}-1)^{1/2}/2 \leqslant \varepsilon$. Since $(\sqrt{5}-1)^{1/2}/2 < 1/\sqrt{2}$, we see that (i) and (iii) is satisfied while (ii) is dissatisfied for $(\sqrt{5}-1)^{1/2}/2 \leqslant \varepsilon < 1/\sqrt{2}$. □

The proof of the following Lemma, which is a kind of the 'fiber principle', is almost standard (cf. [21]). Suppose that $c := (c_j)_{j=1}^n \in \ell^2(\mathbb{Z}^d)^n$. We let $\hat{c}_j(x) := \sum_{k\in\mathbb{Z}^d} c_j(k)\mathrm{e}^{-2\pi \mathrm{i}k\cdot x}$ to be the Fourier series with coefficients $c_j \in \ell^2(\mathbb{Z}^d)$, and let $\hat{c} := (\hat{c}_j)_{j=1}^n \in L^2(\mathbb{T}^d)^n$. Note that $\|c\|^2_{\ell^2(\mathbb{Z}^d)^n} = \int_{\mathbb{T}^d} \|\hat{c}(x)\|^2_{\mathbb{C}^n}$.

Lemma 3.6. *Let $\Phi := \{\varphi_j\}_{j=1}^n \subset L^2(\mathbb{R}^d)$. Suppose that $E(\Phi)$ is a Bessel sequence with its pre-frame operator T_Φ. Then, $c := (c_j)_{j=1}^n \in \ell^2(\mathbb{Z}^d)^n$ belongs to* $\ker T_\Phi$ *if and only if $\hat{c}(x) \in \ker G_\Phi(x)$ for a.e. $x \in \mathbb{T}^d$. Moreover, for any $c \in \ell^2(\mathbb{Z}^d)^n$, $(P_{\ker T_\Phi}c)^\wedge(x) = (P_{\ker G_\Phi(x)})(\hat{c}(x))$ for a.e. $x \in \mathbb{T}^d$. In particular, $c \perp \ker T_\Phi$ if and only if $\hat{c}(x) \perp \ker G_\Phi(x)$ for a.e. $x \in \mathbb{T}^d$.*

Proof. A direct calculation shows that $\|f\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{T}^d} \|\hat{f}_{\|x}\|^2_{\ell^2(\mathbb{Z}^d)}\,\mathrm{d}x$ for $f \in L^2(\mathbb{R}^d)$. Let $c = (c_i)_{i=1}^n \in \ell^2(\mathbb{Z}^d)^n$. Then

$$\begin{aligned}\|T_\Phi c\|^2 &= \left\|\sum_{j=1}^n \sum_{k\in\mathbb{Z}^d} c_j(k) T_k\varphi_j\right\|^2\\ &= \left\|\sum_{j=1}^n \sum_{k\in\mathbb{Z}^d} c_j(k)\mathrm{e}^{-2\pi \mathrm{i}k\cdot x}\hat{\varphi}_j(x)\right\|^2 = \left\|\sum_{j=1}^n \hat{c}_j(x)\hat{\varphi}_j(x)\right\|^2\\ &= \int_{\mathbb{T}^d} \left\|\sum_{j=1}^n \hat{c}_j(x)\hat{\varphi}_{j\,\|x}\right\|^2_{\ell^2(\mathbb{Z}^d)} \mathrm{d}x = \int_{\mathbb{T}^d} \left\langle \sum_{l=1}^n \hat{c}_l(x)\hat{\varphi}_{l\,\|x}, \sum_{j=1}^n \hat{c}_j(x)\hat{\varphi}_{j\,\|x}\right\rangle_{\ell^2(\mathbb{Z}^d)} \mathrm{d}x\\ &= \int_{\mathbb{T}^d} \sum_{l=1}^n \sum_{j=1}^n \hat{c}_l(x)\overline{\hat{c}_j}(x)\langle\hat{\varphi}_{l\,\|x}, \hat{\varphi}_{j\,\|x}\rangle_{\ell^2(\mathbb{Z}^d)}\,\mathrm{d}x\end{aligned}$$

$$= \int_{\mathbb{T}^d} \sum_{j=1}^{n} \overline{\hat{c}_j}(x) \sum_{l=1}^{n} (G_\Phi(x))_{jl} \hat{c}_l(x)\,\mathrm{d}x = \int_{\mathbb{T}^d} \langle G_\Phi(x)\hat{c}(x), \hat{c}(x)\rangle_{\mathbb{C}^n}\,\mathrm{d}x.$$

