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Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations

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ABSTRACT

This paper is concerned with the existence and uniqueness of a weighted pseudo almost automorphic mild solution to the semilinear fractional equation: $D_t^{\alpha}u(t) = Au(t) + D_t^{\alpha-1}f(t, u(t))$, $t \in \mathbb{R}$, $1 < \alpha < 2$ in complex Banach spaces with S^p-weighted pseudo almost automorphic coefficients, where A is a linear densely defined operator of sectorial type on a complex Banach space $\mathbb{X}$. Moreover, we present an application to a fractional wave equation.

1. Introduction

Weighted pseudo almost automorphic functions have many applications in several problems for example in the theory of functional differential equations, integral equations and abstract evolution equations [1]. The concept of weighted pseudo almost automorphic functions was first introduced by Blot et al. [2]. In [3–6], a new generalization of the concept of almost automorphic functions was introduced by Chang et al. [7]; such a new concept is called Stepanov-like weighted pseudo almost automorphic functions, which generalizes the concept of Stepanov-like weighted pseudo almost periodic functions introduced by Diagana [5] as well as weighted pseudo almost automorphic functions. Very recently, we in [8,9] established new compositions for S^p-weighted pseudo almost automorphic functions and applications to nonautonomous evolution equations and integral equations.

In recent years, fractional equations have gained considerable importance due to their applications in various fields of the science, such as physics, mechanics, chemistry engineering, etc. Significant development has been made in ordinary and partial differential equations involving fractional derivatives, we refer to the monographs of Kilbas et al. [10,11], Diethelm [12], Hilfer [13], Podlubny [14], and the Refs. [15–18]. The study of almost automorphic solutions to fractional differential equation were initiated by Araya and Lizama [19]. In their work, the authors investigated the existence and uniqueness of an almost automorphic mild solution of the semilinear equation

$$D_t^{\alpha}u(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R}, \ 1 < \alpha < 2, \tag{1.1}$$

where A is a generator of an α-resolvent family and D_t is a Riemann Liouville fractional derivative. In [20] Cuevas and Lizama considered the following fractional differential equation:

$$D_t^{\alpha}u(t) = Au(t) + D_t^{\alpha-1}f(t, u(t)), \quad t \in \mathbb{R}, \ 1 < \alpha < 2, \tag{1.2}$$

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where A is a linear operator of sectorial negative type on a complex Banach space. Under suitable conditions on f, the authors proved the existence and uniqueness of an almost automorphic mild solution to (1.2). See also [21], a new and general existence and uniqueness theorem of almost automorphic solutions is obtained for (1.1) with S^p almost automorphic coefficients. Recently, Agarwal et al. [22,23] studied the existence and uniqueness of a weighted pseudo almost periodic mild solution and pseudo almost periodic solutions to the semilinear fractional equation (1.2).

Motivated by the above works, we study in this paper the existence and uniqueness of weighted pseudo almost automorphic solutions to (1.2) with S^p-weighted pseudo almost automorphic coefficients, where $1 < \alpha < 2$, $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a linear densely defined operator of sectorial type on a complex Banach space $(\mathbb{X}, \|\cdot\|)$. The application of the paper follows the extended results in [22] and can be seen as a contribution to this emerging field.

The work is organized as follows. In Section 2, we recall some definitions, lemmas and preliminary results. In Section 3, we prove the existence and uniqueness of weighted pseudo almost automorphic mild solutions for the fractional differential equation (1.2). An example is given in Section 4 to illustrate the results obtained.

2. Preliminaries

In this section, we introduce definitions, notations, lemmas and preliminary facts which are used throughout this work. In the paper, we assume that $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ are two Banach spaces. Let $BC(\mathbb{R}, \mathbb{X})$ (respectively, $BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) stands for the class of all $\mathbb{X}$-valued bounded continuous functions from $\mathbb{R}$ into $\mathbb{X}$ (respectively, the class of all jointly bounded continuous functions from $\mathbb{R} \times \mathbb{Y}$ into $\mathbb{X}$). The space $BC(\mathbb{R}, \mathbb{X})$ equipped with the sup norm defined by $\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|$ is a Banach space. The notation $\mathfrak{B}(\mathbb{X}, \mathbb{Y})$ stands for the space of bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$ endowed with the uniform operator topology, and we abbreviate to $\mathfrak{B}(\mathbb{X})$, whenever $\mathbb{X} = \mathbb{Y}$.

Now we give some necessary definitions.

First, let us recall that a closed and linear operator A is said to be sectorial of type ω if there exist $0 < \theta < \frac{\pi}{2}, M > 0$ *and* $\omega \in \mathbb{R}$ such that its resolvent exists outside the sector $\omega + \Sigma_\theta := \{\omega + \lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \theta\}$ and $\|(\lambda - A)^{-1}\| \le \frac{M}{|\lambda-\omega|}$, $\lambda \notin \omega + \Sigma_\theta$. Sectorial operators are well studied in the literature (for more details, see [24–26]).

Definition 2.1 (*[22,27]*). Let $1 < \alpha < 2$. Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space $\mathbb{X}$. We say that A is the generator of a solution operator if there exists $\omega \in \mathbb{R}$ and a strongly continuous function $E_\alpha : \mathbb{R}_+ \to \mathfrak{B}(\mathbb{X})$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and $\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{-\lambda t} E_\alpha(t)x dt,\ Re\lambda > \omega,\ x \in \mathbb{X}$.

We note that if A is sectorial of type $\omega \in \mathbb{R}$ with $0 \le \theta < \pi(1 - \frac{\alpha}{2})$, then A is the generator of a solution operator given by

$$E_\alpha(t) := \frac{1}{2\pi i}\int_\zeta e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1} d\lambda, \quad t \ge 0$$

where ζ is a suitable path lying outside the sector $\omega + \Sigma_\theta$.

