2-List-coloring planar graphs without monochromatic triangles

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Abstract
We prove that, for every list-assignment of two colors to every vertex of any planar graph, there is a list-coloring such that there is no monochromatic triangle. This proves and extends a conjecture of B. Mohar and R. Škrekovski and a related conjecture of A. Kündgen and R. Ramamurthi.

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1. Introduction

Let \( G \) be a graph. For every vertex \( v \) of \( G \), let \( L(v) \) be a list of colors which we call available colors for \( v \). An \( L \)-coloring of \( G \) is a coloring of the vertex set such that every vertex \( v \) receives a color from \( L(v) \). If it is clear what \( L \) is, we just call it a list-coloring of \( G \). Usually one assumes that neighbors always have distinct colors. We do not assume that here. Instead we try to find a list-coloring such that there is no monochromatic triangle. The theorem we prove implies the following conjecture of Mohar and Škrekovski [2]: If \( G \) is a planar graph such that every edge is in a triangle, and \( L \) is a list assignment such that each list has two colors, then there is a list-coloring such that no maximal complete subgraph is monochromatic.

Every planar graph is 5-list-colorable [4], and Voigt [6] showed that a planar graph need not be 4-list-colorable. In an attempt to find a list-color version that would imply the 4-color theorem, Kündgen and Ramamurthi [1] made the following conjecture: If \( G \) is a planar graph, and \( L \) is a list assignment such that each list has two colors, then there is a list-coloring such that...
no odd cycle is monochromatic. This is still open. Kündgen and Ramamurthi [1] also made the following related conjecture: If $G$ is a planar, connected graph with at least one edge and $L$ is a list assignment such that each list has two colors, then there is a list-coloring such that no face boundary is monochromatic. This conjecture is weaker than the former conjecture of Kündgen and Ramamurthi because it suffices to verify both conjectures for triangulations. For $K_4$-free graphs the weaker conjecture of Kündgen and Ramamurthi is also weaker than that of Mohar and Škrekovski. The result stated in the Abstract implies the weaker conjecture of Kündgen and Ramamurthi as well as that of Mohar and Škrekovski.

The proof of the present paper has a striking similarity with the list-color proof of Grötzsch’s theorem presented in [5]. In fact, the main challenge seems to be to find a formulation which allows us to use the technique of [4,5]. This raises the question if there is a formal connection between Grötzsch’s theorem and the result of the present paper. And perhaps it gives some hope of finding a list-color version of the 4-color theorem as suggested by Kündgen and Ramamurthi [1].

The notation and terminology are the same as in [3–5]. In addition, if $xyz$ is a path such that the ends $x, z$ are precolored with distinct colors $\alpha, \beta$, respectively, and the list $L(y)$ of available colors at the midvertex $y$ consists of the two colors $\alpha, \beta$, then we say that the path $xyz$ is a bad 2-path. A monochromatic edge, respectively dichromatic edge, is an edge joining vertices of the same, respectively distinct, colors.

2. 2-List-colorings with no monochromatic triangles

We now prove that, for every list-assignment of two colors to every vertex of any planar graph, there is a list-coloring such that there is no monochromatic triangle. It suffices to prove this result for triangulations and hence also for the more general class of near-triangulations. For technical reasons we prove the stronger result below.

**Theorem 1.** Let $G$ be a plane near-triangulation with outer cycle $C : v_1v_2 \ldots v_kv_1$. Let $c$ be a coloring of a nonempty vertex set $A$ on $C$ such that the subgraph $G(A)$ induced by $A$ either equals the outer cycle $C$ or else the edges in $G(A)$ (if any) induce a path which is denoted by $P$. (If $G(A) = C$, then we also write $P = C$. If $G(A)$ has no edge, then $P$ consists of precisely one vertex.) We let $A_i$ denote the set of vertices of degree $i$ in the subgraph $G(A)$ for $i = 0, 1, 2$. So, if $G(A)$ has no edge, then $A_0 = A$ and $A_1$, $A_2$ are empty.

For each vertex $v$ in $G$, let $L(v)$ be a list of colors. If $v$ is in $A$, then $L(v)$ consists of $c(v)$ only. Otherwise, $L(v)$ has precisely two colors. Assume that $G$ has no bad 2-path with both ends in $A_0 \cup A_1$. (Such a path will be called a forbidden bad 2-path.) In other words, we do allow a bad 2-path if at least one of its ends is in $A_2$. (Such a path will be called an allowed bad 2-path.)

Let $q$ be the number of vertices of $P$, and let $m$ be the number of monochromatic edges of $P$. Assume that $q + m \leq 6$ if $P$ is a path, and that $q + m \leq 5$ if $P = C$.

Then $c$ can be extended to an $L$-coloring of $G$ such that no triangle in $G$ is monochromatic.

**Proof.** The proof is by induction on the number of edges of $G$. Suppose (reductio ad absurdum) that $G$ is a counterexample with as few edges as possible.

If $P$ is a path contained in $C$, then we choose the notation such that $P$ is of the form $v_1v_2 \ldots v_q$, $q < k$. But, at this stage, it is possible that $P$ has an edge which is a chord of $C$.

**Claim 1.** $G$ has no separating triangle.
Proof. Suppose (reductio ad absurdum) that $G$ has a separating cycle $C'$ of length 3. We apply the induction hypothesis first to the exterior of that cycle, and then to its interior. This contradiction proves Claim 1. □

The purpose of the next claim is to prove that $G$ cannot be separated by a path of length 1 or 2 except in a very special way: There may be a chord which separates a vertex in $A_0$ from the rest of the graph. And there may be a separating path of length 2, but every such path must join a vertex in $A_2$ with a vertex not in $A$, or it joins two vertices not in $A$. All other separating paths of length 1 or 2 will be disposed of in the technical Claim 2 below. Due to the many cases we shall do this in several steps.

