A new Turán-type theorem for cliques in graphs

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Abstract

Turán's theorem (Mat. Fiz. Lapok 48 (1941) 436) (or rather its extension by Zykov (Mat. Sbornik 24 (66) (1949) 163) answers the following question: For \( k = 2, \ldots, r \), what is the maximum number of \( k \)-cliques (i.e., subgraphs on \( k \) vertices) in a finite graph \( G \), given the clique number \( r \) and the number of vertices of \( G \)? Here we address—and answer—the following closely related question: For \( k = 3, \ldots, r \), what is the maximum number of \( k \)-cliques in \( G \), given the clique number \( r \) and the number of edges of \( G \)? We also prove a "stability theorem" which shows that our result is best possible in a strong sense.

Keywords: Turán's theorem; Zykov's theorem; Clique vectors

1. Introduction and statement of results

Let \( G \) be a finite graph. (All graphs in this paper will be without loops or multiple edges.) Following Bollobás [3] and others, we call the number of vertices of \( G \) the order and the number of edges the size of \( G \). A clique in \( G \) is a complete subgraph of \( G \). More precisely, a \( k \)-clique is a clique of order \( k \). The number of \( k \)-cliques in \( G \) will be denoted by \( c_k(G) \). Thus \( c_1(G) \) is the order and \( c_2(G) \) the size of \( G \). The clique number of \( G \) is the largest \( k \) for which \( c_k(G) > 0 \). It is usually denoted by \( \omega(G) \). Finally, the clique vector of \( G \) is the sequence

\[
c(G) = (c_1(G), c_2(G), \ldots, c_r(G)),
\]

where \( r = \omega(G) \).

How large can the entries of \( c(G) \) be, given the order and the clique number of \( G \)? This question is answered by Turán's theorem, a cornerstone of extremal graph theory (see [3]). Recall that for \( n \geq r \), the Turán graph \( T_r(n) \) is the complete \( r \)-partite graph of order \( n \) whose vertex classes are as nearly equal in size as possible. In other words, the vertex set of \( T_r(n) \) splits into \( r \) subsets of sizes

\[
\left\lceil \frac{n}{r} \right\rceil, \left\lceil \frac{n+1}{r} \right\rceil, \ldots, \left\lceil \frac{n+r-1}{r} \right\rceil,
\]

the vertex classes of \( G \), such that two vertices are joined by an edge if and only if they belong to different vertex classes. The clique number of \( T_r(n) \) is clearly \( r \).

Then we have

Turán's theorem (Turán [13]). If a graph \( G \) has order \( n \) and clique number \( r \), then for \( k = 2, \ldots, r \),

\[
c_k(G) \leq c_k(T_r(n)).
\]

Moreover, this inequality is strict for all such \( k \) unless \( G \) is isomorphic to \( T_r(n) \).
Explicit expressions for the numbers $c_k(T_r(n))$ will be given shortly.

To be precise, Turán [13] established the case $k=2$ of the result; the more general version above is an easy consequence. It was first proved by Zykov [14] and later, independently, by Sauer [12], Erdős [7], Roman [11], Hadzhiivanov [10] and others.

Our aim in the present paper is to prove an analogue of Turán’s theorem where instead of the order of $G$, the size of $G$ is given. What is the maximum number of $k$-cliques that $G$ can have under these circumstances? This question is answered in Theorem 1 below. The main ingredients of the proof will be Turán’s theorem just stated, Brook’s [4] classical theorem on graph colorings, and the more recent “colored Kruskal–Katona theorem” of Frankl et al. [8].

In the sequel, we shall use the convenient notation
\[
\binom{n}{k}_r := c_k(T_r(n))
\]
introduced in [8]. By definition, $(\binom{n}{k}_r)$ is the $k$th elementary symmetric function of the numbers appearing in (1). More explicitly, if $n = pr + q$ with integers $p$ and $q$ satisfying $0 \leq q < r$, then $q$ of the numbers are equal to $p + 1$ and $r - q$ are equal to $p$. This yields
\[
\binom{n}{k}_r = \sum_{i=0}^{k} \binom{q}{i} \binom{r-i}{k-i} (p+1)^{p-k-i}.
\]

We adopt the convention that $(\binom{n}{k}_r) = 0$ if $k > r$, and $(\binom{n}{0}_r) = 1$. Note that $(\binom{n}{1}_r) = n$.

All these statements remain valid for $n \leq r$. In that case, $n$ of the entries in (1) are equal to 1 and $r - n$ are equal to 0. Hence $T_r(n)$ is the complete graph on $n$ vertices and $(\binom{n}{k}_r) = (\binom{n}{r})$.

For this reason, the numbers $(\binom{n}{k}_r)$ are sometimes called “generalized binomial coefficients”. They satisfy the recurrence relation
\[
\binom{n}{k}_r = \binom{n-1}{k}_r + \binom{\left\lfloor \frac{r-1}{n} \right\rfloor}{k-1}_{r-1}
\]
in which $n$, $k$ and $r$ can be any positive integers (with $r > 1$). The easy proof will be given in Section 3.