Hence, $c \in \ker T_\Phi$ if and only if $\hat{c}(x) \in \ker G_\Phi(x)$ for a.e. $x \in \mathbb{T}^d$ by a routine argument.

Now, suppose that $c := (c_j)_{j=1}^n \in \ell^2(\mathbb{Z}^d)$, and let $d := (d_j)_{j=1}^n \in \ell^2(\mathbb{Z}^d)$ be such that $\hat{d}(x) := P_{\ker G_\Phi(x)}\hat{c}(x)$ a.e. Then $d \in \ker T_\Phi$ by what we have just shown. On the other hand, suppose that $a := (a_j)_{j=1}^n \in \ker T_\Phi$. Then

$$\|c - a\|^2_{\ell^2(\mathbb{Z}^d)^n} = \int_{\mathbb{T}^d} \|\hat{c}(x) - \hat{a}(x)\|^2_{\mathbb{C}^n}\,\mathrm{d}x \geqslant \int_{\mathbb{T}^d} \|\hat{c}(x) - \hat{d}(x)\|^2_{\mathbb{C}^n}\,\mathrm{d}x = \|c - d\|^2_{\ell^2(\mathbb{Z}^d)^n}$$

since $\hat{a}(x) \in \ker_{G_\Phi(x)}$ a.e. This shows that $d = P_{\ker T_\Phi}$.

Finally, suppose that $c := (c_j)_{j=1}^n \in \ell^2(\mathbb{Z}^d)$. Then,

$$\begin{aligned} c \perp \ker T_\Phi &\Leftrightarrow P_{\ker T_\Phi} c = 0 \Leftrightarrow (P_{\ker T_\Phi} c)^\wedge(x) = (P_{\ker G_\Phi(x)})(\hat{c}(x)) = 0 \text{ a.e.} \\ &\Leftrightarrow \hat{c}(x) \perp \ker G_\Phi(x) \text{ a.e.} \qquad \square \end{aligned}$$

We now show that the kind of perturbations in Proposition 3.3 preserves the rank of the Gramian.

Proposition 3.7. *If* (3.1) *and any one of* (i)–(iii) *in Proposition* 3.3 *are satisfied, then* $\dim(\mathscr{S}(\Phi))^\wedge_{\|x} = \dim(\mathscr{S}(\Psi))^\wedge_{\|x}$, *and, in particular,* $\sigma(\mathscr{S}(\Phi)) = \sigma(\mathscr{S}(\Psi))$ *for a.e.* $x \in \mathbb{T}^d$.