Claim 2.

(i) $P$ has no edge which is a chord of $C$. In particular, we may now choose the notation such that either $P = C$, or $P$ is the path $v_1v_2\ldots v_q$.

(ii) If $Q$ is a chord of the form $Q : v_i v_j$ where $v_j$ belongs to $A$, or if $Q$ is a path of the form $Q : v_i u v_j$ where $v_i$ belongs to $A$ and $u$ is inside $C$ or $u$ is one of $v_{i-1}, v_{i+1}$, then $Q$ divides $G$ into two parts, and none of these two parts contains $P$ unless $Q$ has length 2 and is either a path in $C$ or forms a facial triangle together with an edge of $C$, or $Q$ is of the form $v_i v_{i+1} v_{i+3}$ (or $v_i v_{i-1} v_{i-3}$) where $v_{i+2}$ (or $v_{i-2}$) is in $A_0$.

(iii) If $Q$ is a bad 2-path in $G$, then $P$ has an edge which together with $Q$ forms a facial triangle.

(iv) If $e$ is a chord of $C$, then there exists a vertex $v_j$ which is in $A_0$ and has degree 2 in $G$ such that $e$ is the edge joining the two neighbors of $v_j$.

(v) $G$ has no path of the form $v_i u v_j$ where $u$ is inside $C$, $v_i, v_j$ are in $A$, and $v_i, v_j$ are non-consecutive on $C$.

Proof. We first prove (i). Suppose therefore (reductio ad absurdum) that $P$ has an edge of the form $v_i v_j$ where $1 \leq i \leq j - 2$. This edge divides $G$ into graphs $G_1, G_2$. We complete the proof by applying the induction hypothesis to each of $G_1, G_2$. We focus on $G_1$ which is bounded by the cycle $v_i v_{i+1} \ldots v_j v_i$, say. (The argument for $G_2$ is similar.) The only problem with induction applied to $G_1$ is that $G_1$ may have a forbidden bad 2-path which is allowed in $G$. If the only forbidden bad 2-path in $G_1$ is of the form $v_j x v_i$, then there is only one such path by Claim 1. Then we color the vertex $x$ and delete the edge $v_i v_j$ before we apply induction to $G_1$. So assume there exists a forbidden bad 2-path of the form $v_i x v_r$, where $i < r < j$. The path $v_i x v_r$ divides $G_1$ into graphs $G_3$ and $G_4$ where $G_4$ is bounded by the cycle $v_i v_{i+1} \ldots v_r x v_i$. We choose $x$ such that $G_4$ has as few edges as possible. Now we add the edge $v_i v_r$ to $G_3$. If there exists a forbidden bad 2-path starting at $v_r$, then we make a similar modification close to $v_r$. Then we apply induction to the modified $G_3$ and then to $G_4$. When we apply induction to $G_4$, there is no problem with forbidden bad 2-paths because of the minimality of $G_4$. However, $x$ may be joined to several vertices which are both in $A_0$ and $G_4$. Then the edges from $x$ to these vertices divide $G_4$ into parts, and we apply induction to each part separately. This proves (i).

We may now choose the notation such that either $P = C$, or $P$ is the path $v_1 v_2 \ldots v_q$.

We now prove (ii). If $P = C$, then (ii) is trivially true. So we may assume that $P \neq C$. Suppose (reductio ad absurdum) that $Q$ is a path of the form $v_i v_j$ or $v_i u v_j$ (where $v_i$ is in $A$ and $u$ is inside $C$ or $u$ is one of $v_{i-1}, v_{i+1}$) dividing $G$ into graphs $G_1, G_2$ having precisely $Q$ in common such that $G_1$, say, contains $P$. We choose $Q$ such that $G_2$ is minimum subject to this condition. Assume that $i < j$ and that $G_2$ contains the path $v_i v_{i+1} \ldots v_j$ (or $v_{i+1} \ldots v_j$ if $u = v_{i+1}$). For
notational convenience we now assume that $u$ is inside $C$, as the proof is similar if $u$ is one of $v_{i-1}, v_{i+1}$. By Claim 1, $i < j - 1$. Then we apply induction first to $G_1$ and then to $G_2$.

When $G_1$ has been colored, then $Q$ has been colored as well. Possibly $v_{j-1}$ is in $A_0$. So $G_2$ may have a precolored path on 4 vertices. If that path is monochromatic, then we cannot apply induction to $G_2$ because its value of $m + q$ is 7. To prevent this, we add the edge from $v_i$ to $v_j$ before we apply induction to $G_1$ (if $v_{j-1}$ is in $A_0$ and has the same color as $v_j$). (If $u = v_{i+1}$ and $v_{i+2}$ is in $A_0$ and has the same color as $v_{j-1}$, then we add a vertex with the color $c(v_{j-1})$ and join it to $v_{i+1}, v_j$ before we use induction to $G_1$.)

Then $G_2$ satisfies the condition on $m + q$. But, there may be another problem with $G_2$: It may contain a forbidden bad 2-path. As this path is not a forbidden bad 2-path in $G$, it must start at $v_j$, and $v_j$ is not in $A$. If the forbidden bad 2-path in $G_2$ is contained in $C$, then the forbidden bad 2-path in $G_2$ must be $v_{j-1}v_{j-2}$. It is possible that $v_{j-2} = v_i$. Now we add the edge $v_{j-2}v_j$ before we apply induction to $G_2$. If the forbidden bad 2-path in $G_2$ is not contained in $C$, then we obtain a contradiction to the minimality of $G_2$.

This proves (ii).