For $k = 2$, relation (2) becomes $(\binom{n}{2}_2) = (\binom{n-1}{2})_2 + [(r-1)/r]n$. From this it follows that every positive integer $N$ is uniquely expressible in the form
\[
N = (\binom{n}{2}_2)_r + m,
\]
where $n$ and $m$ are integers satisfying $0 \leq m < (r-1)/r)n$. We call (3) the $r$-canonical representation of $N$. Given this representation we define, for each integer $k \geq 3$,$
\tilde{c}_k(N) := (\binom{n}{k}_r) + \binom{m}{k-1}_{r-1}.
\]

We also define $\tilde{c}_k(0) := 0$.

In what follows, the functions $\tilde{c}_k$ will play the role of the generalized binomial coefficients appearing in Turán’s theorem. Note that $\tilde{c}_k(N)$ is an increasing function of $N$, and that $\tilde{c}_k(N) = 0$ if $k > r$.

We can now state our main results. The first is the proposed analogue of Turán’s theorem.

**Theorem 1.** Let $G$ be a graph with clique number $r$ ($r \geq 3$). Then
\[
c_k(G) \leq \tilde{c}_k(c_2(G)), \quad k = 3, \ldots, r.
\]

Moreover, if $v$ and $e$ are integers with $(\binom{v}{e}) \leq e \leq (\binom{v}{e})_r$, then there exists a graph of order $v$, size $e$ and clique number $r$ for which equality holds in (4) for all $k$.

We postpone the proof of Theorem 1 until Section 4, except that the graphs for which equality is attained in (4) will be described in Section 2. For $r = 3$, the result reduces to Theorem 1 in [5]. Notice that the upper bound in (4) does not depend on the order of $G$.

In contrast to what is true for Turán’s theorem, equality in (4) for some $k \geq 3$ does not imply equality for all such $k$. This can be seen from easy examples. Nevertheless, there is a strong “stability theorem” associated with Theorem 1. This is the content of the second result we are going to prove here.

As before, let $G$ be a graph with clique number $r$. Assume that we have $c_2(G) \leq (\binom{v}{e})_r$, where $n \leq c_1(G)$ is some integer. Then Erdős [7, p. 463] conjectured that $c_k(G) \leq (\binom{v}{e})_r$, $k = 3, \ldots, r$. Since $\tilde{c}_k$ maps $\binom{v}{e}$ to $\binom{v}{e}$, Theorem 1 shows that this is indeed true. The result is a sharpening of Turán’s theorem in that $n$ need not be the order of $G$. In fact, much more can be said.
For convenience, the following abbreviation will be used throughout the rest of the paper:

\[ n||r := \left[ \frac{r - 2}{r - 1} \frac{r - 1 - n}{r} \right]. \]

Notice that \([((r - 1)/r)n] \] is the sum of the first \(r - 1\) numbers in (1), and so \(n||r \) is the sum of the first \(r - 2\) of these numbers.

Then we have

**Theorem 2.** Let \( G \) be a graph with clique number \( r \) \((r \geq 3)\). Suppose \( c_2(G) \leq \left( \frac{n}{k} \right)_r \) for some integer \( n \leq c_1(G) \). Then either

\[ c_k(G) = \left( \frac{n}{k} \right)_r, \quad k = 2, \ldots, r, \]

in which case \( G \) is obtained from \( T_r(n) \) by adding \( c_1(G) - n \) isolated vertices, or

\[ c_k(G) \leq \left( \frac{n}{k} \right)_r - \left( \frac{n}{k - 2} \right)_r, \quad k = 3, \ldots, r. \] (5)

Moreover, given integers \( v, n \) and \( r \) with \( v \geq n > r \), there exists a graph of order \( v \), size \( \left( \frac{n}{k} \right)_r - 1 \) and clique number \( r \) for which equality holds in (5) for all \( k \). If \( v > n \), then there is such a graph of size \( \left( \frac{n}{k} \right)_r \) as well.

An isolated vertex of a graph is a vertex not adjacent to any other vertex.

We point out that Theorem 2 reduces to Theorem 2 in our earlier paper [5] when \( r = 3 \). In that case, \( n||r = [n/3] \). The proof of Theorem 2 will also be given in Section 4. However, the graphs achieving equality in (5) are already constructed in Section 2.

**2. Examples**

In this section, we shall describe the graphs alluded to above which show that Theorems 1 and 2 are best possible in the sense stated there. We also illustrate the conclusions of Theorems 1 and 2 with two concrete examples, leaving the necessary computations to the reader.

We begin with Theorem 1.

Let \( v, e \) and \( r \) be given integers satisfying \( r \geq 3 \) and \( \left( \frac{n}{k} \right)_r \leq e \leq \left( \frac{m}{k} \right)_r \). Then we have \( e = \left( \frac{n}{k} \right)_r + m \), for some uniquely determined integers \( n \) and \( m \) with \( 0 \leq m < ((r - 1)/r)n \). Since \( (((r - 1)/r)n) = n - ((n + r - 1)/r) \), the total number of vertices in the \( r - 1 \) smaller vertex classes of \( T_r(n) \) is at least \( m \). (We think of the classes as arranged in increasing order.) Hence the subgraph of \( T_r(n) \) induced by the union of these classes contains a graph \( T_{r-1}(m) \). Now add \( v - n \) new (isolated) vertices to \( T_r(n) \) and join one of them to every vertex of the induced subgraph \( T_{r-1}(m) \). (The latter is required for \( m > 0 \) only.) The resulting graph \( G \) has the desired properties, that is, \( c_1(G) = v \), \( c_2(G) = e \) and \( c_k(G) = \left( \frac{n}{k} \right)_r + \left( \frac{m}{k - 1} \right) \), \( k = 3, \ldots, r \).

