Boolean Packings in Dowling Geometries

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An explicit Boolean packing of the Dowling lattice is constructed.

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1. INTRODUCTION

A Boolean packing of a finite graded poset $P$ is a partitioning of $P$ into disjoint Boolean algebras subject to the conditions that the difference between the ranks, in $P$, of the top and bottom elements of each Boolean algebra is equal to the rank of the Boolean algebra and that the sum of the ranks, in $P$, of the top and bottom elements of each Boolean algebra is greater than or equal to the rank of $P$. Björner [1, Exercise 7.36(b)] asked whether every geometric lattice has a Boolean packing. In this paper we give explicit Boolean packings for the class of Dowling lattices. As a consequence we obtain an identity expanding the rank numbers of Dowling lattices in terms of the binomial coefficients.

2. BOOLEAN PACKINGS

For nonnegative integers $n, k$, let $B_n$ denote the graded poset of subsets of $[n] (= \{1, 2, \ldots, n\})$ (ordered by inclusion) and let $\binom{n}{k}$ denote the number of rank-$k$ subsets (i.e., subsets of cardinality $k$) in $B_n$. Extend the definition of $\binom{n}{k}$ to all pairs of integers by defining $\binom{n}{k}$ to be 0 when $n < 0$ or $k < 0$.

Let $P$ be a finite rank-$n$ graded poset with rank function $\rho : P \to \{0, 1, \ldots, n\}$. For $0 \leq k \leq n$, let $P_k$ denote the set of elements of $P$ of rank $k$ and let $W_k = \text{Card } P_k$.

We say that the elements $x_1, x_2, \ldots, x_h$ of $P$ form a symmetric chain if $x_{i+1}$ covers $x_i$ for every $i < h$ and $\rho(x_1) + \rho(x_h) = n$. A symmetric chain decomposition of $P$ is a covering of $P$ by pairwise disjoint symmetric chains.

We say that a subset $U \subseteq P$ is upper Boolean if

(i) $U$, under the induced order, has a minimum element, say $x$, and a maximum element, say $x'$.

(ii) $U$ is order-isomorphic to $B_{\rho(x')-\rho(x)}$.

(iii) $\rho(x') + \rho(x) \geq n$.

A Boolean packing of $P$ is a covering of $P$ by pairwise disjoint upper Boolean subsets.

EXAMPLE 2.1. Let $G$ be the graph obtained from the complete graph $K_4$ by dropping an edge. Denote the edge set of $G$ by $\{a, b, c, d, e\}$, where $\{a, b, e\}$ and $\{c, d, e\}$ are the two 3-cycles in $G$. Let $P$ denote the geometric lattice of flats of the cycle matroid of $G$. Then the elements of $P$ are: $\emptyset, a, b, c, d, e, abc, ac, ad, bc, bd, cde, abcde$. As the edges $a,b,c$ form a spanning tree of $G$, the set $U_1 = \{\emptyset, a, b, c, abcde\}$ of flats generated by the eight subsets of $\{a, b, c\}$ is upper Boolean in $P$. Now $P - U_1$ can be further partitioned into upper Boolean subsets $U_2 = \{e, cde\}, U_3 = \{d, ad\}$, and $U_4 = \{bd\}$. 
Let $P$ be Boolean packable. Partition $P$ into upper Boolean subsets $U_1, U_2, \ldots, U_h$. Let $x_i$ and $x'_i$ denote, respectively, the minimum and maximum elements of $U_i$, $i = 1, \ldots, h$. Put $\tau(i) = \rho(x'_i) + \rho(x_i) - n$. We see that (ii) above implies the identity

$$W_k = \sum_{i=1}^{h} \binom{\rho(x'_i) - \rho(x_i)}{k} = \sum_{i=1}^{h} \binom{n - 2\rho(x_i) + \tau(i)}{k - \rho(x_i)}.$$ 

It follows from (iii) above that $\tau(i) \geq 0$ for all $i$.