We now prove (iii). Suppose (reductio ad absurdum) that $Q : v_iuv_j$ is a bad 2 path violating (iii). Then $v_i, v_j$ belong to $A$ but $u$ does not. Since there is no forbidden bad 2-path, we may assume that $v_i$ is in $A_2$. We may also assume that $v_i$ has color 1, $v_j$ has color 2, and $L(u)$ consists of the colors 1, 2. By (ii), $u$ is inside $C$ unless both of $v_i, v_j$ are in $A_2$, or $v_j$ is not in $A_2$ and $u$ equals one of $v_{j-1}, v_{j+1}$. We first claim that $u$ can be chosen such that it is inside $C$. For otherwise, the chord $uv_j$ divides $G$ into two parts. We apply induction first to the part with the largest $m + q$ (and we assume that the chord $uv_j$ has been chosen such that this part is minimum) and then to the other part. (Either part may contain a forbidden bad 2-path, but we dispose of that by adding an edge between its ends. If this is not possible, then the reason is that the forbidden bad 2-path starts in $v_i$ and its mid-vertex is inside $C$. But then we have proved what we want to prove now, namely that $u$ can be chosen to be inside $C$.) So we may assume that $u$ can be chosen to be inside $C$.

If $P = C$, then $P \cup Q$ divides $G$ into two parts $G_1, G_2$ such that $G_2$, say, is bounded by a 4-cycle with no monochromatic edge. Then we apply induction first to $G_1$ (with the edge $v_iu$ added) and then to $G_2$. So assume that $P \neq C$.

Consider first the case where $u$ is joined to no vertex in $A_0$. In particular, $v_j$ is in $P$. Assume that $i, j$ are chosen such that $i < j$, the path $Q$ is not necessarily bad, and $j - i$ is maximum subject to these assumptions.

Let $C'$ denote the unique cycle in $P \cup Q$. Then $C'$ has length at least 4. We now give $u$ a color and try to extend that coloring to each connected component of uncolored vertices, if possible. Clearly, it is possible to color $u$ without creating a monochromatic triangle. For otherwise, $u$ would be joined to two pairs of consecutive precolored vertices with colors 1, 2, respectively. As $u$ is not the midvertex of a forbidden bad 2-path, there must be a fifth vertex of $P$, but that contradicts the assumption $q + m \leq 6$. If we cannot apply induction to the exterior of $C'$ after having colored $u$, then the reason is that $j = i + 2$ and $C'$ is the cycle $v_i v_{i+1} v_j u v_i$ of length 4, and $v_{i+1}$ has a color distinct from 1, 2, or $C'$ is the cycle $v_i v_{i+1} v_{i+2} v_j u v_i, v_j, v_{j-1}$, all have color 1, one of $v_{i+1}, v_{i+2}$ has the color 2, and the other has a third color. (Only in those cases the condition on $q + m$ could be satisfied in $G$ but violated in the exterior of $C'$.) But then we uncolor $u$ and apply instead induction first to the exterior of $C'$ after having added the edge $v_iu$, and then we apply induction to the interior of $C'$. So we may assume that we can apply induction to the exterior of $C'$ after having colored $u$. If we cannot apply induction to the interior of $C'$ after having colored $u$, then the reason is that $C'$ is too long and has too many monochromatic edges
(or both). In that case we uncolor $u$, we apply induction to $C′$ and its interior. If $C′$ and its interior has a forbidden bad 2-path, then its ends must be $v_i, v_j$ (by the definition of “forbidden”). In that case we add the edge $v_i v_j$ before we apply induction to $C′$ and its interior. The induction is then possible because $P$ has at least one vertex not in $C′$, and the edge $v_i v_j$ is not monochromatic which implies that the condition on the new $q + m$ is satisfied. Then we apply induction to the exterior of $C′$. This proves (iii) in the case where $u$ is joined to no vertex in $A_0$. So assume that $u$ is also joined to some vertex in $A_0$. If possible, we choose $u$ such that $v_j$ is in $A_0$.

Then $P, u$ and the edges from $u$ to $A$ divide $G$ into parts. Assume first that $u$ is not joined to two consecutive vertices on $P$ which have the same color 1 or the same color 2. Then we first apply induction to the part which has the largest contribution to $q + m$. If that part contains a forbidden bad 2-path, say $v_j u v_i$, then we add the edge between its ends before we use induction. It is also possible that $v_j u v_i$ is not a forbidden bad 2-path but instead there is a forbidden bad 2-path of the form $v_i u v_s$. In that case we consider $u′$ instead of $u$ and add the edge $v_i v_s$. The important thing is that we first apply induction to a part which has the largest contribution to $q + m$. Then $u$ or $u′$ (say $u$) receives a color, and we apply induction to the remaining uncolored parts (after having deleted the edge $v_i v_j$). Those uncolored parts whose precolored paths have three vertices (with $u$ being a mid-vertex) are easy to dispose of. However, there might be an uncolored part which is bounded by the colored path $v_1 v_2 v_3 u v_r$, say, such that $v_3, u, v_r$ all have color 1, say. This would be problematic because the value of $m + q$ is greater than 6 in this case. And this situation might occur if we apply induction first to the cycle $v_3 v_4 \ldots v_j u v_3$ and its interior. Since this part is the one that has the largest contribution to $m + q$, none of the edges $v_1 v_2, v_2 v_3$ are monochromatic, and $j < r$. Then we change the color of $u$ to 2. This might create a problem with the uncolored part bounded by $v_3 v_4 \ldots v_j u v_3$ because $v_j$ has color 2. But, this will not occur because $G$ has no forbidden bad 2-path $v_j u v_r$.