Lemma 2.1 ([*28*]). *Let $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ be a sectorial operator in a complex Banach space $\mathbb{X}$. Satisfying hypothesis $\omega + \Sigma_\theta := \{\omega+\lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \theta\}$ and $\|(\lambda-A)^{-1}\| \le \frac{M}{|\lambda-\omega|}$, $\lambda \notin \omega+\Sigma_\theta$, for some $M > 0$, $\omega < 0$ and $0 \le \theta < \pi(1-\frac{\alpha}{2})$. Then there exists $C > 0$ such that*

$$\|E_\alpha(t)\|_{\mathfrak{B}(\mathbb{X})} \le \frac{CM}{1 + |\omega|t^\alpha}, \quad t \ge 0. \tag{2.1}$$

Definition 2.2 (*[29–32]*). A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$ there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$. The collection of all such functions will be denoted by $AA(\mathbb{X})$.

Now, let $\mathbb{U}$ denote the set of all functions $\rho : \mathbb{R} \to (0, \infty)$, which are locally integrable over $\mathbb{R}$ such that $\rho > 0$ almost everywhere. For a given $r > 0$ and for each $\rho \in \mathbb{U}$, we set $m(r, \rho) := \int_{-r}^{r} \rho(t)dt$.

Thus the space of weights $\mathbb{U}_\infty$ is defined by

$$\mathbb{U}_\infty := \{\rho \in \mathbb{U} : \lim_{r\to\infty} m(r, \rho) = \infty\}.$$

Now for $\rho \in \mathbb{U}_\infty$, we define

$$PAA_0(\mathbb{X},\rho) := \left\{ f \in BC(\mathbb{R},\mathbb{X}) : \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \|f(t)\|\rho(t)dt = 0 \right\};$$

$$PAA_0(\mathbb{Y},\mathbb{X},\rho) := \left\{ f \in C(\mathbb{R}\times\mathbb{Y},\mathbb{X}) : f(\cdot,y) \quad \text{is bounded for each } y \in \mathbb{Y} \right.$$
$$\left. \text{and} \quad \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \|f(t,y)\|\rho(t)dt = 0 \ \text{ uniformly in } y \in \mathbb{Y} \right\}.$$

Definition 2.3 ([2]). Let $\rho \in \mathbb{U}_\infty$. A function $f \in BC(\mathbb{R},\mathbb{X})$ (respectively, $f \in BC(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) is called weighted pseudo almost automorphic if it can be expressed as $f = g + h$, where $g \in AA(\mathbb{X})$ (respectively, $AA(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) and $h \in PAA_0(\mathbb{X},\rho)$ (respectively, $PAA_0(\mathbb{Y},\mathbb{X},\rho)$). We denote by $WPAA(\mathbb{X})$ (respectively, $WPAA(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) the set of all such functions.

Lemma 2.2 ([6, Theorem 3.4]). *Let $\rho \in \mathbb{U}_\infty$. Suppose that $PAA_0(\mathbb{X},\rho)$ is translation invariant. Then the decomposition of weighted pseudo almost automorphic functions is unique.*

Lemma 2.3 ([33, Theorem 2.15]). *Let $\rho \in \mathbb{U}_\infty$. If $PAA_0(\mathbb{X},\rho)$ is translation invariant, then $(WPAA(\mathbb{X}), \|\cdot\|_\infty)$ is a Banach space.*

Definition 2.4 ([4,34]). The Bochner transform $f^b(t,s)$, $t \in \mathbb{R}$, $s \in [0,1]$, of a function $f : \mathbb{R} \to \mathbb{X}$ is defined by

$$f^b(t,s) := f(t+s).$$

Definition 2.5 ([4]). The Bochner transform $f^b(t,s,u)$, $t \in \mathbb{R}$, $s \in [0,1]$, $u \in \mathbb{X}$ of a function $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ is defined by

$$f^b(t,s,u) := f(t+s,u) \quad \text{for each } u \in \mathbb{X}.$$

Definition 2.6 ([4,34]). Let $p \in [1,\infty)$. The space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to \mathbb{X}$ such that $f^b \in L^\infty(\mathbb{R}, L^p(0,1;\mathbb{X}))$. This is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p d\tau \right)^{\frac{1}{p}}.$$

Definition 2.7 ([34,35]). The space $AS^p(\mathbb{X})$ of Stepanov-like almost automorphic (or S^p-almost automorphic) functions consists of all $f \in BS^p(\mathbb{X})$ such that $f^b \in AA(L^p(0,1;\mathbb{X}))$. In other words, a function $f \in L^p_{loc}(\mathbb{R},\mathbb{X})$ is said to be S^p-almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p(0,1;\mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $g \in L^p_{loc}(\mathbb{R},\mathbb{X})$ such that

$$\lim_{n\to\infty} \left(\int_t^{t+1} \|f(s+s_n) - g(s)\|^p ds \right)^{\frac{1}{p}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \left(\int_t^{t+1} \|g(s-s_n) - f(s)\|^p ds \right)^{\frac{1}{p}} = 0$$

pointwise on $\mathbb{R}$.

Definition 2.8 ([34,35]). A function $f : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$, $(t,u) \to f(t,u)$ with $f(\cdot,u) \in L^p_{loc}(\mathbb{R},\mathbb{X})$ for each $u \in \mathbb{Y}$, is said to be S^p-almost automorphic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{Y}$ if $t \to f(t,u)$ is S^p-almost automorphic for each $u \in \mathbb{Y}$. That means, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $g(\cdot,u) \in L^p_{loc}(\mathbb{R},\mathbb{X})$ such that

$$\lim_{n\to\infty} \left(\int_t^{t+1} \|f(s+s_n,u) - g(s,u)\|^p ds \right)^{\frac{1}{p}} = 0,$$

and

$$\lim_{n\to\infty} \left(\int_t^{t+1} \|g(s-s_n,u) - f(s,u)\|^p ds \right)^{\frac{1}{p}} = 0,$$

pointwise on $\mathbb{R}$ and for each $u \in \mathbb{Y}$. We denote by $AS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ the set of all such functions.

