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Second Method of Lyapunov and Existence of Integral Manifolds for Impulsive Differential-Difference Equations

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By means of piecewise continuous functions which are analogues of Lyapunov's functions, sufficient conditions are obtained for the existence of integral manifolds for impulsive differential-difference equations with variable impulsive perturbations. © 2001 Academic Press

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1. INTRODUCTION

The impulsive differential-difference equations describe processes with after-effect and state changing by jumps. These equations are an adequate mathematical apparatus for simulation in physics, chemistry, biology, population dynamics, biotechnologies, control theory, industrial robotics, etc.

In spite of the great possibilities for application, the theory of the impulsive differential-difference equations is developing rather slowly [2].

In the present paper, by means of piecewise continuous auxiliary functions which are analogues of the classical Lyapunov's functions, sufficient conditions are obtained for the existence of integral manifolds for impulsive differential-difference equations with variable impulsive perturbations. The investigations are carried out by using minimal subsets of a suitable space of piecewise continuous functions, by the elements of which the derivatives of Lyapunov's functions are estimated [3].

Results related to the study of the existence of integral manifolds for impulsive differential equations without delay have been obtained [1, 4–6].

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2. STATEMENT OF THE PROBLEM: PRELIMINARY NOTES

Let R^n be the n-dimensional Euclidean space with norm $|x| = (\sum_{k=1}^n x_k^2)^{1/2}$, let Ω be a domain in R^n, $\Omega \neq \phi$, $h > 0$, $t_0 \in R$, $\varphi_0 \in C[[t_0 - h, t_0], \Omega]$, $R_+ = [0, \infty)$.

Consider the initial value problem

$$x(t) = f(t, x(t), x(t-h)), \qquad t > t_0, \quad t \neq \tau_k(x(t)), \tag{1}$$

$$x(t) = \varphi_0(t), \qquad t \in [t_0 - h, t_0], \tag{2}$$

$$\Delta x(t) = I_k(x(t)), \qquad t = \tau_k(x(t)), \quad t > t_0, \quad k = 1, 2, \ldots, \tag{3}$$

where f: $(t_0, \infty) \times \Omega \times \Omega \to R^n$, τ_k: $\Omega \to (t_0, \infty)$, I_k: $\Omega \to R^n$, $k = 1, 2, \ldots, \Delta x(t) = x(t+0) - x(t-0)$.

Let $\tau_0(x) \equiv t_0$ for $x \in \Omega$.

Introduce the notations

$$G_k = \{(t, x) \in [t_0, \infty) \times \Omega : \tau_{k-1}(x) < t < \tau_k(x)\}, \qquad k = 1, 2, \ldots$$

$$\sigma_k = \{(t, x) \in [t_0, \infty) \times \Omega : t = \tau_k(x)\};$$

i.e., σ_k, $k = 1, 2, \ldots,$ are hypersurfaces with equations $t = \tau_k(x(t))$, $C_0 = C[[t_0 - h, t_0], \Omega]$, and K is the class of all continuous and strictly increasing functions a: $R_+ \to R_+$ such that $a(0) = 0$. By $x(t) = x(t; t_0, \varphi_0)$ we denote the solution of the problem (1), (2), (3); $J^+(t_0, \varphi_0)$ is the maximal interval of the type $[t_0, \beta)$ in which the solution $x(t; t_0, \varphi_0)$ is defined, and by $\theta_+(t_0, \varphi_0(t))$ denote the integral orbit of the solution $x(t; t_0, \varphi_0)$ for $t \in J^+(t_0, \varphi_0)$.

We shall make a description of the solution $x(t) = x(t; t_0, \varphi_0)$ of the problem (1), (2), (3):

1. For $t_0 - h \leq t \leq t_0$ to the solution $x(t)$ coincides with the function $\varphi_0 \in C_0$.

2. Let $t_1, t_2, \ldots$ $(t_0 < t_1 < t_2 < \cdots)$ be the moments at which the integral curve $(t, x(t))$ of problem (1), (2), (3) meets the hypersurfaces $\{\sigma_k\}_{k=1}^\infty$; i.e., each of the points $t_1, t_2, \ldots$ is a solution of one of the equations $t = \tau_k(x(t))$, $k = 1, 2, \ldots$. Let $t_l^h = t_l + h$, $l = 0, 1, 2, \ldots$.