Assume next that $u$ is joined to two endvertices of a monochromatic edge of color 1 or 2 on $P$. There is only one such monochromatic edge or two consecutive monochromatic edges on $P$ because of the condition on $m + q$ and the assumption that $G$ has no forbidden bad 2-path. Then we give $u$ another color and use induction to each uncolored part. Before we argue formally we describe a couple of extreme cases. One extreme case is where $u$ is joined to $v_i, v_i+1, v_j$, where $v_j$ is in $A_0, v_i, v_j$ have color 2, and $v_i, v_i+1$ have color 1. Then we are forced to give $u$ the color 2 and now one could be concerned about the uncolored part containing the monochromatic path $v_{i-1} uv_j$. However, the bad 2-path $v_{i+1} u v_j$ is allowed which implies that $q \geq i + 2$, and hence the precolored path containing $v_{i-1} u v_j$ does not have a larger value of $q + m$ than that of $P$. Another extreme case is where $u$ is joined to $v_r, v_i, v_i+1, v_j$, where $r$ is small, $i$ is large, $v_r, v_j$ have color 2, and $v_i, v_i+1$ have color 1. Again, we are forced to give $u$ the color 2 and now one could be concerned about the cycle $v_r \ldots v_i u v_r$ and its interior. However, we must have $r > 1$ or $q \geq i + 2$ because of (ii), and now the assumption that $P$ satisfies the assumption on $m + q$ implies that also $v_r v_i+1 \ldots v_i u v_r$ satisfies that assumption.

Formally, we may argue as follows: If $P$ has a monochromatic edge $v_i v_{i+1}$ whose ends are joined to $u$, and $1 < i < q - 1$, then it is easy to see that, after having colored $u$ with another color, each uncolored component satisfies the assumption on $m + q$. On the other hand, if $u$ is joined to $v_1, v_2$ each of which has color 1 and $u$ is given the color 2, then an uncolored component could be problematic only if it is bounded by the cycle $uv_2 \ldots v_q u$ or if it contains the path $v_j u v_2 \ldots v_q$. In either case the induction is possible because there is no forbidden bad 2-path.

This proves (iii).

We now prove (iv), (v) simultaneously.
Suppose therefore (reductio ad absurdum) that either \( Q : v_iv_j \) is a chord of \( C \) which contradicts (iv), or \( Q : v_iuv_j \) is a path contradicting (v) where \( 1 \leq i < j \leq k \). Then \( Q \) divides \( G \) into near-triangulations \( G_1, G_2 \). Consider first the case where \( Q \) must be of the form \( v_iuv_j \) where \( i < j, u \) is inside \( C \), and \( v_i, v_j \) are both in \( P \). Choose \( i, j \) such that \( j - i \) is maximum. As \( Q \) is not a bad 2-path, we can give \( u \) a color distinct from those of \( v_i, v_j \). If this creates no monochromatic triangle, then we apply induction to \( G_1, G_2 \) unless \( i = 1, j = q - 1 \) or \( i = 2, j = q \). On the other hand, if the coloring of \( u \) creates a monochromatic triangle, then this triangle contains a monochromatic edge in \( P \) but not incident with any of \( v_i, v_j \). In this case (and also in the cases \( i = 1, j = q - 1 \) or \( i = 2, j = q \)) we apply first induction to the cycle \( v_juv_i, v_{i+1} \ldots v_j \) and its interior and then to the rest of the graph which is possible because the number of monochromatic edges increases by at most 1 and the number of vertices in the precolored path decreases. (If \( i = 2, j = q \) and \( v_1, v_2, v_q \) all have the same color, then we add the edge \( v_2v_q \) before we use induction to \( G_1 \).) We may therefore assume that some \( Q \) contradicting (iv) or (v) contains an end-vertex not in \( P \). Then we choose \( Q \) such that the sum of the number of vertices of \( P \) in \( G_2 \) and the number of monochromatic edges of \( P \) in \( G_2 \) is smallest possible. Subject to this we choose \( Q \) such that the number of edges in \( G_2 \) is smallest possible. We obtain a contradiction by applying the induction hypothesis first to \( G_1 \) and then to \( G_2 \). It only remains to argue that the induction hypothesis really can be applied to \( G_1, G_2 \). Clearly, we can apply induction to \( G_1 \). The only possible problem is that \( G_1 \) might have a forbidden bad 2-path which is allowed in \( G \). But this is not possible by (iii) unless \( i = 2 \) and there is a bad 2-path \( v_1wv_2 \). In this case we color \( w \) and delete the edge \( v_1v_2 \) before we use induction to \( G_1 \).

We shall now argue why we can apply induction to \( G_2 \). Before we prove this we first dispose of the special case where the coloring of \( G_1 \) results in a monochromatic triangle in \( G_2 \). This can occur only if \( Q \) is the chord \( v_{q-1}v_{q+1}, \) and \( v_{q-1}, v_q \) have the same color. In this case we give \( v_{q+1} \) another color and delete \( v_q \) before we use induction. (If we create a new forbidden bad 2-path, then we add the edge between its ends before we use induction. The other end of this edge may be \( v_1 \) in which case we create a precolored cycle. However, the condition on \( m + q \) is still satisfied because the deletion of \( v_q \) reduces the value of \( m + q \) by 2.) So we may assume that \( G_2 \) has no monochromatic triangle after we have colored \( G_1 \).

Consider first the case where \( Q \) is a chord \( v_iv_j \). Assume the notation is such that the outer cycle of \( G_2 \) is \( v_i, v_{i+1}, \ldots, v_j, v_i \).