Definition 2.9 ([7]). Let $\rho \in \mathbb{U}_\infty$. A function $f \in BS^p(\mathbb{X})$ is said to be Stepanov-like weighted pseudo almost automorphic (or S^p-weighted pseudo almost automorphic) if it can be expressed as $f = g + h$, where $g \in AS^p(\mathbb{X})$ and $h^b \in PAA_0(L^p(0,1;\mathbb{X}),\rho)$. In other words, a function $f \in L^p_{loc}(\mathbb{R},\mathbb{X})$ is said to be Stepanov-like weighted pseudo almost automorphic relatively to the weight $\rho \in \mathbb{U}_\infty$, if its Bochner transform $f^b : \mathbb{R} \to L^p(0,1;\mathbb{X})$ is weighted pseudo almost automorphic in the sense that there exist two functions $g, h : \mathbb{R} \to \mathbb{X}$ such that $f = g + h$, where $g \in AS^p(\mathbb{X})$ and $h^b \in PAA_0(L^p(0,1;\mathbb{X}), \rho)$. We denoted by $WPAAS^p(\mathbb{X})$ the set of all such functions.

Definition 2.10 ([7]). Let $\rho \in \mathbb{U}_\infty$. A function $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$, $(t,u) \to f(t,u)$ with $f(\cdot,u) \in L^p_{loc}(\mathbb{R},\mathbb{X})$ for each $u \in \mathbb{Y}$, is said to be Stepanov-like weighted pseudo almost automorphic (or S^p-weighted pseudo almost automorphic) if it can be expressed as $f = g + h$, where $g \in AS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ and $h^b \in PAA_0\,(\mathbb{Y}, L^p(0,1;\mathbb{X}), \rho)$. We denoted by $WPAAS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ the set of all such functions.

Remark 2.1. It is clear that if $1 \le p < q < \infty$ and $f \in L^q_{loc}(\mathbb{R},\mathbb{X})$ is S^q-almost automorphic, then f is S^p-almost automorphic. Also if $f \in AA(\mathbb{X})$, then f is S^p-almost automorphic for any $1 \le p < \infty$.

Lemma 2.4 ([7]). *Let $\rho \in \mathbb{U}_\infty$. Assume that $PAA_0(L^p(0,1;\mathbb{X}),\rho)$ is translation invariant. Then the decomposition of a S^p-weighted pseudo almost automorphic function is unique.*

Lemma 2.5 ([7]). *If $f \in WPAA(\mathbb{X})$, then $f \in WPAAS^p(\mathbb{X})$ for each $1 \le p < \infty$. In other words, $WPAA(\mathbb{X}) \subset WPAAS^p(\mathbb{X})$.*

Lemma 2.6 ([7]). *Let $\rho \in \mathbb{U}_\infty$. The space $WPAAS^p(\mathbb{X})$ equipped with the norm $\|\cdot\|_{S^p}$ is a Banach space.*

Theorem 2.1 ([7]). *Let $\rho \in \mathbb{U}_\infty$ and let $f = g + h \in WPAAS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$ with $g \in AS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$, $h^b \in PAA_0(\mathbb{X}, L^p(0,1;\mathbb{X}),\rho)$. Assume that the following conditions* (i) *and* (ii) *are satisfied:*

(i) *$f(t,x)$ is Lipschitzian in $x \in \mathbb{X}$ uniformly in $t \in \mathbb{R}$; that is, there exists a constant $L > 0$ such that*

$$\|f(t,x) - f(t,y)\| \le L\|x - y\|$$

for all $x, y \in \mathbb{X}$ and $t \in \mathbb{R}$.

(ii) *$g(t,x)$ is uniformly continuous in any bounded subset $K' \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$.*

If $u = u_1 + u_2 \in WPAAS^p(\mathbb{X})$, with $u_1 \in AS^p(\mathbb{X})$, $u_2^b \in PAA_0(L^p(0,1;\mathbb{X}),\rho)$ and $K = \overline{\{u_1(t) : t \in \mathbb{R}\}}$ is compact, then $\Lambda : \mathbb{R} \to \mathbb{X}$ defined by $\Lambda(\cdot) = f(\cdot, u(\cdot))$ belongs to $WPAAS^p(\mathbb{X})$.

Theorem 2.2 ([8]). *Let $\rho \in \mathbb{U}_\infty$ and let $f = g + h \in WPAAS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$ with $g \in AS^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$, $h^b \in PAA_0(\mathbb{X}, L^p(0,1;\mathbb{X}),\rho)$. Assume that the following conditions* (i) (ii) *and* (iii) *are satisfied:*

(i) *there exists a nonnegative function $L \in BS^p(\mathbb{R})$ with $p > 1$ such that for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$*

$$\left(\int_t^{t+1} \|f(s,u) - f(s,v)\|^p ds\right)^{\frac{1}{p}} \le L(t)\|u - v\|$$

(ii) *$\rho \in L^q_{loc}(\mathbb{R})$ satisfies $\lim_{T\to\infty} \sup \frac{T^{\frac{1}{p}} m_q(T,\rho)}{m(T,\rho)} < \infty$.*

(iii) *$g(t,x)$ is uniformly continuous in any bounded subset $K \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$.*

If $u = u_1 + u_2 \in WPAAS^p(\mathbb{X})$, with $u_1 \in AS^p(\mathbb{X})$, $u_2^b \in PAA_0(L^p(0,1;\mathbb{X}),\rho)$ and $K = \overline{\{u_1(t) : t \in \mathbb{R}\}}$ is compact, then $\Lambda : \mathbb{R} \to \mathbb{X}$ defined by $\Lambda(\cdot) = f(\cdot, u(\cdot))$ belongs to $WPAAS^p(\mathbb{X})$.