We form the sequence $\{\sigma_i\}_{i=0}^\infty$ observing the following rules:

(a) $\{\tau_i\}_{i=0}^\infty = \{t_k\}_{k=0}^\infty \cup \{t_l^h\}_{l=0}^\infty$.

(b) $\tau_0 \equiv t_0$.

(c) The sequence $\{\tau_i\}_{i=0}^\infty$ is monotone increasing.

We shall note that in general it is possible that

$$\{t_k\}_{k=1}^\infty \cap \{t_l^h\}_{l=0}^\infty \neq \varnothing.$$

2.1. For $\tau_0 < t \le \tau_1$ the solution of the problem (1), (2), (3) coincides with the solution of the problem (1), (2).

2.2. For $\tau_i < t \le \tau_{i+1}$, $i - 1, 2, \dots,$ one of the following three cases may occur:

(a) If $\tau_i \in \{t_k\}_{k=1}^{\infty} \setminus \{t_l^h\}_{l=0}^{\infty}$, $\tau_i - t_k$, and j_k is the number of the hypersurface met by the integral curve $(t, x(t))$ at the moment t_k, then the solution $x(t)$ coincides with the solution of the problem

$$\dot{y}(t) = f(t, y(t), x(t-h)), \tag{4}$$

$$y(t_k) = x(t_k) + I_{j_k}(x(t_k)). \tag{5}$$

(b) If $\tau_i \in \{t_l^h\}_{l=0}^{\infty} \setminus \{t_k\}_{k=1}^{\infty}$, then the solution $x(t)$ of problem (1), (2), (3) coincides with the solution of the problem

$$\dot{y}(t) = f(t, y(t), x(t-h+0)), \tag{6}$$

$$y(\tau_i) = x(\tau_i). \tag{7}$$

(c) If $\tau_i \in \{t_k\}_{k=1}^{\infty} \cap \{t_l^h\}_{l=0}^{\infty}$, $\tau_i = t_k$, then the solution $x(t)$ of the problem (1), (2), (3) coincides with the solution of problem (5), (6).

3. If the point $x(t_k) + I_{j_k}(x(t_k)) \notin \Omega$, then the solution $x(t)$ is not defined for $t > t_k$.

4. The function $x(t)$ is piecewise continuous on $J^+(t_0, \varphi_0)$, continuous from the left at the points $t_1, t_2, \dots$ of $J^+(t_0, \varphi_0)$ and $x(t_k + 0) = x(t_k) + I_{j_k}(x(t_k))$, $k = 1, 2, \dots$.

Definition 1. We call an arbitrary manifold M in the extended phase space $[t_0 - h, \infty) \times \Omega$ of (1), (2), (3) an *integral manifold* if from $(t, \varphi_0(t)) \in M$ for $t \in [t_0 - h, t_0]$ it follows that $\theta_+(t_0, \varphi_0(t)) \subset M$.

In what follows we shall use the class V_M of piecewise continuous auxiliary functions $V\colon [t_0, \infty) \times \Omega \to R_+$ which are analogues of Lyapunov's functions [1].

Definition 2. We shall say that the function $V\colon [t_0, \infty) \times \Omega \to R_+$ belongs to the class V_M which kernel is the manifold M in the extended phase space of (1), (2), (3) if the following conditions hold:

1. The function V is continuous in G and locally Lipschitz continuous with respect to its second argument x in each of the sets G_k, $k = 1, 2, \dots$.

2. $V(t, x) = 0$ for $(t, x) \in M$, $t \ge t_0$, and $V(t, x) > 0$ for $(t, x) \in [t_0, \infty) \times \Omega \setminus M$.

3. For each $k = 1, 2, \ldots$ and $(t_0^*, x_0^*) \in \sigma_k$ there exist the finite limits

$$V(t_k^* - 0, x_0^*) = \lim_{\substack{(t, x) \to (t_0^*, x_0^*) \\ (t, x) \in G_k}} V(t, x),$$

$$V(t_0^* + 0, x_0^*) = \lim_{\substack{(t, x) \to (t_0^*, x_0^*) \\ (t, x) \in G_{k+1}}} V(t, x)$$

and the equality $V(\iota_0^* - 0, x_0^*) = V(\iota_0^*, x_0^*)$ is valid.