From the formula above and elementary properties of the binomial coefficients we see that if $P$ admits a Boolean packing then $W_k \leq W_{k+1}, k < n/2$ and $W_k \leq W_{n-k}, k < n/2$. De Bruijn, Tengbergen, and Kruyswijk [2] constructed a symmetric chain decomposition of $B_k$, for all $l \geq 0$. It follows that if $P$ has a Boolean packing then the bottom half of $P$ can be covered by pairwise disjoint symmetric chains. In particular, there exist order-matchings of $P_k$ into $P_{k+1}, k < n/2$ and of $P_k$ into $P_{n-k}, k < n/2$.

Several recent papers have dealt with order-matchings in partition and Dowling lattices. Kung [7] proved that the incidence matrix of rank $k$ vs. rank $k+1$ elements ($k < n/2$) in a rank-$n$ Dowling lattice has full row rank. Thus there are order-matchings from rank $k$ into rank $k+1$, $k < n/2$. In [3] Canfield proved the stronger result that, for large $n$, there are order-matchings in the partition lattice of an $n$-element set from rank $k$ into rank $k+1$ for $k < K_n$, where $K_n \sim n(1 - (\log 4/\log n))$. Loeb, Damiani, and D’Antona [5] gave an explicit covering of the bottom half of the partition lattice by pairwise disjoint symmetric chains. An inductive proof of this result was independently given in [9]. Mason [8] (see also Kung [6]) proved that the points of any geometric lattice can be covered by pairwise disjoint symmetric chains. In Section 2 we construct an explicit Boolean packing of the Dowling lattice.

### 3. Partial $G$-Partitions

Let $G$ be a finite (multiplicative) group with identity $1$ and let $m = \text{Card } G$. A partial partition of $[n]$ is a set of nonempty pairwise disjoint subsets of $[n]$. A partial $G$-partition of $[n]$ is a set $\pi = \{a_1, \ldots, a_t\}$ of functions satisfying

(i) For $j = 1, \ldots, t$, $a_j$ is a function from a nonempty subset $D(a_j)$ of $[n]$ to the group $G$.

(ii) $\{D(a_1), \ldots, D(a_t)\}$ is a partial partition of $[n]$.

The sets $D(a_j)$, $j = 1, \ldots, t$, are called the blocks of $\pi$ and we say that $\pi$ has $t$ blocks. Let $Q(G)$ denote the set of all partial $G$-partitions of $[n]$.

Let $\pi = \{a_1, \ldots, a_t\} \in Q(G)$. A function $a : B \to G$, $B \subseteq [n]$ is said to be a (left) linear combination (over $G$) of $\pi$ if $B = D(a_1) \cup \cdots \cup D(a_t)$ and there exist $g_1, \ldots, g_t \in G$ such that, for all $j = 1, \ldots, t$, we have $a(x) = g_j a_j(x), x \in D(a_j)$.

Define a binary relation $\leq_d$ on $Q(G)$ as follows: $\pi \leq_d \sigma$ if for every $b_j$ is a linear combination of some subset of $\pi$. The relation $\leq_d$ is easily seen to be reflexive and transitive, hence is a preorder on $Q(G)$. Define an equivalence relation $\sim$ on $Q(G)$: $\pi \sim \sigma$ if and only if $\pi \leq_d \sigma$ and $\sigma \leq_d \pi$. Let $[\pi]$ denote the $\sim$-class containing $\pi$. If $\pi = [\sigma]$ then it is easily seen that $\pi$ and $\sigma$ have the same blocks.

Let $Q_{\ast}(G) = Q(G)/\sim$ and define a partial order $\leq$ on $Q_{\ast}(G)$ as follows: $\pi \leq \sigma$ if and only if $\pi \leq_d \sigma$. $Q_{\ast}(G)$ is called a Dowling lattice. It is shown in [4] that $Q_{\ast}(G)$ is a rank-$n$ geometric lattice with rank function given by: rank of $\pi = n - \text{number of blocks of } \pi$.