**Example 1.** Take \( r = 6 \) and \( e = 30 \). The 6-canonical representation of 30 is easily seen to be \( \left( \frac{3}{6} \right)_6 + 4 \). Hence, by Theorem 1, every graph with 30 edges and clique number 6 has at most

\[ \vec{c}_3(30) = \left( \frac{8}{3} \right)_6 + \left( \frac{4}{2} \right)_5 = 50, \]

\[ \vec{c}_4(30) = \left( \frac{8}{4} \right)_6 + \left( \frac{4}{3} \right)_5 = 45, \]

\[ \vec{c}_5(30) = \left( \frac{8}{5} \right)_6 + \left( \frac{4}{4} \right)_5 = 21, \]

\[ \vec{c}_6(30) = \left( \frac{8}{6} \right)_6 + \left( \frac{4}{5} \right)_5 = 4 \]

3-, 4-, 5- and 6-cliques, respectively. Moreover, for every \( v \geq 9 \) there is a graph of order \( v \), size 30 and clique number 6 for which these upper bounds are attained. It suffices to add \( v - 8 \) isolated vertices to the Turán graph \( T_6(8) \) and join one of them to each of the four vertices forming the one-element vertex classes of \( T_6(8) \).

Next we consider Theorem 2.
Suppose \(v, n\) and \(r\) are integers with \(v \geq n > r \geq 3\). Take the union of \(T_r(n)\) with \(v - n\) isolated vertices and delete one of the edges joining vertices of the two largest vertex classes of \(T_r(n)\). Call the resulting graph \(G\). By definition, \(n||r\) is the total number of vertices in the \(r - 2\) smaller vertex classes of \(T_r(n)\). Hence these classes induce a Turán graph \(T_{r-2}(n||r)\). Clearly, \(G\) has the required properties, i.e., \(c_1(G) = v\) and \(c_k(G) = \left(\binom{n}{k} - \binom{n||r}{k-2}\right) - k = 2, \ldots, r\). In particular, \(c_2(G) = \left(\binom{n}{2}\right) - 1\). If \(v > n\), then the edge removed above can be replaced by a new edge incident with one of the isolated vertices. The graph \(H\) so obtained has the same clique vector as \(G\), except that \(c_2(H) = \left(\binom{n}{2}\right)\).

**Example 2.** Take \(r = 5\) and \(n = 12\). Since \(\binom{12}{2} = 57\) and \(12||5 = \left[\binom{12}{4}12\right] = 6\), Theorem 2 asserts that every graph with at most 57 edges and clique number 5 either has exactly \(\binom{12}{3} = 134\) 3-cliques, \(\binom{12}{4} = 156\) 4-cliques and \(\binom{12}{5} = 72\) 5-cliques, or at most

\[
\begin{align*}
\left(\binom{12}{3}\right) - \left(\binom{6}{3}\right) &= 128, \\
\left(\binom{12}{4}\right) - \left(\binom{6}{2}\right) &= 144, \\
\left(\binom{12}{5}\right) - \left(\binom{6}{3}\right) &= 64
\end{align*}
\]

3-, 4- and 5-cliques, respectively.

For each \(v \geq 12\), the graph consisting of \(T_5(12)\) and \(v - 12\) isolated vertices is the unique graph having clique vector \((v, 57, 134, 156, 72)\). The clique vector of any other graph of order \(v\), size at most 57 and clique number 5 is dominated (componentwise) by \(c(G) = (v, 56, 128, 144, 64)\) or by \(c(H) = (v, 57, 128, 144, 64)\). Here \(G\) and \(H\) are the two graphs described above, with \(H\) arising for \(v > 12\) only.

### 3. Preliminaries

Let \(G\) be a graph, and let \(v\) be a vertex of \(G\). The degree of \(v\) in \(G\) is the number of edges of \(G\) incident with \(v\). In particular, \(v\) is isolated if its degree in \(G\) is 0. The minimum, resp., maximum degree of all vertices of \(G\) will be denoted by \(\delta(G)\), resp., \(\Delta(G)\). If \(\bar{G}\) is the complement of \(G\), then clearly \(c_1(G) = \delta(G) + \Delta(\bar{G}) + 1\).

Two important graph invariants associated with \(G\) are the clique number and the chromatic number of \(G\). The clique number \(\omega(G)\) was already defined in Section 1. The chromatic number, denoted by \(\chi(G)\), is the smallest number of colors that can be assigned to the vertices of \(G\) so that no two adjacent vertices receive the same color.

By definition, \(\omega(G)\) is the independence (or stability) number of \(G\), that is, the largest number of vertices of \(G\) no two of which are adjacent. This yields the simple bound \(c_1(G) \leq \omega(G)\chi(G)\) which will be applied in the proof of Theorem 1. A much deeper result is the classical theorem of Brooks that we shall also use in that proof.

**Brook’s theorem** (Brooks [4]). If a graph \(G\) is connected and not a complete graph or an odd cycle (i.e., a cycle of odd order), then \(\chi(G) \leq \Delta(G)\).