Theorem 2.3 ([8]). *Let $\rho \in \mathbb{U}_\infty$ and let $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ be a S^p-weighted pseudo almost automorphic function. suppose that f satisfies the following conditions :*

(i) *$f(t,x)$ is uniformly continuous in any bounded subset $K' \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$,*

(ii) *$g(t,x)$ is uniformly continuous in any bounded subset $K' \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$.*

(iii) *For every bounded subset $K' \subset \mathbb{X}$, $\{f(\cdot,x) : x \in K'\}$ is bounded in $WPAAS^p(\mathbb{X})$.*

If $x = \alpha + \beta \in WPAAS^p(\mathbb{X})$, with $\alpha \in AS^p(\mathbb{X})$, $\beta^b \in PAA_0(L^p(0,1;\mathbb{X}),\rho)$ and $K = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact, then defined by $f(\cdot, x(\cdot))$ belongs to $WPAAS^p(\mathbb{X})$.

Now, we recall a useful compactness criterion.

Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $h(t) \ge 1$ for all $t \in \mathbb{R}$ and $h(t) \to \infty$ as $|t| \to \infty$. We consider the space

$$C_h(\mathbb{X}) = \left\{u \in C(\mathbb{R},\mathbb{X}) : \lim_{|t|\to\infty} \frac{u(t)}{h(t)} = 0\right\}.$$

Endowed with the norm $\|u\|_h = \sup_{t\in\mathbb{R}} \frac{\|u(t)\|}{h(t)}$, it is a Banach space (see [36]).

Lemma 2.7 ([36]). *A subset $R \subseteq C_h(\mathbb{X})$ is a relatively compact set if it verifies the following conditions:*

(c-1) *The set $R(t) = \{u(t) : u \in R\}$ is relatively compact in $\mathbb{X}$ for each $t \in \mathbb{R}$.*
(c-2) *The set R is equicontinuous.*
(c-3) *For each $\epsilon > 0$ there exists $L > 0$ such that $\|u(t)\| \le \epsilon h(t)$ for all $u \in R$ and all $|t| > L$.*

Lemma 2.8 ([37]Leray–Schauder Alternative Theorem). *Let D be a closed convex subset of a Banach space $\mathbb{X}$ such that $0 \in D$. Let $F : D \to D$ be a completely continuous map. Then the set $\{x \in D : x = \lambda F(x), 0 < \lambda < 1\}$ is unbounded or the map F has a fixed point in D.*

3. Weighted pseudo almost automorphic mild solutions

Before starting our main results in this subsection, we recall the definition of the mild solution to Eq. (1.2).

Definition 3.1 ([22]). Assume that A generates an integrable solution operator $E_\alpha(t)$. A continuous function $u : \mathbb{R} \to \mathbb{X}$ satisfying the integral equation

$$u(t) = \int_{-\infty}^{t} E_\alpha(t-s) f(s, u(s)) ds, \quad t \in \mathbb{R} \tag{3.1}$$

is called a mild solution on $\mathbb{R}$ to Eq. (1.2).

We make the following assumption:

(H1) A is a sectorial operator of type $\omega < 0$.
(H2) $f \in WPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, there exists a constant $L_f > 0$, such that

$$\|f(t,x) - f(t,y)\| \le L_f \|x - y\|$$

for all $t \in \mathbb{R}$ and each $x, y \in \mathbb{X}$.
(H3) $f \in WPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, there exists a nonnegative function $L_f(\cdot) \in BS^p(\mathbb{R})$, with $p > 1$ such that

$$\|f(t,x) - f(t,y)\| \le L_f(t) \|x - y\|$$

for all $t \in \mathbb{R}$ and each $x, y \in \mathbb{X}$.
(H4) Let $\rho \in L^q_{loc}(\mathbb{R})$ satisfies

$$\lim_{T\to\infty} \frac{T^{\frac{1}{p}} m_q(T,\rho)}{m(T,\rho)} < \infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $m_q(T,\rho) = (\int_{-T}^{T} \rho^q(t) dt)^{\frac{1}{q}}$.
(H5) the function $f = g + h \in WPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ where $g \in AS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ is uniformly continuous in any bounded subset $M \subset \mathbb{X}$ uniformly in $t \in \mathbb{R}$ and $h^b \in PAA_0(\mathbb{X}, L^p(0,1;\mathbb{X}), \rho)$.

Lemma 3.1. *Let $S(t)_{t\ge 0} \subset \mathfrak{B}(\mathbb{X})$ be a strongly continuous family of bounded and linear operators such that*

$$\|S(t)\| \le \varpi(t), \quad t \in \mathbb{R}^+,$$

where $\varpi \in L^1(\mathbb{R}^+)$ is nonincreasing. Then, for each $f \in AS^p(\mathbb{X})$,

$$\int_{-\infty}^{t} S(t-s) f(s) ds \in AA(\mathbb{X}).$$

Proof. Let $p = 1$. The conclusion was given in [21, lemma 2.2]. Let $p > 1$. It follows from Remark 2.1 that $f \in AS^1(\mathbb{X})$. This completes the proof. □

Lemma 3.2. *Let $\rho \in \mathbb{U}_\infty$. Let also $f = g + h \in WPAAS^p(\mathbb{X}, \rho)$ with $g \in AS^p(\mathbb{R}, \mathbb{X})$ and $h^b \in PAA_0(\mathbb{X}, L^p(0,1;\mathbb{X}), \rho)$. Then $F(t) := \int_{-\infty}^{t} E_\alpha(t-s) f(s) ds \in WPAA(\mathbb{X}, \rho)$.*

Proof. Let $F(t) = G(t) + H(t)$ where

$$G(t) := \int_{-\infty}^{t} E_\alpha(t-s) g(s) ds \qquad H(t) := \int_{-\infty}^{t} E_\alpha(t-s) h(s) ds.$$

By (2.1), we have

$$\|E_\alpha(t)\|_{\mathfrak{B}(\mathbb{X})} \le \frac{CM}{1 + |\omega| t^\alpha}, \quad t \ge 0$$

since $1 < \alpha < 1$, $\frac{CM}{(1+|\omega|t^{\alpha})} \in L^1(\mathbb{R}^+)$ and is nonincreasing. So Lemma 3.1 yields that $G(t) \in AA(\mathbb{X})$. Finally, we prove that $H(t) \in PAA_0(\mathbb{X}, \rho)$.