For nonnegative integers $n, k$, let $T_m(n, k)$ denote the number of $A$-classes of partial $G$-partitions in $Q_{\ast}(G)$ having $k$ blocks (note that this number depends only on the cardinality $m$ of $G$).
We need to make a few more definitions. A \emph{composition} is a finite sequence \( \lambda = (\lambda_1, \ldots, \lambda_t) \) of positive integers. The terms \( \lambda_i \) are called the \emph{parts} of \( \lambda \) and we say that \( \lambda \) has \( t \) parts. The sum of the parts of a composition \( \lambda \) is denoted \( |\lambda| \). We assume the existence of a unique composition of 0, with zero parts.

Let \( \lambda = (\lambda_1, \ldots, \lambda_t) \) be a composition and let \( t \) be the number of parts of \( \lambda \) that are \( \geq 2 \).

Define the \emph{rank}, \emph{bias}, and \emph{weight} of \( \lambda \), denoted \( r(\lambda), b(\lambda) \) and \( w(\lambda) \), respectively, as follows:

\[
\begin{align*}
    r(\lambda) &= |\lambda| - t, \\
    b(\lambda) &= |\lambda| - t - 1, \\
    w(\lambda) &= \prod_{i=1}^{t} \left( |\lambda| - \lambda_{i+1} - \lambda_{i+2} - \cdots - \lambda_t - 1 \right),
\end{align*}
\]

where the empty product is taken to be 1. Note that, for all compositions \( \lambda \), \( r(\lambda), b(\lambda) \geq 0 \). For a nonnegative integer \( n \), let \( C(n) \) denote the set of all compositions \( \lambda \) satisfying \( |\lambda| = n \).

We now construct a Boolean packing of \( Q_n(G) \). By a nontrivial (resp. trivial) \emph{block} of a partial \( G \)-partition we mean a block of cardinality \( \geq 2 \) (resp. \( = 1 \)). Define

\[
\begin{align*}
    QB_n(G) &= \{ \pi \in Q_n(G) : \text{Union of the blocks of } \pi \text{ is } [n] \}, \\
    QT_n(G) &= \{ \pi \in Q_n(G) : \pi \text{ has no trivial blocks} \}.
\end{align*}
\]

(Here, and below, it is easily seen that the definitions make sense).

Given \( \pi \in Q_n(G) \) define \( R(\pi) \in QB_n(G) \), called the \emph{restriction} of \( \pi \), and \( E(\pi) \in QT_n(G) \), called the \emph{extension} of \( \pi \), as follows: there is a unique \( \pi' \in QB_n(G) \) having the same nontrivial blocks as \( \pi \) and satisfying \( \pi' \leq \pi \). Define \( R(\pi) = \pi' \). Similarly, there is a unique \( \pi' \in QT_n(G) \) having the same nontrivial blocks as \( \pi \) and satisfying \( \pi' \leq \pi \). Define \( E(\pi) = \pi' \). Informally, \( E(\pi) \) is obtained from \( \pi \) by dropping the trivial blocks and \( R(\pi) \) is obtained from \( \pi \) by adding all possible trivial blocks. Note that if two elements of \( Q_n(G) \) have the same restrictions then they also have the same extensions.

**Example 3.1.** When \( G \) is the one element group we write \( Q_n, QB_n(G) \), and \( QT_n(G) \) for \( Q_n(G), QB_n(G) \), and \( QT_n(G) \) respectively.

(i) The elements of \( QB_3(G) \) are (in obvious notation): \( 1|2|3|4, 12|3|4, 2|13|4, 2|3|14, 1|23|4, 1|3|24, 1|2|34, 12|34, 13|24, 23|14, 123|4, 2|134, 3|124, 1234, 1234 \). (Here \( \emptyset \) denotes the maximum element of \( Q_n \), with zero blocks.)