The following ways of decomposing a graph will be needed in the sequel.

First, let \(v\) be a vertex of \(G\). Denote by \(G - v\) the subgraph obtained from \(G\) by deleting \(v\) and all the edges incident with \(v\), and by \(G[N_v]\) the subgraph induced by the set \(N_v\) of vertices adjacent to \(v\). Then for all \(k > 0\),

\[
c_k(G) = c_k(G - v) + c_{k-1}(G[N_v]),
\]

where \(c_0(G[N_v]) = 1\).

Second, let \(H\) and \(K\) be two graphs having disjoint vertex sets. The join of \(H\) and \(K\) is the graph \(G\) whose vertex set is the union of the vertex sets of \(H\) and \(K\) and whose edges are the edges of \(H\) and \(K\) and, in addition, all possible edges joining a vertex of \(H\) to a vertex of \(K\). We express this by writing \(G = H + K\). If \(\omega(H) = s\) and \(\omega(K) = t\), then clearly \(\omega(G) = s + t\) and

\[
c_k(G) = \sum_{i+j=k} c_i(H)c_j(K), \quad k = 1, \ldots, s + t.
\]

Here it is understood that \(c_0(H) = c_0(K) = 1, c_i(H) = 0\) if \(i > s\) and \(c_j(K) = 0\) if \(j > t\).
We now establish some properties of the functions \( \tilde{c}_k \) defined in Section 1. As a first step, let us verify the recurrence relation (2).

**Lemma 1.** For all positive integers \( r, n \) and \( k \),
\[
\binom{n}{k}_r = \left( \binom{n-1}{k} \right)_r + \left( \binom{\lceil \frac{n}{r} \rceil}{k-1} \right)_{r-1}.
\]

**Proof.** Let \( G \) be the Turán graph \( T_r(n) \), and let \( v \) be a vertex of \( G \) contained in a largest vertex class of \( G \) (of order \( [(n+r-1)/r] \)). Then \( G-v \) is isomorphic to \( T_r(n-1) \) and \( G[N_v] \) is isomorphic to \( T_{r-1}([(r-1)/r)n]) \), the latter because of \( n - [(n+r-1)/r] = [(r-1)/r)n]. \) As \( \left( \binom{n}{k} \right)_r = c_k(T_r(n)) \) and similarly for the other summands, the assertion follows from (6). 

For a direct (arithmetical) proof, see [6]. Note that Lemma 1 reduces to the familiar recurrence relation for binomial coefficients when \( r \geq n \).

The following is a simple consequence of Lemma 1. Suppose \( a, b, c \) and \( d \) are nonnegative integers satisfying \( a \leq b \leq c \leq d \) and \( a + d = b + c \). Then for \( k = 2, \ldots, r \),
\[
\binom{a}{k}_r + \binom{d}{k}_r \geq \binom{b}{k}_r + \binom{c}{k}_r. \tag{8}
\]

It suffices to note that \( \binom{n}{k}_r - \binom{n-1}{k}_r \) is a non-decreasing function of \( n \).

Then we have

**Lemma 2.** Let \( a \) and \( b \) be nonnegative integers such that, for some positive integer \( n \), \( \binom{a}{2}_r \leq a + b < \binom{n+1}{2}_r \), and \( b \leq [(r-1)/r)n] \). Then for \( k = 3, \ldots, r \),
\[
\tilde{c}_k(a + b) \geq \tilde{c}_k(a) + \binom{b}{k-1}_{r-1}.
\]

**Proof.** Suppose first that \( a > \binom{b}{2}_r \), say \( a = \binom{c}{2}_r + c \) with \( c \geq 0 \). By Lemma 1 and the assumption on \( a + b \), we have \( b + c < [(r-1)/r)n \). Hence, by the definition of \( \tilde{c}_k \),
\[
\tilde{c}_k(a) = \binom{n}{k}_r + \binom{c}{k-1}_{r-1}
\]
and
\[
\tilde{c}_k(a + b) = \binom{n}{k}_r + \binom{b+c}{k-1}_{r-1}.
\]

Thus the assertion reduces to proving that
\[
\binom{b+c}{k-1}_{r-1} \geq \binom{b}{k-1}_{r-1} + \binom{c}{k-1}_{r-1}.
\]

But this is the case \( a = 0 \) of (8) (with \( k \) and \( r \) replaced by \( k-1 \) and \( r-1 \)).

Suppose next that \( a < \binom{b}{2}_r \). Then Lemma 1 and the assumptions on \( a + b \) and \( b \) imply that \( a = \binom{a-1}{2}_r + c \), where \( 0 \leq c \leq [(r-1)/r)n \), and \( a + b = \binom{c}{2}_r + b + c + [(r-1)/r)n \) with \( 0 \leq b + c - [(r-1)/r)n \) < \((r-1)/r)n \). These expressions are thus \( r \)-canonical, whence
\[
\tilde{c}_k(a) = \binom{n-1}{k}_r + \binom{c}{k-1}_{r-1}
\]
and
\[
\tilde{c}_k(a + b) = \binom{n}{k}_r + \binom{b + c - \lceil \frac{n-1}{r} \rceil}{k-1}_{r-1}.
\]
It remains to show that
\[
\left( \frac{b + c - \left\lfloor \frac{-1}{r} \right\rfloor}{k - 1} \right)_{r-1} + \left( \frac{-1}{k - 1} \right)_{r-1} \geq \left( \frac{b}{k - 1} \right)_{r-1} + \left( \frac{c}{k - 1} \right)_{r-1}.
\]
Again, this follows from (8) if we set \(d = [(r - 1)/r]n\).