Proof. Note that if (i) or (ii) or (iii) is satisfied, then $\mu < \sqrt{A}$. Hence, $R(\mathscr{S}(\Phi), \mathscr{S}(\Psi)) > 0$ by Theorem 2.1. If (ii) or (iii) is satisfied, then $R(\mathscr{S}(\Psi), \mathscr{S}(\Phi)) > 0$. Now, Lemma 3.4 implies the dimension conclusion for (ii) or (iii). On the other hand, suppose that (3.1) and (i) is satisfied. Since $R(\mathscr{S}(\Phi), \mathscr{S}(\Psi)) > 0$, $R((\mathscr{S}(\Phi))^\wedge_{\|x}, (\mathscr{S}(\Psi))^\wedge_{\|x}) > 0$ for a.e. $x \in \sigma(\mathscr{S}(\Phi))$ by Proposition 3.1. Hence $P_{(S(\Psi))^\wedge_{\|x}} : (S(\Phi))^\wedge_{\|x} \to (S(\Psi))^\wedge_{\|x}$ is one-to-one for a.e. $x \in \sigma(\mathscr{S}(\Phi))$. This shows that $\dim(\mathscr{S}(\Phi))^\wedge_{\|x} \leqslant \dim(\mathscr{S}(\Psi))^\wedge_{\|x}$ for a.e. $x \in \sigma(\mathscr{S}(\Phi))$. On the other hand, if $x \in \mathbb{T}^d \setminus \mathscr{S}(\Phi)$, then clearly $\dim(\mathscr{S}(\Phi))^\wedge_{\|x} = 0$. Therefore, $\dim(\mathscr{S}(\Phi))^\wedge_{\|x} \leqslant \dim(\mathscr{S}(\Psi))^\wedge_{\|x}$ for a.e. $x \in \mathbb{T}^d$. Now, suppose that there is $C \subset \mathbb{T}^d$ with positive Lebesgue measure such that $\dim(\mathscr{S}(\Phi))^\wedge_{\|x} < \dim(\mathscr{S}(\Psi))^\wedge_{\|x}$ for each $x \in C$. Recall that $\dim(\mathscr{S}(\Phi))^\wedge_{\|x} = \operatorname{rank} G_\Phi(x)$, $\dim(\mathscr{S}(\Psi))^\wedge_{\|x} = \operatorname{rank} G_\Psi(x)$ and $G_\Phi(x)$ and $G_\Psi(x)$ are all $n \times n$ matrices. Hence $\dim\ker G_\Psi(x) < \dim\ker G_\Phi(x)$ for each $x \in C$. Now, $P_{\ker G_\Psi(x)}|_{\ker G_\Phi(x)} : \ker G_\Phi(x) \to \ker G_\Psi(x)$ cannot be one-to-one since the dimension of the domain is greater than the dimension of the range. Therefore, for each $x \in C$, there exists $\gamma_x := (\gamma_{x,1}, \ldots, \gamma_{x,n}) \in \mathbb{C}^n$ such that $\gamma_x \in \ker G_\Phi(x) \ominus \ker G_\Psi(x)$ and $\gamma_x \neq 0$. Define $c := (c_j)_{j=1}^n \in \ell^2(\mathbb{Z}^d)^n$ via $\hat{c}(x) := \chi_C(x) \cdot \gamma_x$. Then, clearly, $c \neq 0$ and $c \in \ker T_\Phi \ominus \ker T_\Psi$ by Lemma 3.6. This shows that $R(\ker T_\Phi, \ker T_\Psi) = 0$, contradicting (iii). The dimension conclusion for (i) follows from this contradiction. The spectrum conclusion for (i) (ii) and (iii) follows from the dimension conclusion and the definition of the spectrum. $\square$

Proposition 3.8. *Suppose that* (3.1) *is satisfied. Then* (i) *implies* (iii), *but not vice versa. Moreover,* (ii) *does not imply* (i).

Proof. If (3.1) and Condition (i) are satisfied, then $\dim(\mathscr{S}(\Phi))^{\wedge}_{\|x} = \dim(\mathscr{S}(\Psi))^{\wedge}_{\|x}$ a.e. by Proposition 3.7. Hence $\operatorname{rank} G_\Phi(x) = \operatorname{rank} G_\Psi(x)$ for a.e. Moreover $R(\mathscr{S}(\Phi), \mathscr{S}(\Psi)) > 0$ by Theorem 2.1 since $\mu < \sqrt{A}$. Therefore $R(\mathscr{S}(\Psi), \mathscr{S}(\Phi)) = R(\mathscr{S}(\Phi), \mathscr{S}(\Psi)) > 0$ by Lemma 3.4. Hence (iii) is satisfied.