Introduce the following conditions:

H1. $f \in C[(t_0, \infty) \times \Omega \times \Omega, R^n]$.

H2. The function f is Lipschitz continuous with respect to its second and third arguments in $(t_0, \infty) \times \Omega \times \Omega$ uniformly on $t \in (t_0, \infty)$.

H3. The functions I_k are Lipschitz continuous in Ω.

H4. The functions $(I + I_k)$: $\Omega \to \Omega$, $k = 1, 2, \ldots,$ where I is the identity in Ω.

H5. $\tau_k \in C[\Omega, (t_0, \infty)]$, $k = 1, 2, \ldots$.

H6. $t_0 < \tau_1(x) < \tau_2(x) < \cdots, x \in \Omega$.

H7. $\tau_k(x) \to \infty$ as $k \to \infty$ uniformly on $x \in \Omega$.

H8. $j_k < j_{k+1} < \cdots < j_{k+p} < \cdots$, where j_k is the number of the hypersurface met by the integral curve $(t, x(t))$ of the problem (1), (2), (3) at the moment t_k; $k, j_k, p = 1, 2, \ldots$.

We shall note that for the impulsive differential equations with variable impulsive perturbations the so called "beating" of the solution may occur, i.e., a phenomenon for which the integral curve $(t, x(t))$ meets several of infinitely many times one and the same hypersurface. In the present paper we shall consider problems of the form (1), (2), (3) for which "beating" of the solutions is absent.

Introduce the following condition:

H9. The integral curve of the solution of the problem (1), (2), (3) meets each of the hypersurfaces σ_k at most once.

Condition H9 is satisfied in the case when $\tau_k \equiv t_k$, $k = 1, 2, \ldots, x \in \Omega$, i.e., when the impulses take place at fixed moments.

Introduce the following classes of functions:

$\mathrm{PC}\big[[t_0,\infty),\Omega\big] = \{x : [t_0,\infty) \to \Omega$, x is piecewise continuous with points of discontinuity of the first kind belonging to the interval (t_0,∞) at which it is continuous from the left$\}$,

$$\Omega_1 = \big\{x \in PC\big[[t_0,\infty),\Omega\big] : V(s, x(s)) \le V(t, x(t)),\\ t - h \le s \le t,\ t \ge t_0,\ V \in V_M\big\}.$$

Introduce the function

$$D_{-}V(t, x(t)) = \liminf_{\sigma \to 0^-} \sigma^{-1}\big[V(t + \sigma, x(t) + \sigma f(t, x(t), x(t-h)))\\ - V(t, x(t))\big].$$

3. MAIN RESULTS

LEMMA 1. *Let the conditions* H1–H9 *hold*. *Then*

1. $t_k \to \infty$.
2. $J^+(\iota_0, \varphi_0) = [\iota_0, \infty)$.

Proof of Assertion 1. From condition H8 we derive the inequalities

$$j_1 < j_2 < \cdots .$$

From the above inequalities, since j_k are positive integers, we conclude that $j_k \to \infty$ as $k \to \infty$. Then by condition H7 we get to the equalities

$$\lim_{k\to\infty} t_k = \lim_{j_k\to\infty} \tau_{j_k}(x_k) = \infty,$$

where $x_k = x(t_k; t_0, \varphi_0)$.

Proof of Assertion 2. Since by the conditions H1, H2, and H4 the solution $x(t; t_0, \varphi_0)$ of the problem (1), (2), (3) is defined on each of the intervals $(t_k, t_{k|1}]$, $k = 1, 2, \ldots,$ then from Assertion 1 we conclude that it is continuable for each $t \ge t_0$.

Introduce the following condition:

H10. The integral curve $(t, x(t))$ of the solution of the problem (1), (2), (3) meets for $t \ge t_0$ successively each one of the hypersurfaces σ_1, σ_2 exactly once.