(ii) The elements of \( QT_3(G) \) are: \( \emptyset, 12, 13, 14, 23, 24, 34, 12|34, 13|24, 23|14, 123, 134, 124, 234, 1234 \). (Here \( \emptyset \) denotes the maximum element of \( Q_n \), with zero blocks.)

(iii) Let \( \pi = 4|25|36|8 \in Q_9 \). Then \( R(\pi) = 1|4|25|36|7|8|9 \) and \( E(\pi) = 25|36 \).

Let \( \pi \in QB_n(G) \). Write \( \pi = \{a_1, \ldots, a_t\} \) with max \( D(a_1) < \max D(a_2) < \cdots < \max D(a_t) \). Let \( \lambda_i = \text{Card } D(a_i), i = 1, \ldots, t \). The composition \( (\lambda_1, \lambda_2, \ldots, \lambda_t) \in C(n) \) is called the \emph{type} of \( \pi \).

**Proposition 3.2.** The number of elements in \( QB_n(G) \) of type \( \lambda \) is \( w(\lambda)m^{w(\lambda)} \), \( \lambda \in C(n) \).

**Proof.** Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t) \). We can construct a \( \pi \in QB_n(G) \) of type \( \lambda \) as follows:

(i) In the first step we choose the blocks of \( \pi \). Put \( C_t = [n] \) and choose a subset \( X_t \) of \( C_t - \{\max C_t\} \) of cardinality \( \lambda_t - 1 \) and put \( B_t = X_t \cup \{\max C_t\} \). Now put \( C_{t-1} = C_t - B_t \) and choose a subset \( X_{t-1} \) of \( C_{t-1} - \{\max C_{t-1}\} \) of cardinality \( \lambda_{t-1} - 1 \) and put \( B_{t-1} = X_{t-1} \cup \{\max C_{t-1}\} \). Continuing in this fashion we obtain \( B_{t-2}, \ldots, B_1 \). The blocks of \( \pi \) are \( B_1, \ldots, B_t \).
(ii) In this step we choose functions \( a_j : B_j \rightarrow G, j \in [l] \).

In step (i) above the blocks can be chosen in
\[
\left( \frac{n-1}{\lambda_1-1} \right) \left( \frac{n-\lambda_2-1}{\lambda_2-1} \right) \cdots \left( \frac{n-\lambda_t-1}{\lambda_t-1} \right) \equiv w(\lambda)
\]
ways. In step (ii) above we need to count the number of \( A \)-classes of functions. Thus we fix \( a_j(\max B_j) = 1 \) for all \( j \in [l] \). With this provision the number of choices in step (ii) is
\[
m^{k-l} = m^{\tau(\lambda)}.
\]
The proposition follows.

\( \square \)

**Proposition 3.3.**

(i) Let \( \pi \in \mathcal{Q}_B(G) \) and let \( \lambda \in C(n) \) be the type of \( \pi \). Then the subset \( \{ \pi \in \mathcal{Q}_n(G) : R(\pi) = \pi \} \) of \( \mathcal{Q}_n(G) \) is upper Boolean with minimum rank \( r(\lambda) \) and maximum rank \( n-r(\lambda)+b(\lambda) \). Denote this subset by \( \langle \pi, \mathcal{E}(\pi) \rangle \).

(ii) The subsets \( \langle \pi, \mathcal{E}(\pi) \rangle, \pi \in \mathcal{Q}_B(G) \) partition \( \mathcal{Q}_n(G) \).