$$\begin{aligned} H(t) &= \int_{-\infty}^{t} E_\alpha(t-s)h(s)ds \\ &= \int_0^{\infty} E_\alpha(s)h(t-s)ds. \end{aligned}$$

Now, we consider for each $n = 0, 1, \ldots,$ the integrals

$$\begin{aligned} H_n(t) &= \int_{t-n-1}^{t-n} E_\alpha(t-\sigma)h(\sigma)d\sigma. \\ \|H_n(t)\| &\le \int_{t-n-1}^{t-n} \|E_\alpha(t-\sigma)h(\sigma)\|d\sigma \\ &\le \int_n^{n+1} \frac{CM}{(1+|\omega|\sigma^\alpha)}\|h(t-\sigma)\|d\sigma \\ &\le \frac{CM}{(1+|\omega|n^\alpha)}\left(\int_n^{n+1}\|h(t-\sigma)\|^p d\sigma\right)^{\frac{1}{p}}. \end{aligned}$$

Then, for $r > 0$, we see that

$$\frac{1}{m(r,\rho)}\int_{-r}^{r}\|H_n(t)\|\rho(t)dt \le \frac{CM}{(1+|\omega|n^\alpha)}\frac{1}{m(r,\rho)}\int_{-r}^{r}\left(\int_n^{n+1}\|h(t-\tau)\|^p d\tau\right)^{\frac{1}{p}}\rho(t)dt.$$

Using the fact that the space $PAA_0(\mathbb{X}, \rho)$ is translation invariant, it follows that $t \to h(t-\sigma)$ belongs to $PAA_0(\mathbb{X}, \rho)$. The above inequality leads to $H_n(t) \in PAA_0(\mathbb{X}, \rho)$ for each $n = 0, 1, \ldots$. The last estimation also leads to

$$\|H_n(t)\| \le \frac{CM}{(1+|\omega|n^\alpha)}\|h\|_{S^p}.$$

Notice that

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{CM}{(1+|\omega|n^\alpha)} &\le \left(CM + \sum_{n=1}^{\infty}\int_{n-1}^{n}\frac{CM}{(1+|\omega|s^\alpha)}ds\right) \\ &\le CM + \int_0^{\infty}\frac{CM}{(1+|\omega|s^\alpha)}ds \\ &\le CM\left(1 + \frac{|\omega|^{-\frac{1}{\alpha}}\pi}{\alpha\sin\left(\frac{\pi}{\alpha}\right)}\right) < \infty. \end{aligned}$$

Then, we deduce from the Weierstrass test that the series $\sum_{n=0}^{\infty} H_n(t)$ is uniformly convergent on $\mathbb{R}$. Moreover,

$$H(t) = \int_{-\infty}^{t} E_\alpha(t-s)h(s)ds = \sum_{n=0}^{\infty} H_n(t).$$

Clearly, $H(t) \in C(\mathbb{R}, \mathbb{X})$ and

$$\|H(t)\| \le \sum_{n=0}^{\infty}\|H_n(t)\| \le CM\left(1 + \frac{|\omega|^{-\frac{1}{\alpha}}\pi}{\alpha\sin\left(\frac{\pi}{\alpha}\right)}\right)\|h\|_{S^p}.$$

Applying $H_n(t) \in PAA_0(\mathbb{X}, \rho)$ and the inequality

$$\begin{aligned} \frac{1}{m(r,\rho)}\int_{-r}^{r}\|H(t)\|\rho(t)dt &\le \frac{1}{m(r,\rho)}\int_{-r}^{r}\left\|H(t) - \sum_{k=0}^{n} H_k(t)\right\|\rho(t)dt \\ &\quad + \sum_{k=0}^{n}\frac{1}{m(r,\rho)}\int_{-r}^{r}\|H_k(t)\|\rho(t)dt, \end{aligned}$$

we deduce that the uniform limit $H(\cdot) = \sum_{n=0}^{\infty} H_n(t) \in PAA_0(\mathbb{X}, \rho)$. Therefore, $F(t) = G(t) + H(t)$ is weighted pseudo almost automorphic. □

Theorem 3.1. *Let $\rho \in \mathbb{U}_\infty$. Assume that* (H1), (H2) *and* (H5) *hold. Then* (1.2) *has a unique mild solution in* $WPAA(\mathbb{X}, \rho)$ *provided*

$$CM|\omega|^{-\frac{1}{\alpha}}\pi L_f < \alpha \sin\left(\frac{\pi}{\alpha}\right). \tag{3.2}$$

Proof. Consider the operator $Q : WPAA(\mathbb{X}, \rho) \to WPAA(\mathbb{X}, \rho)$ such that

$$(Qu)(t) := \int_{-\infty}^{t} E_\alpha(t-s)f(s, u(s))ds, \quad t \in \mathbb{R}.$$

First let us prove that $Q(WPAA(\mathbb{X}, \rho)) \subset WPAA(\mathbb{X}, \rho)$. For each $u \in WPAA(\mathbb{X}, \rho)$, by using the fact that the range of almost automorphic functions is relatively compact with Lemma 2.5 and Theorem 2.1 one can easily see that $f(\cdot, u(\cdot)) \in WPAAS^p(\mathbb{X}, \rho)$. Hence from the proof of Lemma 3.2, we know that $(Qu)(\cdot) \in WPAA(\mathbb{X}, \rho)$. That is Q maps $WPAA(\mathbb{X}, \rho)$ into $WPAA(\mathbb{X}, \rho)$.