LEMMA 2. *Let the following conditions hold*:

1. *Conditions* H1–H10 *are met*.
2. $g \in PC[[t_0, \infty) \times R_+, R]$ *and* $g(t, 0) = 0$ *for* $t \in [t_0, \infty)$.
3. $B_k \in C[R_+, R_+]$ *and* $B_k(0)$, $k = 1, 2, \ldots$.
4. $t_0 < t_1 < t_2 < \cdots$ *are the moments at which the integral curve* $(t, x(t))$ *of the problem* (1), (2), (3) *meets the hypersurfaces* $\{\sigma_k\}_{k=1}^{\infty}$.
5. *The maximal solution* $r(t; t_0, u_0)$ *of the problem*

$$\begin{aligned} &\dot{u} = g(t, u), \qquad t > t_0, \quad t \neq t_k, \quad k = 1, 2, \ldots, \\ &u(t_0 + 0) = u_0 \geq 0, \\ &\Delta u(t_k) = B_k(u(t_k)), \qquad k = 1, 2, \ldots \end{aligned} \tag{8}$$

is defined in the interval $[t_0, \infty)$.

6. *The functions* ψ_k: $R_+ \to R_+$, $\psi_k(u) = u + B_k(u)$, $k = 1, 2, \ldots,$ *are nondecreasing with respect to* u.
7. *The functions* $V \in V_M$ *is such that*

$$V(t_0, \varphi_0(t_0)) \leq u_0$$

and the inequalities

$$\begin{aligned} &D_-V(t, x(t)) \leq g(t, V(t, x(t))), \\ &\qquad\qquad\qquad t \neq \tau_k(x(t)), \quad k = 1, 2, \ldots, \\ &V(t + 0, x(t) + I_k(x(t))) \leq \psi_k(V(t, x(t))), \\ &\qquad\qquad\qquad t = \tau_k(x(t)), \qquad k = 1, 2, \ldots \end{aligned} \tag{9}$$

are valid for each $t > t_0$ *and* $x \in \Omega_1$.

Then

$$V(t, x(t; t_0, \varphi_0)) \leq r(t; t_0, u_0), \qquad \textit{for } t \in [t_0, \infty). \tag{10}$$

Proof. From Lemma 1 it follows that $J^+(t_0, \varphi_0) = [t_0, \infty)$ and from condition H10 it follows that for $t \in [t_0, \infty)$, $(t, x(t))$ meets successively the hypersurfaces $\sigma_1, \sigma_2, \ldots$. Since in the interval $(t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots,$ $x(t)$coincides with the solution of the problem (1), (5) ($j_k = k$), we conclude that for $t_k < t \leq t_{k+1}$ the function $x(t)$ satisfies the integral equation

$$x(t) = x(t_k) + I_k(x(t_k)) + \int_{t_k}^{t} f(\tau, x(\tau), x(\tau - h))\, d\tau.$$

On the other hand, the maximal solution $r(t; t_0, u_0)$ of the problem (8) is defined by the equality

$$r(t; t_0, u_0) = \begin{cases} r_0(\iota; \iota_0, u_0^+), & \iota_0 < \iota < \iota_1, \\ r_1(t; t_1, u_1^+), & t_1 < t \le t_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots & \\ r_k(t; t_k, u_k^+), & t_k < t \le t_{k+1}, \\ \dots\dots\dots\dots\dots\dots\dots\dots & \end{cases}$$

where $r_k(t; t_k, u_k^+)$ is the maximal solution of the equation without impulses $\dot{u} - g(t, u)$ in the interval $(t_k, t_{k+1}]$, $k = 0, 1, 2, \dots,$ for which $u_k^+ = \psi_k(r_{k-1}(t_k; t_{k-1}, u_{k-1}^+))$, $k = 1, 2, \dots$ and $u_0^+ - u_0$.