If \( G_2 \) contains no edge of \( P \), then it follows easily from (ii) that we can apply induction to \( G_2 \). (We may have to add one or two new edges in case there are bad 2-paths. Also, if \( v_i, v_j \) are not in \( A \), then there could be a bad 2-path \( v_iuv_j \) which we dispose of by coloring \( u \) and deleting the edge \( v_iv_j \).) So assume that \( v_i \) is in \( A_2 \). Then the precolored vertices of \( G_2 \) may contain a path with as many as 5 vertices and 2 monochromatic edges (or 4 vertices and 3 monochromatic edges). If that happens, then \( v_{j-1} \) is in \( A \) and has the same color, say 1, as \( v_j, v_i \). In that case we give \( v_j \) the available color distinct from 1 before we apply induction to \( G_1 \). The new precolored path in \( G_1 \) may have as many as 6 vertices. However, if \( v_{j+1} \) is a precolored vertex, then the edge \( v_jv_{j+1} \) cannot be monochromatic because \( G \) has no bad 2-path from \( v_{j-1} \) to \( v_{j+1} \). The minimality of \( G_2 \) implies that every edge from \( v_j \) to \( P \) must have an end of the form \( v_p \), where \( 2 \leq p \leq i \). The edges from \( v_j \) to \( P \) divide \( G_1 \) into subgraphs. We apply the induction hypothesis to each of those subgraphs after having colored \( v_j \). (If \( v_j \) is the end of a new forbidden bad 2-path, then we add the edge between its ends before we use induction. As we are considering a case where the value of \( m + q \) gives us trouble in \( G_2 \), it does not give us trouble in \( G_1 \) even when the addition of a new edge creates a precolored cycle.) This shows that we can apply induction to \( G_1, G_2 \). This contradiction proves the statement (iv) in Claim 2.
We finally prove (v). That is, we assume that \( Q : v_iu_jv_j \) is a path contradicting (v) where \( 1 \leq i < j \leq k \). By (ii) we may assume that \( v_i \) is in \( A_2 \). We assume that \( v_j \) is in \( A_0 \), as we have already disposed of the case where \( v_j \) must be in \( P \). We repeat the proof in the previous paragraph where \( Q \) is a chord. We define \( G_1, G_2 \) such that \( G_2 \) has the same minimality property as in that case, we apply induction first to \( G_1 \) and then to \( G_2 \). If the value of \( m + q \) for \( G_2 \) is 7, then the vertices of \( Q \) get the same color, and in that case we change the color of the midvertex of \( Q \) before we apply induction to \( G_1, G_2 \). \( \square \)

Claim 3. \( P \) has no monocromatic edge \( v_i v_{i+1} \).

**Proof.** Suppose (reductio ad absurdum) that \( P \) has an edge \( v_iv_{i+1} \) such that \( c(v_i) = c(v_{i+1}) = 1 \). By Claim 1, this edge is in precisely one triangle \( v_iv_{i+1}wv_j \) which is facial. By Claim 2, \( v \) is inside \( C \). We may assume that \( v \) has the available colors 1, 2 since otherwise, we just contract the edge \( v_iv_{i+1} \) and apply the induction hypothesis.

Now we give \( v \) the color 2, delete the edge \( v_iv_{i+1} \), and apply the induction hypothesis. This is possible because the new precolored path or cycle has one more vertex than \( P \) but fewer monocromatic edges. This contradiction proves Claim 3. \( \square \)

Claim 4. \( q \geq 4 \).

**Proof.** Suppose (reductio ad absurdum) that \( q < 4 \). Let \( w \) be the unique common neighbor of \( v_1, v_2 \) (not in \( C \) if \( C \) is a triangle). If \( q > 1 \), then we color \( w \) and delete the edge \( v_1v_2 \). If \( q = 1 \), then we give \( v_2 \) a color distinct from the color of \( v_1 \), we color \( w \) and delete the edge \( v_1v_2 \) (and we also delete \( v_1 \) if \( w = v_k \)). If \( v_3 \) is in \( A \), we choose the color of \( v_2 \) to be distinct from \( c(v_3) \), too. Then we apply the induction hypothesis to the resulting graph. If \( w \) is inside \( C \), there is no edge from \( w \) to a precolored vertex distinct from \( v_1, v_2 \), by Claim 2. Claim 2 also implies that, if \( q = 1 \), then the coloring of \( v_2 \) does not create a bad 2-path from \( v_2 \) to \( A \setminus \{v_1, v_2\} \) except possibly the path \( v_2v_3v_4 \) (which would be allowed) or the path \( v_2v_3v_4 \) in which case we recolor \( v_2, w \) without creating a monotromatic triangle.

If \( w \) is not inside \( C \), then \( w = v_k \) in which case we argue similarly. The coloring of \( v_2, v_k \) may create two forbidden bad 2-paths, namely \( v_2v_3v_4 \) and \( v_kv_{k-1}v_{k-2} \). We can dispose of one of those by recoloring \( v_2, v_k \) and we can dispose of the other by adding an edge between the ends.

This contradiction proves Claim 4. \( \square \)

Claim 5. If \( P \) is a path, and if \( Q \) is a path of length 2 or 3 joining two vertices in \( A_0 \cup A_1 \), then \( Q \) is of length 3 and is contained in the outer cycle \( C \).

**Proof.** Suppose (reductio ad absurdum) that \( Q \) exists such that \( Q \) either contains an edge not in \( C \) or has length 2 and is contained in \( C \). By Claim 4, \( Q \) is distinct from the precolored path \( P \). By Claim 2, \( Q \) does not intersect \( A_2 \). If \( Q \) is contained in \( C \) and of length 2, and the ends of \( Q \) have the same color, then we identify the ends of \( Q \) and apply the induction hypothesis, a contradiction. If \( Q \) is contained in \( C \) and of length 2, and the ends of \( Q \) have distinct colors, then one end of \( Q \) has a color which is not an available color of the midvertex of \( Q \) because \( Q \) is not a bad 2-path. We delete the edge of \( Q \) between these two vertices and apply the induction hypothesis, a contradiction. (The edge-deletion may create a vertex of degree 1. In that case we also delete that vertex.) So we may assume that \( Q \) is not contained in \( C \) and divides \( G \) into parts \( G_1, G_2 \) where \( G_1 \) contains \( P \). By Claim 2, \( Q \) cannot have length 2, so \( Q = v_iuv_j \). We
may assume that both of \( u, v \) are in inside \( C \). (Otherwise, we are back to the case where \( Q \) has length 2, by Claim 2.) We apply induction first to \( G_1 \) and then to \( G_2 \). If \( v_i, v_j \) have the same color, say 1, then we identify \( v_i, v_j \) before we apply induction to \( G_1 \).