We now construct an example such that (3.1) and (ii) are satisfied, but (i) is not satisfied. Since (ii) implies (iii) by Proposition 3.5, the proof is complete once such an example is constructed. As before we let the spatial dimension $d = 1$. Let $\{e_k\}_{k\in\mathbb{Z}}$ be the standard orthonormal basis for $\ell^2(\mathbb{Z})$. Let us define $\Phi := \{\varphi_i\}_{i=1}^3$ as follows:

$$\hat{\varphi}_1 := \chi_{(\mathbb{T}+1)}, \quad \hat{\varphi}_2 := \frac{1}{2}\chi_{(\mathbb{T}+1)}, \quad \varphi_3 := \chi_{(\mathbb{T}+2)}.$$

Then, for each $x \in \mathbb{T}$

$$\hat{\varphi}_{1\|x} = e_1, \quad \hat{\varphi}_{2\|x} = \frac{1}{2}e_1, \quad \hat{\varphi}_{3\|x} = e_2.$$

Therefore, for each $x \in \mathbb{T}$,

$$G_\Phi(x) = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

whose rank is 2, and whose eigenvalues are 0, 1, 5/4 and $\ker G_\Phi(x) = \operatorname{span}\left\{\frac{1}{\sqrt{5}}(-1, 2, 0)\right\}$. Proposition 3.2 implies that $E(\Phi)$ is a frame sequence with frame bounds $A = 1$ and $B = 5/4$. Note that $(\mathscr{S}(\Phi))^{\wedge}_{\|x} = \operatorname{span} \hat{\Phi}_{\|x} = \operatorname{span}\{e_1, e_2\}$. We then define $\Psi := \{\psi_i\}_{i=1}^3$ as follows:

$$\hat{\psi}_1 := \hat{\varphi}_1, \quad \hat{\psi}_2 := \hat{\varphi}_2 + \frac{1}{2}\chi_{(\mathbb{T}+2)}, \quad \psi_3 := \hat{\varphi}_3 + \frac{1}{2}(\chi_{(\mathbb{T}+1)} - \chi_{(\mathbb{T}+2)}).$$

Then,

$$\hat{\psi}_{1\|x} = e_1, \quad \hat{\psi}_{2\|x} = \frac{1}{2}(e_1 + e_2), \quad \hat{\psi}_{3\|x} = \frac{1}{2}(e_1 + e_2).$$

Therefore, for each $x \in \mathbb{T}$,

$$G_\Psi(x) = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

whose rank is also 2 and $\ker G_\Psi(x) = \operatorname{span}\left\{\frac{1}{\sqrt{2}}(0, -1, 1)\right\}$. Note that, for each $x \in \mathbb{T}$, $(\mathscr{S}(\Psi))^{\wedge}_{\|x} = \operatorname{span} \hat{\Psi}_{\|x} = \operatorname{span}\{e_1, e_2\} = (\mathscr{S}(\Phi))^{\wedge}_{\|x}$. Now, let $\Xi := \{\xi_i\}_{i=1}^3$, where $\xi_i := \varphi_i - \psi_i$. Then, for each $x \in \mathbb{T}$,

$$\hat{\xi}_{1\|x} = 0, \quad \hat{\xi}_{2\|x} = -\frac{1}{2}e_2, \quad \hat{\xi}_{3\|x} = -\frac{1}{2}(e_1 - e_2).$$

Hence, for each $x \in \mathbb{T}$,

$$G_\Xi(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{2} \end{pmatrix},$$

whose eigenvalues are 0, $\left(3 - \sqrt{5}\right)/8$, $\left(3 + \sqrt{5}\right)/8$. Therefore $\mu = \left(\left(3 + \sqrt{5}\right)/8\right)^{1/2} < 1 = \sqrt{A}$, and (3.1) is satisfied. Now, let us define $c = (c_1, c_2, c_3) \in \ell^2(\mathbb{Z})^3$ via $(\hat{c}_1(x), \hat{c}_2(x), \hat{c}_3(x)) =$