Now if $u, v \in WPAA(\mathbb{X}, \rho)$, using (2.1) and (H2), we have

$$\begin{aligned}
\|(Qu)(t) - (Qv)(t)\|_\infty &= \sup_{t\in\mathbb{R}} \left\| \int_{-\infty}^{t} E_\alpha(t-s)[f(s, u(s)) - f(s, v(s))]ds \right\| \\
&\le L_f \sup_{t\in\mathbb{R}} \int_0^\infty \|E_\alpha(s)\|_{\mathfrak{B}(\mathbb{X})} \|u(t-s) - v(t-s)\| ds \\
&\le L_f \|u - v\|_\infty CM \left(\int_0^\infty \frac{1}{(1 + |\omega| s^\alpha)} ds \right) \\
&\le \frac{CML_f |\omega|^{-\frac{1}{\alpha}}\pi}{\alpha \sin\left(\frac{\pi}{\alpha}\right)} \|u - v\|_\infty.
\end{aligned}$$

This proves that Q is a contraction, so by the Banach fixed point theorem Q has a unique fixed point, which gives rise to a unique $u \in WPAA(\mathbb{X}, \rho)$. This completes the proof. □

A different Lipschitz condition is considered in the following result.

Theorem 3.2. *Let $\rho \in \mathbb{U}_\infty$. Assume that* (H1), (H3), (H4) *and* (H5) *are satisfied. Then* (1.2) *has a unique mild solution in* $WPAA(\mathbb{X}, \rho)$ *whenever*

$$\|L_f\|_{S^p} CM \left(1 + \frac{|\omega|^{-\frac{1}{\alpha}}\pi}{\alpha \sin\left(\frac{\pi}{\alpha}\right)} \right) < 1. \tag{3.3}$$

Proof. Consider the nonlinear operator Q given by

$$(Qu)(t) := \int_{-\infty}^{t} E_\alpha(t-s)f(s, u(s))ds, \quad t \in \mathbb{R}.$$

Let $u \in WPAA(\mathbb{X}, \rho)$, with Lemma 2.5 and Theorem 2.2, it follows that the function $s \to f(s, u(s))$ is in $WPAAS^p(\mathbb{R}, \mathbb{X})$. Moreover from Lemma 3.2 we infer that $Qu \in WPAA(\mathbb{X}, \rho)$. That is Q maps $WPAA(\mathbb{X}, \rho)$ into itself. Next, we prove that the operator Q has a unique fixed point in $WPAA(\mathbb{X}, \rho)$. For each $t \in \mathbb{R}$, $u, v \in WPAA(\mathbb{X}, \rho)$, we have

$$\begin{aligned}
\|(Qu)(t) - (Qv)(t)\| &= \left\| \int_{-\infty}^{t} E_\alpha(t-s)[f(s, u(s)) - f(s, v(s))]ds \right\| \\
&\le \int_{-\infty}^{t} \|E_\alpha(t-s)\|_{\mathfrak{B}(\mathbb{X})} \|f(s, u(s)) - f(s, v(s))\| ds \\
&\le \int_{-\infty}^{t} \frac{CM}{1 + |\omega|(t-s)^\alpha} (L_f(s)\|u(s) - v(s)\|) ds \\
&\le \int_0^\infty \frac{CM}{1 + |\omega| s^\alpha} L_f(t-s) \parallel ds \|u - v\|_\infty \\
&= \sum_{k=0}^{\infty} \int_k^{k+1} \frac{CM}{1 + |\omega| s^\alpha} L_f(t-s) ds \|u - v\|_\infty \\
&\le \sum_{k=0}^{\infty} \frac{CM}{1 + |\omega| k^\alpha} \int_k^{k+1} L_f(t-s) ds \|u - v\|_\infty
\end{aligned}$$

$$\leq \sum_{k=0}^{\infty} \frac{CM}{1+|\omega|k^{\alpha}} \left(\int_{t-k-1}^{t-k} \|L_f(s)\|^p ds \right)^{\frac{1}{p}} \|u-v\|_{\infty}$$
$$\leq \sum_{k=0}^{\infty} \frac{CM}{1+|\omega|k^{\alpha}} \|L_f\|_{S^p} \|u-v\|_{\infty}$$
$$\leq \left(CM + \sum_{k=1}^{\infty} \int_{k-1}^{k} \frac{CM}{1+|\omega|s^{\alpha}} ds \right) \|L_f\|_{S^p} \|u-v\|_{\infty}$$
$$= \left(CM + \int_0^{\infty} \frac{CM}{1+|\omega|s^{\alpha}} ds \right) \|L_f\|_{S^p} \|u-v\|_{\infty}$$
$$= CM \left(1 + \frac{|\omega|^{-\frac{1}{\alpha}} \pi}{\alpha \sin\left(\frac{\pi}{\alpha}\right)} \right) \|L_f\|_{S^p} \|u-v\|_{\infty}.$$

Which gives

$$\|(Qu) - (Qv)\|_{\infty} \leq CM \left(1 + \frac{|\omega|^{-\frac{1}{\alpha}} \pi}{\alpha \sin\left(\frac{\pi}{\alpha}\right)} \right) \|L_f\|_{S^p} \|u-v\|_{\infty}.$$

In view of (3.3), Q is a contraction mapping. On the other hand, it is well know that $WPAA(\mathbb{X}, \rho)$ is a Banach space under the supremum norm. Thus, Q has a unique fixed point $u \in WPAA(\mathbb{X}, \rho)$, which satisfies

$$u(t) = \int_{-\infty}^{t} E_{\alpha}(t-s)\,(f(s, u(s)))ds,$$

for all $t \in \mathbb{R}$. Thus Eq. (1.2) has a unique weighted pseudo almost automorphic mild solution. □

We next study the existence of weighted pseudo almost automorphic mild solutions of Eq. (1.2) when the perturbation f is not Lipschitz continuous. For that, we require the following assumptions:

(H6) $f \in WPAAS^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ and $f(t, x)$ is uniformly continuous in any bounded subset $M \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$ and for every bounded subset $M \subset \mathbb{X}$, $\{f(\cdot, x) : x \in M\}$ is bounded in $WPAAS^p(\mathbb{X})$.