This contradiction proves Claim 5. \( \square \)

We now focus on the edge \( vqvq_1 \). If \( P = C \), then we let \( vqvq_1 \) denote any edge of \( C \). By Claim 3, \( vqvq_1 \) have distinct colors. Assume that the notation is such that \( c(vq) = 1, c(vq_1) = 2 \). Let \( wvqvq_1w \) be the unique triangle containing the edge \( vqvq_1 \).

**Claim 6.** \( G \) has a 4-cycle \( vvqvq_1usvq \) whose interior has precisely one vertex \( w \). Moreover, \( L(s) = \{1, 3\} \), \( L(u) = \{2, 3\} \), and \( L(w) = \{1, 2\} \). Finally, \( u \) is not joined to any precolored vertex distinct from \( vqvq_1 \) except possibly \( vq_2 \), and if we give \( u \) the color 3, then \( usvq \) is the only bad 2-path from \( u \) to a vertex in \( A_0 \cup A_1 \).

**Proof.** We may assume that \( L(w) = \{1, 2\} \) since otherwise we delete the edge \( vqvq_1 \) and use induction. The edge \( wvq_1 \) is contained in precisely one more triangle, say \( uvqvq_1w \). By Claim 2, \( u \) is not in \( C \).

We may assume that \( L(u) \) contains the color 2, say \( L(u) = \{\alpha, 2\} \), since otherwise we delete the edges \( vqvq_1, wvq_1 \) and use induction. Now we give \( u \) the color \( \alpha \), we delete the edges \( vqvq_1, wvq_1 \) and we obtain a contradiction using the induction hypothesis, if possible. So we shall discuss the cases where it is not possible to use induction.

By Claim 2, \( u \) is not joined to any vertex of \( A \) except \( vqvq_1 \) and possibly \( vq_2 \).

So assume that, after having given \( u \) the color \( \alpha \), there is a forbidden bad 2-path \( usvq \) where \( v_i \) is in \( A_0 \cup A_1 \) (where we now refer to the graph obtained from \( G \) by deleting the edge \( vqvq_1 \)) and has color \( \beta \), and \( L(s) = \{\alpha, \beta\} \). For convenience we assume that \( s \) is inside \( C \). (The case where \( s \) is in \( C \) is similar. A slight complication would occur if \( s = v_i-1 \) and \( v_i-2 \) is in \( A \). But, this does not occur by Claim 5.) Then we apply induction as in the proof of Claim 5. When we apply induction to \( G_1 \) we do not color \( u \) in advance. When we apply induction to \( G_2 \), then the precolored path has 5 vertices. So the only problem that may occur is that two of the edges are monochromatic. If \( \beta = 2 \), we prevent this by adding a new vertex of color 2 joined to \( u, s \) before we apply induction to \( G_1 \). So assume that \( \beta \neq 2 \). Then the only problem that may occur is that \( s, v_i \) are both colored \( \beta \), while \( u, vqvq_1 \) are both colored 2. We prevent this by adding a vertex with the two available colors 2, \( \beta \) joined to \( vqvq_1, u, s, v_i \) and also adding the edge \( vqvq_1v_i \) before we apply induction to \( G_1 \). This leaves only one problem: This extension of \( G_1 \) equals \( G \). In particular, \( \beta = 1 \). We can choose the notation such that \( \alpha = 3 \). As the 2-path \( usvq \) is bad when we give \( u \) the color 3, we conclude that \( L(s) = \{1, 3\} \). As the above extension of \( G_1 \) equals \( G \), it follows that \( G \) has only one vertex inside the 4-cycle \( vqvq_1usvq \). That vertex must be \( w \). This proves Claim 6. \( \square \)

Let \( vq_{q+1}, u_1, u_2, \ldots, u_p, s, w, vq_1 \) be the neighbors of \( vq \) in clockwise order. Put \( u_0 = vq_{q+1} \), and \( u_{p+1} = s \).

**Claim 7.** \( p \geq 1 \), and \( L(u_p) = \{1, 3\} \).

**Proof.** Suppose (reductio ad absurdum) that Claim 7 is false. If \( s = vq_{q+1} \), then we give \( u \) the color 3, we give \( w \) the color 2, and we delete the vertices \( w, vq \). So, the induction hypothesis leads to a contradiction because there is no forbidden bad 2-path \( uzvq \) from \( u \) to \( A_0 \cup A_1 \). (Indeed,
the last statement in Claim 6 says that such a path would end in \( v_q \), but \( v_q \) has been deleted.) Hence \( s \neq v_q + 1 \). We now give \( s \) the color 3, we give \( w \) the color 1 and we delete \( w \) and the edges \( sv_q, v_qv_{q-1} \) and add instead the edge \( sv_{q-1} \). There must be a new bad 2-path which prevents us from using the induction hypothesis. (Otherwise, we obtain a contradiction.) By Claim 5, there can be only one such bad 2-path, namely \( su_pv_q \). (By Claim 2, there is no edge from \( s \) to \( P - v_q \).) Now \( u_p \neq v_q + 1 \), since otherwise, we would also delete \( v_q \) and obtain a contradiction. Hence \( p \geq 1 \). Since \( su_pv_q \) is bad, we conclude that \( L(u_p) = \{1, 3\} \).