Our last auxiliary result is Lemma 3 below. It will be used to show that if the assertion of Theorem 1 is true for each of two graphs \(H\) and \(K\), then it is also true for their join \(H + K\).

Let \(d = (d_1, \ldots, d_s)\) and \(e = (e_1, \ldots, e_t)\) be two finite sequences of positive integers. The convolution of \(d\) and \(e\) is the sequence \(c = (c_1, \ldots, c_r)\), defined by \(r := s + t\) and
\[
c_k := \sum_{i+j=k, i,j \geq 0} d_i e_j, \quad k = 1, \ldots, r.
\]

Here we adopt the convention that \(d_0 = e_0 = 1, d_i = 0\) if \(i > s\) and \(e_j = 0\) if \(j > t\). Extending the above definition, it then follows that \(c_0 = 1\) and \(c_k = 0\) if \(k > r\). We write \(c = d * e\) to indicate that \(c\) is the convolution of \(d\) and \(e\). In particular, relation (7) amounts to saying that \(c(G) = c(H) * c(K)\). Clearly, this composition of sequences is commutative and associative.

**Lemma 3.** Let \(d\) and \(e\) be sequences of integers and let \(c = d * e\) be their convolution, as above. If \(d_2 \leq \left( \frac{d_i}{2} \right)_{i=1}^r, e_2 \leq \left( \frac{e_j}{2} \right)_{j=1}^t\) and
\[
d_i \leq \bar{\ell}_i(d_2), \quad i = 3, \ldots, s,
\]
\[
e_j \leq \bar{\ell}_j(e_2), \quad j = 3, \ldots, t,
\]
then \(c_2 \leq \left( \frac{c_k}{2} \right)_{k=1}^r\) and \(c_k \leq \bar{\ell}_k(c_2),\quad k = 3, \ldots, r\).

Note that \(d_2 \leq \left( \frac{d_i}{2} \right)_{i=1}^r\) and \(e_2 \leq \left( \frac{e_j}{2} \right)_{j=1}^t\) are automatically satisfied when \(d\) and \(e\) are clique vectors of graphs having clique numbers \(s\) and \(t\).

Although the statement of Lemma 3 is purely arithmetical, we have been unable to establish it by direct computation. Instead, we refer the reader to Lemma 3 in our recent article [6] where a more general result is obtained. The main ingredient there is the “colored Kruskal–Katona theorem” of Frankl et al. [8]. Notice that in order to prove Lemma 3 above, it suffices to assume that \(d_i = \bar{\ell}_i(d_2), i = 3, \ldots, s\) and \(e_j = \bar{\ell}_j(e_2), j = 3, \ldots, t\). The reason is that \(c_2\) depends on \(d_1, d_2, e_1\) and \(e_2\) only. It follows that
\[
d_k = \bar{\ell}_{ik}(d_i), \quad 2 \leq i < k \leq s,
\]
\[
e_k = \bar{\ell}_{jk}(e_j), \quad 2 \leq j < k \leq t,
\]
where \(\bar{\ell}_{ik}\) and \(\bar{\ell}_{jk}\) are the functions defined in [6] which generalize \(\bar{\ell}_k\) and \(\bar{\ell}_k\) in the present paper. (See the second remark in Section 5 below.) Since it is also true that \(d_2 \leq \bar{\ell}_i(d_2) \leq \left( \frac{d_i}{2} \right)_{i=1}^r\) and \(e_2 \leq \bar{\ell}_j(e_2) \leq \left( \frac{e_j}{2} \right)_{j=1}^t\), the hypothesis of Lemma 3 in [6] is fulfilled. We deduce that \(c_2 \leq \left( \frac{c_k}{2} \right)_{k=1}^r\) and \(c_k \leq \bar{\ell}_k(c_2), k = 3, \ldots, r\), as claimed.

**4. Proofs**

We are now ready to prove our main results.

**Proof of Theorem 1.** We first show that every graph \(G\) with clique number \(r \geq 3\) satisfies \(c_k(G) \leq \bar{\ell}_k(c_2(G)), k = 3, \ldots, r\).

The proof is by induction on the order of \(G\). For \(c_1(G) = 1\), there is nothing to show. Hence assume that \(c_1(G) > 1\) and suppose the assertion of Theorem 1 holds for all graphs with clique number at most \(r\) and order less than \(c_1(G)\).

As in Section 2, we write \(c_2(G) = \left( \frac{c_i}{2} \right)_{i=1}^r + m\) with \(0 \leq m < ((r - 1)/r)n\). Note that \(c_1(G) \geq n\), by Turán’s theorem. We distinguish two cases.