(H7) There exists a continuous nondecreasing function $W : [0, \infty) \to (0, \infty)$ such that

$$\|f(t, x)\| \leq W(\|x\|) \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \mathbb{X}.$$

The following existence result is based upon the nonlinear Leray–Schauder alternative theorem. It corresponds to an extension of [22, Theorem 3.3].

Theorem 3.3. *Let $\rho \in \mathbb{U}_{\infty}$. Assume that A is sectorial of type $\omega < 0$. Let $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ be a function that satisfies assumptions* (H5)– (H7), *and the following additional conditions:*

(i) *For each $z \geq 0$,*

$$\lim_{|t|\to\infty} \frac{1}{h(t)} \int_{-\infty}^{t} \frac{W(zh(s))}{1+|\omega|(t-s)^{\alpha}} ds = 0,$$

where h is the function given in Lemma 2.7.

We set

$$\beta(z) = CM \left\| \int_{-\infty}^{t} \frac{W(zh(s))}{1+|\omega|(t-s)^{\alpha}} ds \right\|_h,$$

where C and M are the constants given in (2.1).

(ii) *For each $\epsilon > 0$ there is $\delta > 0$ such that for every $u, v \in C_h(\mathbb{X})$, $\|u-v\|_h \leq \delta$ implies that*

$$\int_{-\infty}^{t} \frac{\|f(s, u(s)) - f(s, v(s))\|}{1+|\omega|(t-s)^{\alpha}} ds \leq \epsilon,$$

for all $t \in \mathbb{R}$.

(iii) $\liminf_{\xi\to\infty} \frac{\xi}{\beta(\xi)} > 1$.

(iv) *For all $a, b \in \mathbb{R}$, $a < b$, and $z > 0$, the set $\{f(s, h(s)x) : a \leq s \leq b, x \in C_h(\mathbb{X}), \|x\|_h \leq z\}$ is relatively compact in $\mathbb{X}$.*

then Eq. (1.2) has a weighted pseudo almost automorphic mild solution.

Proof. We define the nonlinear operator $Q : C_h(\mathbb{X}) \to C_h(\mathbb{X})$ by

$$(Qu)(t) := \int_{-\infty}^{t} E_\alpha(t-s) f(s, u(s)) ds, \quad t \in \mathbb{R}.$$

We will show that Q has a fixed point in $WPAA(\mathbb{X}, \rho)$. For the sake of convenience, we divide the proof into several steps.

(I) For $u \in C_h(\mathbb{X})$, we have that

$$\frac{\|(Qu)(t)\|}{h(t)} \le \frac{CM}{h(t)} \int_{-\infty}^{t} \frac{W(\|u\|_h h(s))}{1+|\omega|(t-s)^\alpha} ds.$$

It follows from condition (i) that Q is well defined.

(II) The operator Q is continuous. In fact, for any $\epsilon > 0$, we take $\delta > 0$ involved in condition (ii). If $u, v \in C_h(\mathbb{X})$ and $\|u - v\|_h \le \delta$, then

$$\|(Qu)(t) - (Qv)(t)\| \le CM \int_{-\infty}^{t} \frac{\|f(s, u(s)) - f(s, v(s))\|}{1+|\omega|(t-s)^\alpha} ds \le \epsilon,$$

which shows the assertion.

(III) We will show that Q is completely continuous. We set $B_z(\mathbb{X})$ for the closed ball with center at 0 and radius z in the space $\mathbb{X}$. Let $V = Q(B_z(C_h(\mathbb{X})))$ and $v = Q(u)$ for $u \in B_z(C_h(\mathbb{X}))$. First, we will prove that $V(t)$ is a relatively compact subset of $\mathbb{X}$ for each $t \in \mathbb{R}$. It follows from condition (i) that for $\epsilon > 0$, we can choose $a \ge 0$ such that $CM \int_a^\infty \frac{W(zh(t-s))}{1+|\omega|s^\alpha} ds \le \epsilon$. Since

$$v(t) = \int_0^a E_\alpha(s) f(t-s, u(t-s)) ds + \int_a^\infty E_\alpha(s) f(t-s, u(t-s)) ds$$

and

$$\left\| \int_a^\infty E_\alpha(s) f(t-s, u(t-s)) ds \right\| \le CM \int_a^\infty \frac{W(zh(t-s))}{1+|\omega|s^\alpha} ds \le \epsilon,$$

we get $v(t) \in \overline{ac_0(N)} + B_\varepsilon(\mathbb{X})$, where $c_0(N)$ denotes the convex hull of N and $N = \{E_\alpha(s) f(\xi, h(\xi)x) : 0 \le s \le a, t-a \le \xi \le t, \|x\|_h \le z\}$. Using the strong continuity of $E_\alpha(\cdot)$ and property (iv) of f, we infer that N is a relatively compact set, and $V(t) \subseteq \overline{ac_0(N)} + B_\varepsilon(\mathbb{X})$, which establishes our assertion.

Second, we show that the set V is equicontinuous. In fact, we can decompose

$$\begin{aligned} v(t+s) - v(t) = & \int_0^s E_\alpha(\sigma) f(t+s-\sigma, u(t+s-\sigma)) d\sigma \\ & + \int_0^a [E_\alpha(\sigma+s) - E_\alpha(\sigma)] f(t-\sigma, u(t-\sigma)) d\sigma \\ & + \int_a^\infty [E_\alpha(\sigma+s) - E_\alpha(\sigma)] f(t-\sigma, u(t-\sigma)) d\sigma. \end{aligned}$$