Let $t \in (t_0, t_1]$. Then from the corresponding comparison lemma for the continuous case [3] we obtain that

$$V(t, x(t; t_0, \varphi_0)) \le r(t; t_0, u_0).$$

Suppose that (10) is satisfied for $t \in (t_{k-1}, t_k]$, $k > 1$. Then, using (9) and the fact that the function ψ_k is nondecreasing, we obtain

$$\begin{aligned} V(t_k + 0, x(t_k + 0; t_0, \varphi_0)) &\le \psi_k(V(t_k, x(t_k; t_0, \varphi_0))) \\ &\le \psi_k(r(t_k; t_0, u_0)) \\ &= \psi_{k-1}(r_k(t_k; t_{k-1}, u_{k-1}^+)) = u_k^+. \end{aligned}$$

We apply again the corresponding comparison lemma [3] and obtain

$$V(t, x(t; t_0, \varphi_0)) \le r_k(t; t_k, u_k^+) = r(t; t_0, u_0);$$

i.e., the inequality (10) is valid for $t \in (t_k, t_{k+1}]$.

The proof is completed by induction.

COROLLARY 1. *Let the following conditions hold*:

1. *Conditions* II1–II10 *are met.*
2. *The function* $V \in V_M$ *is such that the inequalities*

$$D_- V(t, x(t)) \le 0, \qquad t \ne \tau_k(x(t)), \quad k = 1, 2, \dots,$$

$$V(t + 0, x(t) + I_k(x(t))) \le V(t, x(t)), \qquad t = \tau_k(x(t)), \quad k = 1, 2, \dots$$

are valid for each $t \ge t_0$ *and* $x \in \Omega_1$.

Then

$$V(t, x(t; t_0, u_0)) \le V(t_0, \varphi_0(t_0)), \qquad t \in [t_0, \infty).$$

THEOREM 1. *Assume that*:

1. *Conditions* H1–H10 *are met*.

2. *For the problem* (1), (2), (3) *there exists a function* $V \in V_M$ *with kernel the manifold* M, *so that the following relations are satisfied*:

$$D_{-}V(t, x(t)) \leq 0, \qquad t \neq \tau_k(x(t)), \quad k = 1, 2, \ldots, \quad t \geq t_0, \quad x \in \Omega_1, \tag{11}$$

$$V(t + 0, x(t) + I_k(x(t))) \leq V(t, x(t)),$$
$$t = \tau_k(x(t)), \quad k = 1, 2, \ldots. \tag{12}$$

Then M *is an integral manifold for* (1), (2), (3).

Proof. Suppose that M is not an integral manifold. Therefore there exists $t', t' > t_0$ such that, if $(t, \varphi_0(t)) \in M$ for $t \in [t_0 - h, t_0]$, $(t, x(t; t_0, \varphi_0)) \in M$ for $t_0 < t \leq t'$, and $(t, x(t; t_0, \varphi_0)) \notin M$ for $t > t'$, $V(t', x') = 0$, where $x' = x(t'; t_0, \varphi_0)$. Moreover $x(t) \in \mathrm{PC}[J^+(t_0, \varphi_0), R^n]$.

We denote that for t' the following two cases are possible:

(a) If $t' = \tau_k(x(t))$, $k = j, j + 1, \ldots, j \geq 1$, then $(t' + 0, x(t' + 0; t_0, \varphi_0)) = (t' + 0, x(t'; t_0, \varphi_0) + I_k(x'))$, $(t' + , x(t' + 0; t_0, \varphi_0)) \notin M$ and from Definition 2 it follows that $V(t' + 0, x(t' + 0; t_0, \varphi_0)) > 0$. Consequently $0 = V(t', x') < V(t' + 0, x(t' + 0; t_0, \varphi_0))$ which is contradiction by (11).

(b) If $t' \nmid \tau_k(x(t))$, $k - j, j + 1, \ldots, j \geq 1$, there exists $t'' > t'$ such that $(t'', x(t''; t_0, \varphi_0)) \notin M$. From (11) and (12) it follows that the function $V(t, x(t))$ is not increasing in $[t_0, \infty)$ and from Definition 2

$$V(t'', x(t''; t_0, \varphi_0)) > 0. \tag{13}$$

Since the conditions of Corollary 1 are met, then

$$V(t, x(t; t_0, \varphi_0)) \leq V(t', x(t'; t_0, \varphi_0)),$$

for $t \in [t', \infty)$ and we obtain that

$$V(t'', x(t''; t_0, \varphi_0)) \leq V(t', x(t'; t_0, \varphi_0)) = 0,$$

which is contradicts (13).

The proof of Theorem 1 is complete.

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