**Claim 8.** Each list \( L(u_i) \), \( 1 \leq i \leq p \), contains the color 1.

**Proof.** Suppose (reductio ad absurdum) that some \( L(u_i) \) does not contain the color 1. Then we delete the edge \( u_iv_q \), and we split \( v_q \) up into two vertices, one of which is joined to all \( u_j \) with \( j < i \). The other is joined to all \( u_j \) with \( j > i \) and also to \( w, v_{q-1} \). The resulting graph has fewer edges than \( G \), and therefore the induction hypothesis leads to a contradiction which proves Claim 8.

Claim 6 says, among other things, that \( u \) is not joined to any precolored vertex distinct from \( v_{q-1} \) except possibly \( v_{q-2} \). Claims 2, 5 imply that

**Claim 9.** None of \( u_0, u_1, \ldots, u_p, s \) is joined to a precolored vertex distinct from \( v_q \).

By Claim 1, there is no edge \( u_iu_j \) with \( 0 \leq i < j - 1 \leq p \). Furthermore,

**Claim 10.** There is no edge \( u_iu \) with \( 0 \leq i \leq p \).

**Proof.** Suppose (reductio ad absurdum) that there is an edge \( u_iu \) with \( 0 \leq i \leq p \). If \( P = C \), then \( i \geq 1 \), by Claim 2. Now we give \( u \) the color 3, we delete the interior of the 4-cycle \( v_qv_{q-1}uuiv_q \), we delete the edge \( v_qv_{q-1} \), and then we apply the induction hypothesis. (This is possible because the last statement of Claim 6 implies that the resulting graph has no bad 2-path starting at \( u \).) After that we apply the induction hypothesis to the interior of the 4-cycle \( v_qv_{q-1}uuiv_q \). This contradiction proves Claim 10.

Now the idea is to give each of \( u_1, u_3, \ldots \) (or each of \( u_2, u_4, \ldots \) if \( P = C \)) a color distinct from 1 and then delete \( v_q \) because it cannot possibly be in a monochromatic triangle. We shall refer to this coloring procedure as the **coloring scheme**. We may create a bad 2-path in this way, and therefore we define a 4-cycle to be **problematic** if it has the form \( v_q u_i v_j v_q \), where \( 1 \leq i < j - 1 \leq p \), and furthermore, \( L(u_i) = \{1, \alpha\} \), \( L(u_j) = \{1, \beta\} \), and \( L(t) = \{\alpha, \beta\} \). (In particular, \( \alpha \neq \beta \).)

We say that the problematic 4-cycle \( v_q u_i v_j v_q \) is of **type 1** if it has the following property: If we give \( u_i \) the color \( \alpha \) (the available color distinct from the color of \( v_q \)), then that coloring can be extended to the interior of \( v_q u_i v_j v_q \) without creating monochromatic triangles for any choice of colors of \( u_j, t \) (chosen among the available colors). Otherwise we say it is of **type 2**.

**Claim 11.** If the problematic 4-cycle \( v_q u_i v_j v_q \) is of type 1, and if we give \( u_j \) the available color distinct from the color of \( v_q \), then that coloring can be extended to the interior of \( v_q u_i v_j v_q \) without creating monochromatic triangles for any choice of available colors of \( u_i, t \).
Proof. Suppose (reductio ad absurdum) that if we give \( u_j \) the color \( \beta \), then there is a coloring of \( u_i, t \) which cannot be extended to the interior of \( v_qu_i tu_j v_q \) without creating a monochromatic triangle. Then this 4-cycle must have two monochromatic edges, by the induction hypothesis. Hence \( u_i \) must have color 1, and \( t \) must have color \( \beta \). The unique vertex which is inside \( v_qu_i tu_j v_q \) and which is joined to \( u_i, t \) must have the available colors 1, \( \beta \), since otherwise we delete the edge \( u_i t \) and color the interior of \( v_qu_i tu_j v_q \) using the induction hypothesis (because we create no bad 2-path). Now we use the assumption that \( v_qu_i tu_j v_q \) is not of type 1. This implies that if we give \( u_i \) the color \( \alpha \), then there is a coloring of \( u_j, t \) which cannot be extended to the interior of \( v_qu_i tu_j v_q \). Then this 4-cycle must have two monochromatic edges, by the induction hypothesis. Hence \( u_j \) must have color 1, and \( t \) must have color \( \alpha \). Now we delete the edge \( u_i t \) and color the interior of \( v_qu_i tu_j v_q \) using the induction hypothesis. Note that when we put the edge \( u_j t \) back we create no monochromatic triangle because \( \alpha \neq \beta \). This contradiction proves Claim 11. \( \square \)

We now return to the coloring scheme, that is, the idea of giving each of \( u_1, u_3, \ldots \) (or \( u_2, u_4, \ldots \) if \( P = C \)) a color distinct from \( 1 \) and then delete \( v_q \). However, we modify the sequence \( u_1, u_3, \ldots \) (or \( u_2, u_4, \ldots \)) as follows. The first vertex we color is \( u_1 \) (or \( u_2 \) if \( P = C \)) unless there exists a problematic 4-cycle \( v_qu_i tu_j v_q \) of type 2 (or \( P = C \) and there exists a 4-cycle \( v_qv_q+1tu_jv_q \)). If such a cycle exists, then we choose one such that \( j \) is maximum. First we color \( u_j \) with the color different from 1. We delete the interior of the 4-cycle \( v_qu_i tu_j v_q \). (Note that it is safe to do so by the definition of type 2.) Then we split \( v_q \) into two vertices. More precisely, we delete the edges \( v_qu_r \) for \( 0 \leq r \leq i \). Then we add a new vertex \( v'_q \) with color 1 and all edges from \( v'_q \) to \( u_i, u_{i-1}, \ldots, u_0 \). The new vertex \( v'_q \) will not be the end of a forbidden bad 2-path. Then we color \( u_j, u_{j+2}, \ldots \) and after that we delete \( v_q \). The only problem is that we may create forbidden bad 2-paths. The path \( u_j u_{j+1} u_{j+2} \) is not bad because \( u_{j+1} \) has the available color \( 1 \) and none of \( u_j, u_{j+2} \) have that color. It is only the problematic 4-cycles that can create new bad 2-paths. They are all of type 1 by the maximality of \( j \). So if we have colored a vertex \( u_a \) (with the available color distinct from 1), then we give \( u_{a+2} \) the color distinct from 1 unless \( u_a \) is part of a problematic 4-cycle \( u_a u_b v_q u_a \) with \( b > a + 1 \) in which case we choose \( b \) to be maximum. In this case we do not color \( u_{a+2} \) but we delete the interior of \( u_a u_b v_q u_a \), we give \( u_{b+1} \) the color distinct from 1, and then we continue the color scheme from there. If \( b < a \), then the reason we gave \( u_a \) a color is that \( u_b \) was not colored, and then the problematic 4-cycle \( u_a u_b v_q u_a \) does not create a new forbidden bad 2-path and we may therefore ignore it. In other words, the problematic 4-cycles may have a complicated intersection pattern, but we only pay attention to those which do not contain any previously colored vertex in the color scheme, and this is sufficient in order to avoid forbidden bad 2-paths.