For each $\epsilon > 0$, we can choose $a > 0$ and $\delta_1 > 0$ such that

$$\begin{aligned} & \left\| \int_0^s E_\alpha(\sigma) f(t+s-\sigma, u(t+s-\sigma)) d\sigma + \int_a^\infty [E_\alpha(\sigma+s) - E_\alpha(\sigma)] f(t-\sigma, u(t-\sigma)) d\sigma \right\| \\ & \le CM \left(\int_0^s \frac{W(zh(t+s-\sigma))}{1+|\omega|\sigma^\alpha} d\sigma + 2 \int_a^\infty \frac{W(zh(t-\sigma))}{1+|\omega|\sigma^\alpha} d\sigma \right) \\ & \le \frac{\epsilon}{2}, \end{aligned}$$

for $s \le \delta_1$. Moreover, since $\{f(t-\sigma, u(t-\sigma)) : 0 \le \sigma \le a, u \in B_z(C_h(\mathbb{X}))\}$ is a relatively compact set and $E_\alpha(\cdot)$ is strongly continuous, we can choose $\delta_2 > 0$ such that $\|[E_\alpha(\sigma+s) - E_\alpha(\sigma)] f(t-\sigma, u(t-\sigma))\| \le \frac{\epsilon}{2a}$ for $s \le \delta_2$. Combining these estimates, we get $\|v(t+s) - v(t)\| \le \epsilon$ for s small enough and independent of $u \in B_z(C_h(\mathbb{X}))$.

Finally, applying condition (i), we can see that

$$\frac{\|v(t)\|}{h(t)} \le \frac{CM}{h(t)} \int_{-\infty}^{t} \frac{W(\|u\|_h h(s))}{1+|\omega|(t-s)^\alpha} ds \to 0, \quad |t| \to \infty,$$

and this convergence is independent of $u \in B_z(C_h(\mathbb{X}))$. Hence, by Lemma 2.7, V is a relatively compact set in $(C_h(\mathbb{X}))$.

(IV) Let us show assume that $u^\lambda(\cdot)$ is a solution of equation $u^\lambda = \lambda Q(u^\lambda)$ for some $0 < \lambda < 1$. We can estimate

$$\begin{aligned}\|u^\lambda(t)\| &= \lambda \left\| \int_{-\infty}^{t} E_\alpha(t-s) f(s, u^\lambda(s)) ds \right\| \\ &\le CM \int_{-\infty}^{t} \frac{W(\|u^\lambda\|_h h(s))}{1+|\omega|(t-s)^\alpha} ds \\ &\le \beta(\|u^\lambda\|_h) h(t).\end{aligned}$$

Hence, we get

$$\frac{\|u^\lambda\|_h}{\beta(\|u^\lambda\|_h)} \le 1$$

and combining with condition (iii), we conclude that the set $\{u^\lambda : u^\lambda = \lambda Q(u^\lambda), \lambda \in (0,1)\}$ is bounded.

(V) It follows from Lemma 2.5, (H5)–(H6) and Theorem 2.3 that the function $t \to f(t, x(t))$ belongs to $WPAAS^p(\mathbb{R}, \mathbb{X})$, whenever $x \in WPAA(\mathbb{X}, \rho)$. Moreover, from Lemma 3.2 we infer that $Q(WPAA(\mathbb{X}, \rho)) \subset WPAA(\mathbb{X}, \rho)$ and noting that $WPAA(\mathbb{X}, \rho)$ is a closed subspace of $C_h(\mathbb{X})$, consequently, we can consider $Q : WPAA(\mathbb{X}, \rho) \to WPAA(\mathbb{X}, \rho)$. Using proposition (I)–(III), we deduce that this map is completely continuous. Applying Lemma 2.8, we infer that Q has a fixed point $x \in WPAA(\mathbb{X}, \rho)$, which completes the proof. □

4. Application

To illustrate Theorem 3.1 we consider the following fractional differential equation given by:

$$\partial_t^\alpha u(t,x) = \partial_x^2 u(t,x) - \mu u(t,x) + \partial_t^{\alpha-1}\left(\beta u(t,x)\left(\sin\frac{1}{2+\cos t+\cos \pi t}\right) + \beta e^{-|t|}\sin(u(t,x))\right), \tag{4.1}$$

$t \in \mathbb{R},\ x \in [0, \pi]$,

with boundary conditions

$$u(t,0) = (t,\pi) = 0, \quad t \in \mathbb{R}.$$

Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}}) = (L^2([0,\pi]), \|\cdot\|_2)$ and the operator A defined on $\mathbb{X}$ by $Au = u' - \mu u$, $(\mu > 0)$ with domain

$$D(A) = \{u \in \mathbb{X} : u' \in \mathbb{X}, u(0) = u(\pi) = 0\}.$$

It is well known that $\triangle u = u'$ is the infinitesimal generator of an analytic semigroup on $L^2[0,\pi]$. Hence, A is sectorial of type $\omega = -\mu < 0$. Eq. (4.1) can be formulated by the inhomogeneous problem (1.2), where $u(t) = u(t, \cdot)$. Let us consider the nonlinearity $f(t,\phi)(s) = \beta\phi(s)\sin\frac{1}{2+\cos t+\cos\pi t} + \beta e^{-|t|}\sin(\phi(s)) = g(t,\phi)(s) + h(t,\phi)(s)$, for all $\phi \in \mathbb{X}$, $t \in \mathbb{R}$, $s \in [0,\pi]$ and $\beta \in \mathbb{R}$. By [34, Example 2.3], $\sin\frac{1}{2+\cos t+\cos\pi t} \in AS^2(\mathbb{R})$. Then $g \in AS^2(\mathbb{R}\times\mathbb{X}, \mathbb{X})$. Let $h_1(t) = e^{-|t|}$, one can easily check that $h_1^b(t)$ belongs to $PAA_0(\mathbb{X}, L^p(0,1;\mathbb{X}), \rho)$. Consequently, f is S^p-weighted pseudo almost automorphic function with weight $\rho(t) = 1+t^2$ for $t \in \mathbb{R}$. Assume that $|\beta| < \frac{\alpha\sin(\frac{\pi}{\alpha})}{3CM|\mu|^{-\frac{1}{\alpha}}\pi}$, then, by Theorem 3.1, Eq. (4.1) has a unique weighted pseudo almost automorphic solution.

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