**Proof.**

(i) Let \( \lambda = (\lambda_1, \ldots, \lambda_t) \), with \( t \) parts \( \geq 2 \). Then \( r(\lambda) = n-t \) and \( b(\lambda) = n-l-t \). Let \( S \) denote the union of the trivial blocks of \( \pi \) and let \( U = \{ \pi \in \mathcal{Q}_n(G) : R(\pi) = \pi \} \). Choose \( \sigma, \tau \in U \). Then \( \sigma \) and \( \tau \) have the same nontrivial blocks and the corresponding functions, from these blocks into \( G \), are \( A \)-equivalent. It follows that \( U \) is order-isomorphic to the poset of subsets, under inclusion, of \( S \) (given \( \sigma \in U \), its image under this isomorphism is \( (S - \text{union of the trivial blocks of } \sigma) \)).

The rank, in \( \mathcal{Q}_n(G) \), of the minimum element \( \pi \) of \( U \) is \( n - \text{number of blocks of } \pi = n - t = r(\lambda) \).

The rank, in \( \mathcal{Q}_n(G) \), of the maximum element \( \mathcal{E}(\pi) \) of \( U \) is \( n - \text{number of nontrivial blocks of } \pi = n - l = n - r(\lambda) + b(\lambda) \).

As \( r(\lambda) + (n-r(\lambda) + b(\lambda)) \geq n \) and \( \text{Card } S = l = (n-r(\lambda) + b(\lambda)) - r(\lambda) \) it follows that \( U \) is upper Boolean.

(ii) Given \( \pi \in \mathcal{Q}_n(G) \), let \( \pi = \mathcal{R}(\pi) \). Then \( \pi \in \langle \pi, \mathcal{E}(\pi) \rangle \). An argument similar to that used in part(i) shows that the subsets \( \langle \pi, \mathcal{E}(\pi) \rangle, \pi \in \mathcal{Q}_B(G) \) are disjoint.

\( \square \)

**Theorem 3.4.** \( \mathcal{Q}_n(G) \) is Boolean packable and
\[
T_m(n, n-k) = \sum_{\lambda \in C(n)} w(\lambda)m^{\tau(\lambda)} \left( \frac{n-2r(\lambda)+b(\lambda)}{k-r(\lambda)} \right).
\]

**Proof.** This follows from Propositions 3.2 and 3.3.

\( \square \)

**Example 3.5.** We see from Example 3.1 that the Boolean packing of \( \mathcal{Q}_4 \) given by the theorem above will have the following 15 upper Boolean subsets: \( 1 \{ 2 \} 3 \langle 4, \emptyset \rangle \), \( 2 \{ 3 \} 4, 12 \), \( 2 \{ 1 \} 3, 4, 13 \), \( 2 \{ 1 \} 3, 4, 14 \), \( 1 \{ 2 \} 3, 4, 23 \), \( 1 \{ 3 \} 2, 4, 24 \), \( 1 \{ 2 \} 3, 4, 34 \), \( 2 \{ 1 \} 3, 4, 12 \), \( 2 \{ 1 \} 3, 4, 23, 14 \), \( 1 \{ 2 \} 3, 4, 12, 23 \), \( 1 \{ 2 \} 3, 4, 12, 23, 14 \), \( 1 \{ 2 \} 3, 4, 12, 23, 14 \), \( 1 \{ 2 \} 3, 4, 12, 23, 14 \), \( 1 \{ 2 \} 3, 4, 12, 23, 14 \).

We list a few of these subsets below:
\[
\begin{align*}
2 \{ 1 \} 3, 4, 13 &= 2 \{ 1 \} 3, 4, 12 \{ 3 \} 4, 13, \\
1 \{ 2 \} 3, 4, 23 &= 1 \{ 2 \} 3, 4, 23, 14, \\
2 \{ 1 \} 3, 4, 12, 23, 14 &= 2 \{ 1 \} 3, 4, 12, 23, 14.
\end{align*}
\]

Keeping track of the ranks of the various Boolean algebras in the packing above we can write it symbolically as \( B_4 \oplus B_2 \oplus B_1 \oplus B_0 \). Thus the number of elements in \( \mathcal{Q}_4 \) (equivalently, the number of partitions of a five element set) is equal to \( 16 + 6.4 + 4.2 + 4.1 = 52 \).
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