$\frac{1}{\sqrt{5}}(-1,2,0)$ for each $x \in \mathbb{T}$. Then $c \in \ker T_\Phi$ by Lemma 3.6, and $\|c\|_{\ell^2(\mathbb{Z})^3} = 1$. Lemma 3.6 also implies that

$$\hat{d}(x) = \left\langle \frac{1}{\sqrt{5}}(-1,2,0), \frac{1}{\sqrt{2}}(0,-1,1) \right\rangle_{\mathbb{C}^3} \frac{1}{\sqrt{2}}(0,-1,1) = -\sqrt{\frac{2}{5}} \cdot \frac{1}{\sqrt{2}}(0,-1,1),$$

where $d := P_{\ker T_\Psi} c$. Hence

$$\|d\|^2_{\ell^2(\mathbb{Z})^3} = \int_{\mathbb{T}} \|\hat{d}(x)\|^2_{\mathbb{C}^3}\, \mathrm{d}x = \frac{2}{5}.$$

This shows that $R(\ker T_\Phi, \ker T_\Psi) \leqslant (2/5)^{1/2}$. Therefore,

$$\mu = \sqrt{\frac{3+\sqrt{5}}{8}} \simeq \sqrt{0.654508} \geqslant \sqrt{0.4} = 1 \cdot \sqrt{\frac{2}{5}} \geqslant \sqrt{A}R(\ker T_\Phi, \ker T_\Psi),$$

and hence (i) is not satisfied.

On the other hand, $R(\mathscr{S}(\Psi), \mathscr{S}(\Phi)) = 1$ by Proposition 3.1 since $(\mathscr{S}(\Psi))^\wedge_{\|x} = (\mathscr{S}(\Phi))^\wedge_{\|x} = \operatorname{span}\{e_1, e_2\}$ for each $x \in \mathbb{T}$. Since $\mu < \sqrt{A}$, $\mu < \sqrt{A}R(\mathscr{S}(\Psi), \mathscr{S}(\Phi))$. Therefore (ii) is satisfied. □

We summarize our findings in the following proposition:

Proposition 3.9. *If* (3.1) *is satisfied*, *then*

- (i) *implies* (iii), *but not vice versa*;
- (ii) *implies* (iii), *but not vice versa*;
- (i) *and* (ii) *are independent*.

Finally, we now consider the case that $n = 1$.

Proposition 3.10. *If* (3.1) *with* $n = 1$ *is satisfied, then* (i) *and* (iii) *are equivalent*, *and* (ii) *implies* (i) *and* (iii) *but not vice versa*.

Proof. Let $\Phi := \{\varphi\}$, $\Psi := \{\psi\}$, $\Xi := \{\xi\} \subset L^2(\mathbb{R}^d)$. Then $G_\Phi(x)$ is the 1×1 matrix $(\sum_{k\in\mathbb{Z}^d} |\hat{\varphi}(x+k)|^2)$, and $G_\Psi(x)$ is the 1×1 matrix $(\sum_{k\in\mathbb{Z}^d} |\hat{\psi}(x+k)|^2)$. We show that, under the assumption that (3.1) is satisfied, (i) and (iii) are equivalent to

(vi) $\mu < \sqrt{A}$ and $\sigma(\mathscr{S}(\Psi)) \subset \sigma(\mathscr{S}(\Phi))$.

(cf. [11, Theorem 3.2]). Since (i) implies (iii), it is enough to show that (vi) implies (i) and (iii) implies (vi). Note that, by Proposition 3.2, $\mu^2 < A$ is equivalent to

$$\operatorname*{ess\text{-}sup}_{x\in\mathbb{T}^d} \sum_{k\in\mathbb{Z}^d} |\hat{\varphi}(x+k) - \hat{\psi}(x+k)|^2 < \operatorname*{ess\text{-}inf}_{x\in\sigma(\mathscr{S}(\Phi))} \sum_{k\in\mathbb{Z}^d} |\hat{\varphi}(x+k)|^2. \tag{3.2}$$

Since (3.2) cannot hold if $\hat{\varphi}_{\|x} \neq 0$ and $\hat{\psi}_{\|x} = 0$ on a subset of $\mathbb{T}^d$ with positive Lebesgue measure, we see that $\sigma(\mathscr{S}(\Phi)) \subset \sigma(\mathscr{S}(\Psi))$.