If \( s = u_{p+1} \) is colored in this way, then we give \( w \) the color 1, we delete \( w, v_q \) and add the edge \( v_q u_{p+1} \), and we color the resulting graph and hence also \( G \), by the induction hypothesis, a contradiction. (If \( P = C \), then \( u_1 \) has degree 4, by Claim 6. If there is no problematic 4-cycle of type 2, then the coloring scheme starts with \( u_2 \), we delete \( u_1 \), and we add the edge \( v_q u_2 \) if we create a forbidden bad 2-path between these vertices. Possibly we also create a forbidden bad 2-path from \( v_{q+1} \) to some \( u_t \) with \( t > 2 \). In that case we choose \( t \) to be maximum and add instead the edge \( v_{q+1} u_t \) and delete the interior of the separating 4-cycle containing the forbidden bad 2-path. The number of vertices in the new precolored path is then at most 5, because \( v_q \) will be deleted, and there is no monochromatic edge. If there is some problematic 4-cycle of type 2, then \( v'_q \) will be present, and the new precolored path has at most 6 vertices, and there is no monochromatic edge.)
So we may assume that $u_{p+1}$ is not colored by the coloring scheme. There are two possible reasons for that: either $u_p$ is colored, or $u_{p+1}$ is part of a problematic 4-cycle of type 1 having a vertex that has already been colored. (The coloring scheme then terminates at that vertex.) Then we give $u$ the color 3, we give $w$ the color 2, and we delete $w$, $v_q$ (but we keep the vertex $v'_q$ in case a problematic 4-cycle of type 2 exists). If the resulting graph can be colored without creating monochromatic triangles, then we would have a contradiction, so we may assume that we have created a forbidden bad 2-path from $u$ to a precolored vertex not in $P$.

By the last statement in Claim 6, this bad 2-path is not of the form $uvr$ where $v_r$ is in $A_0 \cup A_1$. So, the new bad 2-path must be of the form $uatu_a$ where $u_a$ has been colored $\alpha$, say, and $L(t) = \{3, \alpha\}$. We choose $a$ to be smallest possible. As $\alpha \neq 3$, we have $a < p$, by Claim 7. We now uncolor all the colored vertices inside the 5-cycle $v_qv_{q-1}atu_av_q$ (or equivalently, we stop the coloring scheme at $u_a$). And, we do not color $u$. We delete the interior of the 5-cycle $v_qv_{q-1}atu_av_q$, and we also delete the edge $v_qv_{q-1}$. We add instead the edge $v_q-1t$. (This does not create a forbidden bad 2-path $v_q-1tv_c$ since otherwise $v_qu_atv_c$ would violate Claim 5.)

We use induction to the resulting graph. Then we focus on the interior of the 5-cycle $v_qv_{q-1}atu_av_q$ and we delete the edge $v_q-1t$. (The purpose of the edge $v_q-1t$ was to ensure that $t, u$ do not both get the color 2.)

If $u$ receives the color 3, then we give $w$ the color 2 and delete $w$ and $v_{q-1}$. We add the edge $uvq$ and apply induction to the interior of the 4-cycle $uvqatu$, a contradiction. So we may assume that $u$ receives the color 2. The edge $v_{q-1}t$ then ensures that $t$ does not have the color 2. Then we give $w$ the color 1 and delete it, and we give $s$ the color 3. If possible, we apply the induction hypothesis to the 5-cycle $usv_quatu$ and its interior. If we can apply the induction hypothesis, we would have a contradiction, so we may assume that it is not possible to apply the induction hypothesis. This implies that the 5-cycle $usv_quatu$ has a monochromatic edge. The only possibility is that $t, u_a$ have the same color $\alpha$, say. As $t$ does not have the color 2, we may assume that $\alpha = 4$.

We now return to the part of the proof where we deleted the interior of the 5-cycle $v_qv_{q-1}atu_av_q$, and we also deleted the edge $v_qv_{q-1}$, and we did not color $u$. We added the edge $v_{q-1}t$ which, however, we do not do this time. Instead we add the edge $v_{q-1}u_a$, and we also add a new vertex $z$ joined to $v_{q-1}, u, t, u_a$ and with the available colors 2, 4. We apply induction to the resulting graph (in which also $v_q$ has been deleted) and repeat the reasoning above. In the reasoning above we used the edge $v_{q-1}t$ to ensure that $t, u$ do not both have the color 2. This is automatically satisfied now because $t$ does not have 2 as an available color. We obtained a contradiction unless $u$ had color 2, and $t, u_a$ both had color 4. Note that this cannot happen now because of the new vertex $z$.

This proves Theorem 1. □

Acknowledgment

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