**Case 1:** \(\delta(G) \leq [(r - 1)/r]n\).

Let \(v\) be a vertex of \(G\) of degree \(\delta(G)\). Set \(G' := G - v\) and \(G'' := G[N_v]\). Then \(\omega(G') \leq r\), \(\omega(G'') \leq r - 1\), and both \(G'\) and \(G''\) have fewer vertices than \(G\). Hence, by the induction hypothesis,
\[
c_k(G') \leq \bar{\ell}_k(c_2(G')), \quad k = 3, \ldots, r,
\]

and by Turán’s theorem,
\[ c_{k-1}(G'') \leq \left( \frac{c_1(G''')}{k-1} \right)_{r}, \quad k = 3, \ldots, r. \]

(Here we have used the fact that the right-hand sides are non-decreasing functions of \( r \), resp., \( r - 1 \).) Since \( c_k(G) = c_k(G') + c_{k-1}(G'') \) and \( c_1(G'') = \delta(G) \leq \lfloor \frac{(r-1)/r) n \rfloor \), Lemma 2 yields, for \( k = 3, \ldots, r \),
\[
c_k(G) \leq \hat{c}_k(c_2(G')) + \left( \frac{c_1(G''')}{k-1} \right)_{r-1} \leq \hat{c}_k(c_2(G')) + c_1(G'') = \hat{c}_k(c_2(G)),
\]
as asserted.

Case 2: \( \delta(G) \geq \lfloor \frac{(r-1)/r) n \rfloor + 1 \).

We first remark that \( c_1(G) \geq n + 1 \). (This assertion is known as Zarankiewicz’s theorem.) Indeed, we would otherwise have \( c_1(G) = n \) and \( c_2(G) = \left( \frac{n}{2} \right) \), by Turán’s theorem. Using the explicit expression for \( \left( \frac{n}{2} \right) \) in Section 1, it is readily seen that \( n \lfloor \frac{(r-1)/r) n \rfloor \geq 2 \left( \frac{n}{2} \right) \). Hence it would follow that \( c_1(G) \delta(G) > 2c_2(G) \), which is clearly false.

Similarly, expressing \( \left( \frac{n}{2} \right) \) in the form \( \lfloor \frac{(r-1)/r) n \rfloor + 1 \) and using the above inequality, we get \( (n+2) \lfloor \frac{(r-1)/r) n \rfloor + 1 \) leads to a contradiction. But the same applies to \( \delta(G) \geq \lfloor \frac{(r-1)/r) n \rfloor + 2 \), since, trivially, \( (n+1) \lfloor \frac{(r-1)/r) n \rfloor + 2 \) is clearly false.

It now follows that \( G \) is the join of two nonempty graphs, or equivalently, \( \tilde{G} \) is disconnected. To see this, we make use of Brook’s theorem. If \( \tilde{G} \) were connected, then \( \gamma(\tilde{G}) = \delta(\tilde{G}) \), or else \( \tilde{G} \) would be a complete graph or an odd cycle. Now \( \tilde{G} \) is certainly not complete since, otherwise, \( \delta(G) = 0 \). If \( \tilde{G} \) were an odd cycle, then it would have \( 2r + 1 \) vertices, in order to satisfy \( \omega(G) = r \). But then \( n = 2r \) and \( \delta(G) = 2r - 2 \), contradicting the fact that \( \lfloor \frac{(r-1)/r) n \rfloor + 1 = 2r - 1 \). Finally, if \( \gamma(\tilde{G}) = \Delta(\tilde{G}) \), then \( \Delta(\tilde{G}) = c_1(G) - \delta(G) - 1 \),
\[
c_1(G) \leq \omega(G) \gamma(\tilde{G}) \leq \omega(G) \Delta(\tilde{G}) = r n - \left[ \frac{r - 1}{r} n \right] - 1 \leq r \left[ \frac{n}{r} \right] \leq n,
\]
contrary to Zarankiewicz’s theorem.

The conclusion is that there exist nonempty vertex-disjoint graphs \( H \) and \( K \) such that \( G = H + K \). Set \( \omega(H) = s \) and \( \omega(K) = t \) whence, in particular, \( r = s + t \). As both \( H \) and \( K \) have fewer vertices than \( G \), the induction hypothesis implies that \( c_k(H) \leq \hat{c}_k(c_2(H)) \), \( k = 3, \ldots, s \) and \( c_k(K) \leq \hat{c}_k(c_2(K)) \), \( k = 3, \ldots, t \). Hence, by Lemma 3 and the fact that \( c(G) = c(H) + c(K) \), we find that \( c_1(G) \leq \hat{c}_k(c_2(G)) \), \( k = 3, \ldots, r \), as desired.

That these inequalities cannot be improved in general was already shown in Section 2. This completes the proof of Theorem 1. \( \square \)

**Proof of Theorem 2.** Again, the proof is by induction on the number of vertices of \( G \). The case \( c_1(G) = 1 \) is trivial. Let \( G \) be a graph with \( \omega(G) \leq r \) and \( c_2(G) \leq \left( \frac{n}{2} \right) \), for some integer \( n \leq c_1(G) \). Suppose the assertion of Theorem 2 holds for all such graphs with fewer than \( c_1(G) \) vertices.