(vi) ⇒ (i): Suppose that (vi) holds. Then, $\sigma(\mathscr{S}(\Psi)) = \sigma(\mathscr{S}(\Phi))$, which follows from what we have just shown. On the other hand, by Lemma 3.6,

$$
\begin{aligned}
c \in \ker T_\Phi &\Leftrightarrow \hat{c}(x) \sum_{k\in\mathbb{Z}^d} |\hat{\varphi}(x+k)|^2 = 0 \Leftrightarrow \operatorname{supp}(\hat{c}) \subset \mathbb{T}^d \setminus \sigma(\mathscr{S}(\Phi)) \\
&\Leftrightarrow \operatorname{supp}(\hat{c}) \subset \mathbb{T}^d \setminus \sigma(\mathscr{S}(\Psi)) \Leftrightarrow \hat{c}(x) \sum_{k\in\mathbb{Z}^d} |\hat{\psi}(x+k)|^2 = 0 \\
&\Leftrightarrow c \in \ker T_\Psi
\end{aligned}
$$

This shows that $\ker T_\Phi = \ker T_\Psi$. Hence $R(\ker T_\Phi, \ker T_\Psi) = 1$, which guarantees that (i) holds since $\mu < \sqrt{A}$.

(iii) $\Rightarrow$ (vi): This follows from Proposition 3.7.

This proves that (i) and (iii) are equivalent if $n = 1$ and (3.1) is satisfied. Since (ii) implies (iii) by Proposition 3.5, (ii) also implies (i).

(i) or (iii) $\not\Rightarrow$ (ii): As before, we let the spatial dimension $d = 1$ and $\{e_k\}_{k\in\mathbb{Z}}$ the standard orthonormal basis for $\ell^2(\mathbb{Z})$. We construct an example satisfying (iv) but not satisfying (ii). Define $\hat{\varphi}_{\|x} := e_0$ and $\hat{\psi}_{\|x} := (1/\sqrt{2}) \cdot e_0 + (1/\sqrt{2}) \cdot e_1$ for each $x \in \mathbb{T}$, i.e., $\hat{\varphi} = \chi_\mathbb{T}$ and $\hat{\psi} = (1/\sqrt{2}) \cdot (\chi_\mathbb{T} + \chi_{(\mathbb{T}+1)})$. Then $A = B = 1$, Let $\xi := \varphi - \psi$. Then $\hat{\xi}_{\|x} = (1 - 1/\sqrt{2}) \cdot e_0 - 1/\sqrt{2} \cdot e_1$ for each $x \in \mathbb{T}$. Hence $\mu^2 = (1 - 1/\sqrt{2})^2 + 1/2 = 2 - \sqrt{2} < 1 = A$. Since $\sigma(\mathscr{S}(\Phi)) = \sigma(\mathscr{S}(\Psi)) = \mathbb{T}$, (iv) is satisfied. Since $(\mathscr{S}(\Phi))^\wedge_{\|x} = \operatorname{span}\{e_0\}$ and $(\mathscr{S}(\Psi))^\wedge_{\|x} = \operatorname{span}\left\{\left(1/\sqrt{2}\right) \cdot e_0 + \left(1/\sqrt{2}\right) \cdot e_1\right\}$ for each $x \in \mathbb{T}$, it is easy to see that $R(\mathscr{S}(\Psi), \mathscr{S}(\Phi)) = 1/\sqrt{2}$ by Proposition 3.1. Hence

$$
\mu = \sqrt{2 - \sqrt{2}} \simeq 0.765367 \geqslant 0.707107 \simeq 1 \cdot 1/\sqrt{2} = \sqrt{A} R(\mathscr{S}(\Psi), \mathscr{S}(\Phi)).
$$

Therefore (ii) is not satisfied. □

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