The case where \( c_2(G) < \left( \frac{n}{2} \right) \) will be considered first. Since \( \left( \frac{n-1}{2} \right) + \lfloor \frac{(r-1)/r) n \rfloor - 1 \) is the \( r \)-canonical representation of \( \left( \frac{n}{2} \right) - 1 \), \( \hat{c}_k \) maps \( \left( \frac{n}{2} \right) - 1 \) to \( \left( \frac{n-1}{k} \right) + \lfloor \frac{(r-1)/r) n \rfloor - 1 \). Applying Lemma 1 twice and using the definition of \( n \lfloor r \rfloor \) from Section 1, the latter sum turns out to be \( \left( \frac{n}{k} \right) - \left( \frac{n}{k+1} \right) \lfloor r \rfloor - 2 \). Hence assertion (5) follows directly from Theorem 1.

It remains to consider the case \( c_2(G) = \left( \frac{n}{2} \right) \), which is much more tedious. If \( n = c_1(G) \), then by Turán’s theorem, \( c_k(G) = \left( \frac{n}{2} \right) \), \( k = 2, \ldots, r \). We therefore assume that \( c_1(G) \geq n + 1 \).

Suppose \( c_1(G) = \left( \frac{n}{2} \right) - m \), for some \( k \geq 3 \) and some \( m \leq \left( \frac{n}{k} \right) \). We must show that \( m = 0 \) and that \( G \) is the union of \( T_2(n) \) and \( c_1(G) - n \) isolated vertices.

The first part of the argument consists in proving that \( \delta(G) < \lfloor \frac{(r-1)/r) n \rfloor \). Assume the contrary. Since \( (n+1) \lfloor \frac{(r-1)/r) n \rfloor + 1 \) can be ruled out immediately. Suppose we have \( \delta(G) = \lfloor \frac{(r-1)/r) n \rfloor \). Then \( c_1(G) \lfloor \frac{(r-1)/r) n \rfloor \leq 2 \left( \frac{n}{2} \right) \), which after writing out \( \left( \frac{n}{2} \right) \) explicitly and simplifying becomes \( c_1(G) \leq n + 1 \) if \( n \equiv 1 \pmod{r} \), and \( c_1(G) \leq n \) otherwise. Hence we find that \( n = pr + 1 \) and \( c_1(G) = pr + 2 \) for some positive integer \( p \). It follows that \( \lfloor \frac{(r-1)/r) n \rfloor = p(r-1) \) and \( n \lfloor r \rfloor = p(r-2) \).
Let \( v \) be a vertex of \( G \) of degree \( \delta(G) = p(r-1) \). Define, as in the proof of Theorem 1, \( G' := G - v \) and \( G'' := G[N_0] \). Then we have \( c_1(G') = pr + 1, c_1(G'') = p(r-1), \) \( \omega(G') \leq r \) and \( \omega(G'') \leq r - 1 \). Furthermore, by (6),

\[
c_2(G') = \left( \frac{pr+1}{2} \right) - \left( \frac{pr}{2} \right)
\]

and

\[
c_1(G') \geq \left( \frac{pr+1}{k} \right) - \left( \frac{p(r-1)}{k} \right)_{r-1} - m = \left( \frac{pr}{k} \right)_{r-1} - m.
\]

Here we have used Lemma 1 and the fact that Turán's theorem applied to \( G'' \) implies \( c_{k-1}(G''') \leq \left( \frac{p(r-1)}{k-1} \right)_{r-1} \).

Considered as a function of \( n \) and \( r \), the parameter \( n/r \) does not change when \( n = pr + 1 \) is replaced by \( n = pr \). Indeed, \([(r-2)/(r-1)][(r-1)/rpr]] \) is also equal to \( p(r-2) \). We can therefore apply the induction hypothesis to \( G' \). It follows that \( m = 0 \) and that \( G' \) consists of \( T_r(pr) \) plus one isolated vertex. In \( G \), this vertex would have degree at most 1, contradicting the fact that \( \delta(G) = p(r-1) > 1 \). Thus the assumption we started with is false; in other words, \( \delta(G) < [(r-1)/rpr] \) is true.

For the second part of the argument, let \( v \) be a vertex of \( G \) of degree \( j \), for some \( j \) with \( 0 \leq j < [(r-1)/rpr] \), and define \( G' \) and \( G'' \) exactly as before. Then

\[
c_2(G') = \left( \frac{n}{2} \right)_{r-1} - j = \left( \frac{n-1}{2} \right)_{r-1} + \left( \frac{r-1}{r} \right)_{r-1} - j.
\]

Assume, for the moment, that \( j > 0 \). Then \([(r-1)/rpr] - j < [(r-1)/rpr] \) and Theorem 1 implies that

\[
c_1(G') \leq \left( \frac{n-1}{k} \right)_{r-1} + \left( \frac{[\frac{r-1}{r}]n - j}{k-1} \right)_{r-1}.
\]

On the other hand, Turán's theorem applied to \( G'' \) yields

\[
c_{k-1}(G'') \leq \left( \frac{j}{k-1} \right)_{r-1}
\]

and so by (6), in view of \( c_k(G) = \left( \begin{array}{c} n \\ k \end{array} \right) - m \),

\[
c_1(G') \geq \left( \frac{n}{k} \right)_{r-1} - \left( \frac{j}{k-1} \right)_{r-1} - m.
\]

Combining (9) and (10) and using the assumption on \( m \), one gets

\[
\left( \frac{j}{k-1} \right)_{r-1} + \left( \frac{[\frac{r-1}{r}]n - j}{k-1} \right)_{r-1} \geq \left( \frac{n}{k} \right)_{r-1} - \left( \frac{n-1}{k} \right)_{r-1} - m
\]

\[
> \left( \frac{[\frac{r-1}{r}]n}{k-1} \right)_{r-1} - \left( \frac{n}{k-2} \right)_{r-2}
\]

\[
= \left( \frac{[\frac{r-1}{r}]n - 1}{k-1} \right)_{r-1}.
\]

Here Lemma 1 has been applied twice.

We now make use of the fact that for all positive integers \( b \) and \( c \),

\[
\left( \frac{b}{k-1} \right)_{r-1} + \left( \frac{c}{k-1} \right)_{r-1} \leq \left( \frac{b+c-1}{k-1} \right)_{r-1}.
\]

This is the case \( a = 1 \) of inequality (8). Setting \( b = j \) and \( c = [(r-1)/rpr] - j \), we find that (11) is violated. Hence the assumption \( j > 0 \) cannot hold and so we have \( j = 0 \), i.e., \( v \) is an isolated vertex of \( G \). But then \( c_1(G') = \left( \begin{array}{c} n \\ 2 \end{array} \right) \) and \( c_1(G') = c_1(G) = \left( \begin{array}{c} n \\ k \end{array} \right) - m \). The induction hypothesis applied to \( G' \) now shows that \( m = 0 \) and that \( G' \) is the union of \( T_{r}(n) \) and \( c_1(G) - n - 1 \) isolated vertices. Therefore, \( G \) itself is the union of \( T_{r}(n) \) and \( c_1(G) - n \) isolated vertices.
Since we have already seen that the inequalities established above are the best possible ones, Theorem 2 is now completely proved.

5. Remarks

(1) As mentioned in Section 1, Theorems 1 and 2 are direct extensions of the corresponding theorems in [5] which treat the case \( r = 3 \). (Note that \( \delta_3 \) is called \( \lambda_3 \) in [5].) While the proof of Theorem 2 in the present paper is a straightforward generalization of that of Theorem 2 in [5], the proof of Theorem 1 is quite different from the corresponding one in that paper.

The difference occurs in Case 2 of the proof. This case is concerned with all graphs \( G \) satisfying the following conditions, for some positive integer \( n \): \( c_1(G) = n + 1 \), \( c_2(G) < \binom{n+1}{2} \), \( c_{r+1}(G) = 0 \) and \( \delta(G) = \lfloor (r - 1)/r \rfloor n + 1 \).

Lemma 2 in [5] shows that there is only one such graph if \( r = 3 \). This graph (called \( \Gamma \) in [5]) can easily be checked directly. For general \( r \), however, the graphs in question are not known.

As it turns out, the main difficulty is to analyze those graphs \( G \) for which, in addition to the above properties, \( \gamma(G) > r \). According to Andrásfai et al. [1], these graphs satisfy \( \delta(G) \leq \lfloor ((3r - 4)/(3r - 1))(n + 1) \rfloor \); in particular, there are only finitely many of them for any given \( r \). (There exist 8 such graphs if \( r = 4 \), and 57 if \( r = 5 \).) The largest one is unique and has order \((r - 2)(3r - 1)\). Since it appears rather hopeless to find—and check—all these graphs, our proof here relies on Brook’s theorem and the crucial Lemma 3.

(2) We conjecture that Theorem 1 can be generalized in the following way:

**Conjecture.** If \( G \) is a graph with clique number \( r \), then for \( 1 \leq l < k \leq r \), \( c_k(G) \leq \delta_{l,k}(c_l(G)) \).

Here \( \delta_{l,k} \) is defined as follows. Given positive integers \( l \) and \( r \), every positive integer \( n \) can be uniquely expressed as

\[
\text{where } a_{j-1} < \left( \frac{(r - l + j - 1)(r - l + j)}{r(l + j)} \right) a_j \text{ for } j = l, l - 1, \ldots, i + 1 \text{ and } a_i \geq i \geq 1. \]

This follows at once from Lemma 1. Given this expression, set

\[
\delta_{l,k}(n) := \left( \frac{a_l}{k} \right)_r + \left( \frac{a_{l-1}}{k-1} \right)_{r-1} + \cdots + \left( \frac{a_i}{k-l+i} \right)_{r-l+i}
\]

and let \( \delta_{l,k}(0) := 0 \).

Since clearly \( \delta_k = \delta_{2,k} \), Theorem 1 establishes the conjecture for \( l = 2 \). Turán’s theorem settles the case \( l = 1 \), in view of \( \delta_{l,k}(n) = \binom{n+1}{2} \). For \( r \)-partite graphs, the conjecture is a consequence of the colored Kruskal–Katona theorem in [8].

For more details, see [6,2]. A “smooth” (asymptotically equivalent but weaker) version of the conjecture appears in [9].

It follows from Lemma 3 in [6] that if the clique vectors of two graphs \( H \) and \( K \) both satisfy the inequalities conjectured above, then so does the clique vector of \( K + H \). However, we do not see how this fact could be exploited for generalizing the proof of Theorem 1. Even if the conjecture turns out to be true, there is still the fundamental problem of characterizing the integer sequences \((c_1, c_2, \ldots, c_r)\) that arise as clique vectors of finite graphs. This problem is completely open.

References